J. Aust. Math. Soc. **95** (2013), 289–296
doi:10.1017/S1446788713000281

THE CLASSIFICATION OF COMMUTATIVE TORSION FILIAL RINGS

R. R. ANDRUSZKIEWICZ and K. PRYSZCZEPKO ✉

(Received 18 December 2012; accepted 6 March 2013; first published online 18 July 2013)

Communicated by B. Gardner

Abstract

The aim of this paper is to give a classification theorem for commutative torsion filial rings.

2010 *Mathematics subject classification*: primary 16D25; secondary 13C05, 13B02.
Keywords and phrases: ideal, filial ring.

1. Introduction

All considered rings are associative but do not necessarily have identity. We say that a ring R is an H-ring if all its subrings are ideals. H-rings were investigated by many authors (see [1, 2, 9–12]). A detailed description of the structure of torsion H-rings turned out to be one of the most difficult problems.

A classification of H-rings was obtained independently by Kruse in his dissertation [9] and by Andrijanov in [2]. Andrijanov showed that there are sixteen types of these rings [2, Theorem 2]. Unfortunately, the still unanswered question is whether there exists any isomorphism between any two rings from the same class. This problem seems complicated because a variety of parameters define these classes.

A ring R is called filial (left filial), if for any ideal (left ideal) J of R, and any ideal (left ideal) I of J, I is an ideal (left ideal) of R. The notion of a filial ring is a natural generalization of the notion of an H-ring. Filial rings and left filial rings were investigated by many authors (see [5–8]). For example, Filipowicz and Puczyłowski in [8] obtained the structure of left filial algebras over a field.

However, the problem of a classification of filial rings is much more complicated and subtle. So far, the most important results concern the case of commutative filial rings (see [3–5]).

The purpose of this paper is to give a complete classification of commutative torsion filial rings. The main theorem of this work (Theorem 4.1) is a surprising analogue of [8, Theorem 4.3]. Nevertheless, our proof is quite different from the one in [8] and requires fundamentally new ideas and methods.

2. Preliminary results

Throughout the paper, $\mathbb{N}$ and $\mathbb{P}$ stand for the set of all positive integers and the set of all primes, respectively. For a ring R, we denote by $N(R)$ the nilradical of R, and by R^+ the additive group of R. We write $o(x)$ for the order of an element x of the group R^+. For $p \in \mathbb{P}$ we let $R_p = \{x \in R : p^k x = 0 \text{ for some } k \in \mathbb{N}\}$ and $R(p) = \{x \in R : px = 0\}$. We say that a ring R is of bounded exponent if there exists $M \in \mathbb{N}$ such that $Mx = 0$ for every $x \in R$, otherwise we say that R is of unbounded exponent. We say that a ring R is a p-ring if R^+ is a p-group for a prime number p. If R is both a p-ring and an H-ring, we shall say that R is an H-p-ring. For a subset S of a ring R, we denote by $\langle S \rangle$ the subgroup of R^+ generated by S, and by $l_R(S)$ the left annihilator of S in R.

The term almost null ring was introduced by Kruse in [10]. These rings play an important role in the study of certain H-rings.

Definition 2.1 [10, Definition 2.1]. We say that a ring R is **almost null** if for every $a \in R$:

(i) $a^3 = 0$;
(ii) $Ma^2 = 0$ for some square-free integer M ($M = M_a$);
(iii) $aR + Ra \subseteq \langle a^2 \rangle$.

Clearly, every almost null ring R is an H-ring such that $R^3 = 0$. Moreover, every homomorphic image and every subring of an almost null ring are almost null. The importance of this notion lies in the following proposition.

Proposition 2.2 [10, Proposition 2.5]. *A nil p-ring R of an unbounded exponent is an H-ring if and only if R is almost null.*

We begin by recalling a few well-known facts.

Proposition 2.3 [7, Corollary 2.3]. *A commutative nil ring R is filial if and only if R is an H-ring.*

Lemma 2.4 (see [8, Theorem 3.3]). *Let R be a nil H-ring such that $pR = 0$ for a prime p. Then R is almost null.*

Remark 2.5. Let C be a commutative ring with identity 1 and let A be a C-algebra. We denote by $(1_C, A)$ the C-algebra obtained from A by adjoining an identity 1 of C. Obviously, $(1_C, A)^+ = C^+ \oplus A^+$. For any $c \in C$, $a \in A$ we write $c + a$ instead of the pair (c, a). According to this notation we have $A \lhd (1_C, A)$ and $(1_C, A)/A \cong C$. It is also clear that if A is commutative, then the algebra $(1_C, A)$ is commutative too. Moreover, if A possesses an identity, then $(1_C, A) \cong C \oplus A$. Note that every ring is a $\mathbb{Z}$-algebra in a natural way.

Definition 2.6. We say that R is a K_0**-ring** if R is a commutative filial ring with identity, such that $N(R) \neq 0$ and $R/N(R)$ is a field.

https://doi.org/10.1017/S1446788713000281 Published online by Cambridge University Press

In [5] we considered K-rings, that is, noetherian K_0-rings. In that paper we proved the following result, which is important in the description of K_0-rings.

THEOREM 2.7 [5, Theorem 4.3]. *For a given ring R with identity 1, the following conditions are equivalent:*

(i) *R is a K_0-ring;*
(ii) *there exists a commutative almost null ring N such that $N \lhd R$, $pN = 0$ for some $p \in \mathbb{P}$, $R = \langle 1 \rangle + N$, $o(1) = p^m$ for some $m \in \mathbb{N}$, and if $m = 1$, then $N \neq 0$.*

A detailed study of a classification of K-rings, (especially the proof of [5, Theorem 4.5]) enables us to obtain a similar classification of K_0-rings.

THEOREM 2.8. *The rings described in Examples 2.9–2.11 are all K_0-rings (up to isomorphism).*

EXAMPLE 2.9 (see [5, Example 1]). Let $n \in \mathbb{N}$, $p \in \mathbb{P}$ and let N be a commutative almost null ring such that $pN = 0$. If $n = 1$ then we additionally assume that $N \neq 0$. Then N is a $\mathbb{Z}_{p^n}$-algebra with a natural external multiplication

$$k \circ a = ka \quad \text{for } k \in \mathbb{Z}_{p^n}, a \in N,$$

and the ring $(1_{\mathbb{Z}_{p^n}}, N)$ is a K_0-ring.

Let $m \in \mathbb{N}$ and let M be a commutative almost null ring such that $pM = 0$. If $m = 1$ then we additionally assume that $M \neq 0$. Then $(1_{\mathbb{Z}_{p^n}}, N) \cong (1_{\mathbb{Z}_{p^m}}, M)$ if and only if $n = m$ and $N \cong M$.

EXAMPLE 2.10 (see [5, Example 2]). Let p be any prime and $m \geq 2$ be a positive integer, and let $t_0 \in \mathbb{Z}_p \setminus \{0\}$. Denote by P the $\mathbb{Z}_{p^m}$-algebra generated by $1, x$ with the relations $px = 0$, $x^2 = t_0 p^{m-1} \cdot 1$. Every element of P can be written as $k + lx$ for uniquely determined $k \in \mathbb{Z}_{p^m}$, $l \in \mathbb{Z}_p$, and P is a filial ring.

Let B be a $\mathbb{Z}_p$-algebra such that $B^2 = 0$. Then B is a P-algebra with external multiplication

$$(k + lx) \circ b = kb \quad \text{for } k \in \mathbb{Z}_{p^m}, l \in \mathbb{Z}_p, b \in B.$$

By Theorem 2.7, the ring $(1_P, B)$ is a K_0-ring. Notice that, if in [5, Example 2] we replace $|B|$ by $\dim_{\mathbb{Z}_p} B$, and use the same arguments, then for $p = 2$, $t_0 = 1$ and for fixed $m \geq 2$ and fixed B there a exists uniquely determined (up to isomorphism) ring $(1_P, B)$, whereas for fixed $p \geq 3$, $m \geq 2$ and B there exist exactly two (up to isomorphism) rings $(1_P, B)$. One of them can be obtained by setting $t_0 = 1$. The other one can be obtained by taking t_0 as an arbitrary nonresidue modulo p.

EXAMPLE 2.11 (see [5, Example 3]). Let p be an odd prime and $m \geq 2$ be a positive integer and let $t_0 \in \mathbb{Z}_p \setminus \{0\}$. Denote by P the $\mathbb{Z}_{p^m}$-algebra generated by elements $1, x, y$ with the relations $xy = yx = px = py = 0$, $x^2 = t_0 p^{m-1} \cdot 1$, $y^2 = \alpha x^2$, where $-\alpha$ is a fixed nonresidue modulo p. Every element of P can be written as $k \cdot 1 + l_1 x + l_2 y$ for uniquely determined $k \in \mathbb{Z}_{p^m}$, $l_1, l_2 \in \mathbb{Z}_p$. From Theorem 2.7 it follows that P is a filial ring.

https://doi.org/10.1017/S1446788713000281 Published online by Cambridge University Press

Let B be a $\mathbb{Z}_p$-algebra such that $B^2 = 0$. Then B is a P-algebra with natural external multiplication

$$(k + l_1 x + l_2 y) \circ b = kb \quad \text{for } k \in \mathbb{Z}_{p^m}, l_1, l_2 \in \mathbb{Z}_p, b \in B.$$

By Theorem 2.7, the ring $(1_P, B)$ is a K_0-ring.

Let C' be a $\mathbb{Z}_p$-algebra with basis $\{x_1, x_1^2, y_1\}$ and the relations $x_1y_1 = y_1x_1 = x_1^3 = 0$, $y_1^2 = \beta x_1^2$ for a nonresidue $-\beta$ modulo p. Let $s_0 \in \mathbb{Z}_p \setminus \{0\}$. Denote by P' the $\mathbb{Z}_{p^{m'}}$-algebra generated by the elements $1, x_1, y_1$ with the relations $x_1y_1 = y_1x_1 = px_1 = py_1 = 0$, $x_1^2 = s_0p^{m'-1} \cdot 1$, $y_1^2 = \beta x_1^2$.

If $(1_P, B) \cong (1_{P'}, B')$, then replacing $|B|$ by $\dim_{\mathbb{Z}_p} B$ in [5, Example 3] and using the same arguments we obtain $m = m'$, $P \cong P'$ and $B \cong B'$.

Conversely, assume that $m = m'$ and let $g\colon B \to B'$ be an isomorphism of rings. Then there exists a nonzero $\gamma \in \mathbb{Z}_p$ such that $\beta = \gamma^2\alpha$, because both $-\alpha$ and $-\beta$ are nonresidues modulo p. It is well known that $\{u^2 + v^2\Delta : u, v \in \mathbb{Z}_p\} = \mathbb{Z}_p$ for a nonzero $\Delta \in \mathbb{Z}_p$. So, there exist $l_1, k_1 \in \mathbb{Z}_p$ such that $t_0 \equiv s_0(l_1^2 + k_1^2\gamma^2\alpha) \mod p$. Moreover, there exists $\gamma' \in \mathbb{Z}_p$ such that $\gamma \cdot \gamma' \equiv 1 \mod p$. Set $l_2 = -\alpha\gamma k_1$, $k_2 = \gamma' l_1$. One can easily check that a function $F\colon (1_P, B) \to (1_{P'}, B')$ given by

$$F(U \cdot 1 + Vx + Wy + b) = U \cdot 1 + (Vl_1 + Wl_2)x_1 + (Vk_1 + Wk_2)y_1 + g(b),$$

where $U \in \mathbb{Z}_{p^m}$, $V, W \in \mathbb{Z}_p$, is an isomorphism of rings.

This shows that for fixed $m \geq 2$ and B there exists a uniquely determined (up to isomorphism) ring $(1_P, B)$. We obtain this ring by setting, for instance, $t_0 = 1$ and taking $-\alpha$ as an arbitrary nonresidue modulo p.

A ring R is strongly regular if $a \in Ra^2$ for every $a \in R$. It is well known that all strongly regular rings are von Neumann regular, and for commutative rings this two properties coincide. The class of all strongly regular rings $\mathbb{S}$ form a radical in the sense of Kurosh and Amitsur. One can easily check that every strongly regular ring is filial.

Lemma 2.12. *Every K_0-ring R is $\mathbb{S}$-semisimple.*

Proof. Assume that $\mathbb{S}(R) \neq 0$. Then $N(R) \cap \mathbb{S}(R) = 0$ and $(N(R) \oplus \mathbb{S}(R))/N(R)$ is a nonzero ideal in the field $R/N(R)$. Hence $N(R) \oplus \mathbb{S}(R) = R$. But R is a ring with identity, so $N(R)$ is also a ring with identity, which is a contradiction. □

3. Useful lemmas concerning idempotents in filial rings

Lemma 3.1. *Let R be a commutative filial ring containing a nil ideal I such that I is a p-ring. Then, for every idempotent $e \in R$, $eI = 0$ or $ei = i$ for every $i \in I$.*

Proof. Suppose the lemma does not hold. Then eI and $J = \{ei - i : i \in I\}$ are nonzero ideals of R contained in I and such that $eI \cap J = 0$. Because I is a nil p-ring, there exist nonzero $a \in eI$ and $b \in J$ such that $a^2 = b^2 = 0$ and $pa = pb = 0$. Hence $ab = 0$, $\langle a\rangle \cap \langle b\rangle = 0$ and this implies $\langle a + b\rangle = [a + b]$. From Proposition 2.3 it follows that I

https://doi.org/10.1017/S1446788713000281 Published online by Cambridge University Press

is an H-ring, so by filiality of R, $\langle a+b\rangle \lhd R$. Therefore, $e(a+b) = k(a+b)$ for some $k \in \mathbb{Z}$. But $e(a+b) = ea + eb = a + 0 = a$, so $a = ka + kb$. Hence $kb \in \langle a\rangle \cap \langle b\rangle = 0$, so $kb = 0$ and, in consequence, $p \mid k$ and $ka = 0$, so $a = 0$. This is a contradiction. □

Lemma 3.2. *Let R be a commutative filial ring such that $N(R)$ is a p-ring and $R/N(R) \in \mathbb{S}$. If $e \in R$ is an idempotent such that $ei \neq i$ for some $i \in N(R)$, then $eN(R) = 0$ and $Re \in \mathbb{S}$.*

Proof. From Lemma 3.1 we get at once that $eN(R) = 0$. Thus $N(R) \subseteq l_R(e)$ and $R = Re \oplus l_R(e)$, so $Re \cong (Re + N(R))/N(R) \lhd R/N(R)$. But $R/N(R) \in \mathbb{S}$ and the radical $\mathbb{S}$ is hereditary, so $Re \in \mathbb{S}$. □

Lemma 3.3. *Let R be a commutative filial ring such that $N(R)$ is a p-ring and $R/N(R) \in \mathbb{S}$. Then for every idempotent $e \in R$, $e \notin \mathbb{S}(R)$ if and only if $ei = i$ for every $i \in N(R)$.*

Proof. $\Rightarrow$. Suppose the assertion of the lemma is false. Then $eN(R) = 0$ by Lemma 3.1, and hence $eR \cap N(R) = 0$, because if $er \in N(R)$ for some $r \in R$, then $0 = e(er) = e^2 r = er$. It follows that $eR \cong (eR + N(R))/N(R) \lhd R/N(R)$. But $R/N(R) \in \mathbb{S}$, so $eR \in \mathbb{S}$. Thus $eR \subseteq \mathbb{S}(R)$ and $e = e^2 \in eR$, so $e \in \mathbb{S}(R)$, which is a contradiction.

$\Leftarrow$. A ring $\mathbb{S}(R)$ is reduced and $N(R)$ is a nil ring, so obviously $\mathbb{S}(R) \cap N(R) = 0$ and $\mathbb{S}(R) \cdot N(R) = 0$. □

Lemma 3.4. *Let R be a commutative filial ring such that $N(R)$ is a p-ring and $R/N(R) \in \mathbb{S}$. If $\mathbb{S}(R) + N(R) \neq R$, then there exists an idempotent $e \in R$ such that $N(R) \subseteq eR$ and $R = eR \oplus l_R(e)$. Moreover, $l_R(e) \in \mathbb{S}$.*

Proof. Take any $x \in R \setminus (\mathbb{S}(R) + N(R))$. Since $R/N(R)$ is a strongly regular ring, there exists $y \in R$ such that $x - x^2 y \in N(R)$ and $yx + N(R)$ is an idempotent in $R/N(R)$. But $N(R)$ is a nil ideal, hence the Köethe-Dickson theorem on lifting idempotents implies $yx - e \in N(R)$ for some idempotent $e \in R$. Therefore, $x - ex = (x - x^2 y) + x(xy - e) \in N(R)$, which yields $x \in eR + N(R)$. But $x \notin \mathbb{S}(R) + N(R)$, so $e \notin \mathbb{S}(R)$. By Lemma 3.3, $ei = i$ for every $i \in N(R)$. Thus $N(R) \subseteq eR$ and $N(eR) = N(R)$. Moreover, $R = eR \oplus l_R(e)$, so $l_R(e) \in \mathbb{S}$. □

Lemma 3.5. *Let R be a commutative filial ring such that $N(R)$ is a p-ring and $R/N(R) \in \mathbb{S}$. If $eN(R) = 0$ for every idempotent $e \in R$, then $R = \mathbb{S}(R) \oplus N(R)$.*

Proof. Take any $a \in R$. Since $R/N(R) \in \mathbb{S}$, there exist $b, e \in R$, $e = e^2$, and $i \in N(R)$ such that $a - ba^2 \in N(R)$ and $ba = e + i$. Hence $a - ae \in N(R)$ and $a \in Re + N(R)$. Lemma 3.2 implies that $Re \in \mathbb{S}$. In consequence, $a \in \mathbb{S}(R) + N(R)$. □

Lemma 3.6. *Let R be a commutative filial p-ring such that $N(R)$ is a ring of unbounded exponent. Then $R = \mathbb{S}(R) \oplus N(R)$.*

Proof. Take any idempotent $e \in R$. If $eN(R) \neq 0$, then $N(R) = N(R)e$ by Lemma 3.1. But $p^n e = 0$ for some $n \in \mathbb{N}$, so $p^n N(R) = 0$, which is a contradiction. We thus get $eN(R) = 0$ and, by Lemma 3.5, $R = \mathbb{S}(R) \oplus N(R)$. □

https://doi.org/10.1017/S1446788713000281 Published online by Cambridge University Press

LEMMA 3.7. *Let R be a commutative filial ring with identity such that $N(R)$ is a p-ring and $R/N(R) \in \mathbb{S}$. Then $R = \langle 1\rangle + \mathbb{S}(R) + N(R)$.*

PROOF. By Proposition 2.3, $N(R)$ is an H-ring. From [2, Lemma 1 and Theorem 2], it follows that $N(R)$ is nilpotent. So, there exists nonzero $i_0 \in l_{N(R)}(N(R))$. Then $\langle i_0\rangle \lhd N(R)$ and $\langle i_0\rangle \lhd R$. Let $r \in R$. Then there exists an integer k such that $ri_0 = ki_0$. Hence, $r - k \cdot 1 \in l_R(i_0)$, and $R = \langle 1\rangle + l_R(i_0)$. Moreover, $\mathbb{S}(R) \cap N(R) = 0$, so $\mathbb{S}(R) \subseteq l_R(i_0)$. Take any $a \in l_R(i_0)$. Then $R/N(R) \in \mathbb{S}$ implies that there exist $b, e \in R$, $e = e^2$, and $i \in N(R)$ such that $a - ba^2 \in N(R)$, and $ba = e + i$. But $ii_0 = 0$, $ai_0 = 0$, so $ei_0 = 0$. Lemma 3.2 now yields $eN(R) = 0$ and $Re \in \mathbb{S}$. But $a - ae \in N(R)$, so $a \in Re + N(R) \subseteq \mathbb{S}(R) + N(R)$. It follows that $l_R(i_0) \subseteq \mathbb{S}(R) + N(R)$ and $l_R(i_0) = \mathbb{S}(R) + N(R)$. Finally, $R = \langle 1\rangle + \mathbb{S}(R) + N(R)$. □

LEMMA 3.8. *Let R be a commutative filial p-ring with identity such that $N(R) \neq 0$. Then $pR \subseteq p \cdot \langle 1\rangle$. In particular, the group $pN(R)^+$ is cyclic and $N(R) = N(R)(p) + p \cdot \langle 1\rangle$.*

PROOF. Since R^+ is a p-group, there exists $n \in \mathbb{N}$ such that $o(1) = p^n$. Hence $p^nR = 0$ and $pR \subseteq N(R)$. By filiality of R and Proposition 2.3, we get that $N(R)$ is an H-ring. But $\langle p \cdot 1\rangle = [p \cdot 1] \lhd N(R)$, so $\langle p \cdot 1\rangle \lhd R$. This means that $pR \subseteq p\langle 1\rangle$. In particular, $pN(R) \subseteq p\langle 1\rangle$, and the group $pN(R)^+$ is cyclic.

If $pN(R) = 0$, then $N(R) \subseteq N(R)(p)$ and $N(R) = N(R)(p) + p\langle 1\rangle$. So, assume that $pN(R) \neq 0$. For every $i \in N(R)$ there exists $k \in \mathbb{Z}$ such that $pi = k(p \cdot 1)$.

If $p \nmid k$ then there exists $l \in \mathbb{Z}$ such that $lk \equiv 1 \mod p^n$, so $p \cdot 1 = lpi$. Thus $pN(R) \subseteq piN(R)$ and $pN(R) \subseteq pN(R)i^m$ for every $m \in \mathbb{N}$. But $N(R)$ is a nil ring, which clearly forces $pN(R) = 0$. This is a contradiction.

Therefore, $p \mid k$. Hence, there exists $k' \in \mathbb{Z}$ such that $k = pk'$. Then $p(i - (pk') \cdot 1) = 0$, $i - (pk') \cdot 1 \in N(R)(p)$. Thus $i = (i - (pk') \cdot 1) + pk' \cdot 1 \in N(R)(p) + p \cdot \langle 1\rangle$, and this leads to $N(R) = N(R)(p) + \langle p \cdot 1\rangle$. □

4. The classification theorem for torsion filial rings

We now state and prove the main theorem of this work.

THEOREM 4.1. *All (up to isomorphism) commutative torsion filial rings are rings of the form $\bigoplus_{p\in\mathbb{P}} R_p$, where every R_p is one of the following rings:*

(i) *$S \oplus N$, where N is a commutative nil H-p-ring and S is a commutative strongly regular p-ring;*

(ii) *$(1_C, S) \oplus S_1$, where S and S_1 are commutative strongly regular p-rings and the p-ring C is a K_0-ring.*

PROOF. Every torsion ring R can be written in the form $R = \bigoplus_{p\in\mathbb{P}} R_p$, where every component R_p of this sum is uniquely determined. From [6, Proposition 2], R is filial if and only if R_p is filial for every $p \in \mathbb{P}$. Therefore, without loss of generality we can assume that R is a commutative p-ring.

Assume that the ring R is filial. From Proposition 2.3, $N(R)$ is an H-p-ring. Moreover, the quotient p-ring $R/N(R)$ is filial and reduced. According to [7, Theorem 4.1] we have $R/N(R) \in \mathbb{S}$. So, if $R = N(R)$ or $N(R) = 0$, then R is like in (i).

Assume now that $0 \neq N(R) \neq R$. If $N(R)$ is a ring of unbounded exponent, then by Lemma 3.6, $R = \mathbb{S}(R) \oplus N(R)$.

It remains to consider the case when $N(R)$ is a ring of bounded exponent. Since $p(R/N(R)) = 0$, we have $p^m R = 0$ for some $m \in \mathbb{N}$. Assume that $R \neq \mathbb{S}(R) \oplus N(R)$. From Lemma 3.4 there exists an idempotent $e \in R \setminus (\mathbb{S}(R) + N(R))$ such that $N(R) \subseteq eR$, $R = eR \oplus l_R(e)$ and $l_R(e) \in \mathbb{S}$. Hence, eR is a commutative filial ring with identity e and $N(eR) = N(R)$, $p^m(eR) = 0$. Moreover, $(eR)/N(R) \in \mathbb{S}$ so, by Lemma 3.7, $eR = \langle e\rangle + \mathbb{S}(eR) + N(R)$. Denote $C = \langle e\rangle + N(R)$. From Lemma 3.8, $N(R) = N(R)(p) + p\langle e\rangle$ and $C = \langle e\rangle + N(R)(p)$. Theorem 2.7 implies that C is a K_0-ring. By Lemma 2.12, $C \cap \mathbb{S}(eR) = 0$. Hence, eR is the direct sum of subrings C and $\mathbb{S}(eR)$. Moreover, for $k \in \mathbb{Z}$, $x \in N$, $s \in \mathbb{S}(eR)$ we have $(ke + x) \cdot s = (ke)s$. This means that $\mathbb{S}(eR)$ is a C-algebra in a natural way. Thus $eR \cong (1_C, \mathbb{S}(eR))$ and, finally, $R \cong (1_C, \mathbb{S}(eR)) \oplus l_R(e)$.

Conversely, if $R \cong S \oplus N$, where N is a nil H-p-ring and S is a strongly regular p-ring, then from [7, Theorem 3.2], it follows that R is filial.

Let $R \cong (1_C, S) \oplus S_1$, where S and S_1 are commutative strongly regular p-rings and the p-ring C is a K_0-ring. The ring R is an extension of the ring $(1_C, S)$ by the ring S_1, so from [7, Theorem 3.2], it is enough to prove that the ring $(1_C, S)$ is filial. But $(1_C, S)$ is an extension of the strongly regular ring S by the filial ring C, so from [7, Theorem 3.2], the ring $(1_C, S)$ is filial.

We will show that the rings described in (i), (ii) are determined uniquely up to isomorphism. Let S_1, S_2, S_3, S_4 be any strongly regular p-rings, let N_1, N_2 be any nil H-p-rings, and finally let the p-rings C_1, C_2 be any K_0-rings. Consider $R_1 = S_1 \oplus N_1$, $R_2 = S_2 \oplus N_2$. Assume that $g : R_1 \to R_2$ is an isomorphism of rings. Clearly, $N(R_1) = N_1$ and $N(R_2) = N_2$, so $g(N_1) = N_2$. Moreover, $\mathbb{S}(R_1) = S_1$, $\mathbb{S}(R_2) = S_2$, so $g(S_1) = S_2$. Hence, $N_1 \cong N_2$ and $S_1 \cong S_2$.

Let $A = (1_{C_1}, S_1) \oplus S_2$, $B = (1_{C_1}, S_3) \oplus S_4$. Assume that $f : A \to B$ is an isomorphism of rings. By Lemma 2.12, $\mathbb{S}(A) = S_1 \oplus S_2$, $\mathbb{S}(B) = S_3 \oplus S_4$. Hence $f(S_1 \oplus S_2) = S_3 \oplus S_4$ and, as a consequence, $C_1 \cong A/\mathbb{S}(A) \cong B/\mathbb{S}(B) \cong C_2$. Next $S_2 = l_A(C_1) \cong l_B(C_2) = S_4$, so $A/S_2 \cong B/S_4$, which yields $S_1 = \mathbb{S}(A/S_2) \cong \mathbb{S}(B/S_4) = S_3$.

Finally, if R is a ring described in (i), then $R/\mathbb{S}(R)$ is a nil ring. But, for every ring T described in (ii), $T/\mathbb{S}(T)$ ia a nonzero ring with an identity as a K_0-ring. This shows that $R \not\cong T$. □

From the classification of nil H-p-rings (see [2, Theorem 2]), it follows that every noetherian nil H-p-ring is finite. It is a well-known fact that every (up to isomorphism) nonzero commutative noetherian strongly regular p-ring is a finite direct sum of fields of characteristic p. Moreover, from Theorem 2.8 and Examples 2.9–2.11 it follows that a K_0-ring is noetherian if and only if it is finite. Hence, by Theorem 4.1 and Remark 2.5, we have the following corollary.

Corollary 4.2. *All (up to isomorphism) commutative torsion noetherian filial rings are rings of the form* $\bigoplus_{p\in\Pi} R_p$, *where* Π *is a finite subset of* $\mathbb{P}$ *and every* R_p *is one of the following rings:*

https://doi.org/10.1017/S1446788713000281 Published online by Cambridge University Press

(i) *$S \oplus N$, where N is a finite commutative nil H-p-ring and S is a commutative strongly regular p-ring and S is a finite direct sum of fields of characteristic p;*
(ii) *$C \oplus S$, where S is a finite direct sum of fields of characteristic p and the p-ring C is a finite K_0-ring.*

Recall that every field which is finitely generated as a ring is finite, and every commutative finitely generated ring is noetherian. Hence, by Corollary 4.2, we have the following corollary.

Corollary 4.3. *All (up to isomorphism) commutative torsion finitely generated filial rings are rings of the form $\bigoplus_{p\in\Pi} R_p$, where Π is a finite subset of $\mathbb{P}$ and every R_p is one of the following rings:*

(i) *$S \oplus N$, where N is a finite commutative nil H-p-ring and S is a commutative strongly regular p-ring and S is a finite direct sum of finite fields of characteristic p;*
(ii) *$C \oplus S$, where S is a finite direct sum of finite fields of characteristic p and the p-ring C is a finite K_0-ring.*

References

[1] V. I. Andrijanov, 'Mixed Hamiltonian nilrings', *Ural. Gos. Univ. Mat. Zap.* **5**(3) (1966), 15–30 (in Russian).
[2] V. I. Andrijanov, 'Periodic Hamiltonian rings', *Mat. Sb. (N.S.)* **74**(116) (1967), 241–261; translation in Mat. Sb. **74**(116) (1967), 225–241.
[3] R. R. Andruszkiewicz, 'The classification of integral domains in which the relation of being an ideal is transitive', *Comm. Algebra.* **31** (2003), 2067–2093.
[4] R. R. Andruszkiewicz and K. Pryszczepko, 'A classification of commutative reduced filial rings', *Comm. Algebra.* **37** (2009), 3820–3826.
[5] R. R. Andruszkiewicz and K. Pryszczepko, 'The classification of commutative noetherian, filial rings with identity', *Comm. Algebra.* **40** (2012), 1690–1703.
[6] R. R. Andruszkiewicz and E. R. Puczyłowski, 'On filial rings', *Port. Math.* **45** (1988), 139–149.
[7] M. Filipowicz and E. R. Puczyłowski, 'Left filial rings', *Algebra Colloq.* **11**(3) (2004), 335–344.
[8] M. Filipowicz and E. R. Puczyłowski, 'The structure of left filial algebras over a field', *Taiwanese J. Math.* **13**(3) (2009), 1017–1029.
[9] R. L. Kruse, 'Rings with periodic additive group in which all subrings are ideals', Dissertation, California Institute of Technology, 1964.
[10] R. L. Kruse, 'Rings in which all subrings are ideals', *Canad. J. Math.* **20** (1968), 862–871.
[11] L. Redei, 'Vollidealringe im weiteren Sinn. I', *Acta Math. Acad. Sci. Hungar.* **3** (1952), 243–268.
[12] L. Redei, 'Die Vollidealringe', *Monatsh. Math.* **56** (1952), 89–95.

R. R. ANDRUSZKIEWICZ, Institute of Mathematics,
University of Białystok, 15-267 Białystok, Akademicka 2, Poland
e-mail: randrusz@math.uwb.edu.pl

K. PRYSZCZEPKO, Institute of Mathematics,
University of Białystok, 15-267 Białystok, Akademicka 2, Poland
e-mail: karolp@math.uwb.edu.pl