REPRESENTATION THEORIES FOR THE LAPLACE TRANSFORM

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1. Introduction. The Widder-Post real inversion operator [4] is defined by

$$L_{k,t}[f] = \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}, \qquad t > 0, \tag{1.1}$$

$k = 1, 2, \ldots$. Utilizing this inversion operator one can obtain the following representation theorem (see e.g. [4] Chapter VII, Theorem 15a) .

THEOREM A. Necessary and sufficient conditions for a function f to have a representation

$$f(x) = \int_0^\infty e^{-xt} F(t)dt, \qquad (x>0) \tag{1.2}$$

where $F(t) \in L_p(0, \infty)$, $p > 1$, are that

(1.3) $f(x)$ has derivatives of all orders in $0 < x < \infty$;

(1.4) $f(x) = o(1) \quad (x \to \infty)$;

$$\int_0^\infty |L_{k,t}[f]|^p \, dt \le M, \qquad k = 1, 2, \ldots . \tag{1.5}$$

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Let φ be a positive function defined on $(0, \infty)$. The spaces $L_p(\varphi)$, $1 \leq p < \infty$, consist of those measurable functions f defined on $(0,\infty)$ such that

$$\|f\|_{L_p(\varphi)} = \{\int_0^{\infty} \varphi(t) \, |f(t)|^p \, dt\}^{1/p} < \infty .$$

Similarly, we define the Lorentz spaces $\Lambda(\varphi, p)$, $p \geq 1$, to consist of those measurable functions f on $(0,\infty)$ for which

$$\|f\|_{\Lambda(\varphi, p)} = \{\int_0^{\infty} \varphi(t) \, f^*(t)^p \, dt\}^{1/p} < \infty ,$$

where f^* is the equimeasurable rearrangement of $|f|$ of decreasing order on $(0, \infty)$. (For a definition of f^* see e.g. [5, Chapter 1, § 13]).

In this paper we generalize Theorem A in the sense that the L_p-spaces are replaced by the $L_p(\varphi)$-spaces and Lorentz spaces $\Lambda(\varphi, p)$, $p > 1$, where φ belongs to a certain general class of functions. If $\varphi \equiv 1$, the $L_p(\varphi)$- and $\Lambda(\varphi, p)$-spaces reduce to the ordinary Lebesgue spaces.

In the next section a number of preliminary results are given. The $L_p(\varphi)$-representation theory is established in Section 3 and the last section contains the $\Lambda(\varphi, p)$-representation theory.

2. Preliminary results. The following theorem is an extension of a result of Widder [4, Chapter VII, Theorem 11b].

THEOREM 2.1. If $f(x)$ has derivatives of all orders in $0 < x < \infty$, then $L_{k,t}[f]$, $t > 0$, exists. If in addition for each positive integer k and some constant $c > 0$

$$\int_0^x L_{k,t}[f]dt = O(e^{cx}) \qquad (x \to \infty) ,$$

then $f(\infty)$ exists and

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$$\lim_{k\to\infty} \int_0^{\infty} e^{-xt} L_{k,t}[f]dt = f(x) - f(\infty) .$$

Proof. The existence of $L_{k,t}[f]$, $t > 0$ is obvious. By (1.1) and hypotheses

$$\int_0^x L_{k,t}[f]dt = \frac{(-1)^k}{k!} \int_0^x (\frac{k}{t})^{k+1} f^{(k)} (\frac{k}{t})dt$$

$$= \frac{(-1)^k}{k!} \int_{k/x}^{\infty} \nu^{k-1} f^{(k)} (\nu)\, d\nu \qquad (\nu = \frac{k}{t})$$

$$= 0(e^{cx}) \ (x\to\infty) ,$$

for $k = 1, 2, \ldots$. Replacing k by $k+1$,

(2.1) $$\int_{1/x}^{\infty} \nu^k f^{(k+1)}(\nu)d\nu = 0(e^{cx}) \ (x\to\infty) , \quad k = 0, 1, \ldots .$$

Let $s > 1/x$, then from (2.1) with $k = 0$

(2.2) $$\int_{1/x}^{s} f'(\nu)d\nu = f(s) - f(\frac{1}{x}) = 0(e^{cx}) \ (x\to\infty) .$$

Since the integral in (2.2) tends to a limit as $s\to\infty$, it follows that $f(\infty)$ exists. Also by (2.1) with $s > 1/x$ and $k = 1, 2, \ldots$

(2.3) $$\int_{1/x}^{s} \nu^k f^{(k+1)}(\nu)d\nu = s^k f^{(k)}(s) - x^{-k} f^{(k)}(\frac{1}{x}) - k \int_{1/x}^{s} \nu^{k-1} f^{(k)}(\nu)d\nu$$

where both integrals exist. It follows, therefore, that

$$f^k(s) = 0(s^{-k}) \quad (s\to\infty)$$

which together with the existence of $f(\infty)$ implies that

$$[f(s) - f(\infty)]^{(k)} = 0(s^{-k}) \quad (s\to\infty)$$

for $k = 0, 1, 2, \ldots$. Hence by Theorem 4.4 of [4, Chapter V]

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(2.4) $$[f(s) - f(\infty)]^{(k)} = o(s^{-k}) \quad (s \to \infty)$$

with $k = 0, 1, 2, \ldots$. From (2.4), $f^{(k)}(s) = o(s^k)$ $(s \to \infty)$, so that by (2.3) with $s \to \infty$

$$x^{-k} f^{(k)} (\frac{1}{x}) = 0(e^{cx}) \; (x \to \infty)$$

i.e.,

(2.5) $$[f(x) - f(\infty)]^{(k)} = f^{(k)}(x) = 0(e^{c/x} x^{-k}) \; (x \to 0+)$$

$k = 1, 2, \ldots$. If $k = 0$, (2.5) is also satisfied, for by (2.2)

$$\int_{1/x}^{\infty} f'(\nu)d\nu = f(\infty) - f(\frac{1}{x}) = 0(e^{cx}) \quad (x \to \infty) \quad ,$$

i.e.,

$$f(x) - f(\infty) = 0(e^{c/x}) \; (x \to 0+) \; .$$

Now Theorem 11a of [4, Chapter VII] holds also if its hypotheses 2 is replaced by

$$f^{(k)}(x) = 0(e^{c/x} x^{-k}) \; (x \to 0+) \quad ,$$

$k = 0, 1, 2, \ldots$. Thus the result follows from (2.4), (2.5) and Theorem 11a with $f(x)$ replaced by $f(x) - f(\infty)$.

LEMMA 2.1. Let ψ and X be non-negative measurable functions on $(0, \infty)$ such that for each $R > 0$

$$\int_0^R \psi(t)\, dt \le \int_0^R X(t)\, dt \; .$$

If φ is a non-negative decreasing function on $(0, \infty)$, then

(2.6) $$\int_0^{\infty} \varphi(t)\, \psi(t)\, dt \le \int_0^{\infty} \varphi(t)\, X(t)\, dt \; .$$

Proof. Assume

$$\int_0^{\infty} \varphi(t)\, X(t)\, dt < \infty \, ,$$

for otherwise (2.6) holds trivially. Define G_1 and G_2 by

$$G_1(t) = \int_0^t \psi(u)\,du\,,\quad G_2(t) = \int_0^t X(u)\,du\,,\quad t \geq 0\,.$$

If $\varphi(0+)$ is finite, then

$$\int_0^R \varphi(t)[\psi(t) - X(t)]dt = \int_0^R \varphi(t)\,d[G_1(t) - G_2(t)]$$

$$= \varphi(R)[G_1(R) - G_2(R)] - \int_0^R [G_1(t) - G_2(t)]d\,\varphi(t) \leq 0$$

since $G_1(t) \leq G_2(t)$ and φ is decreasing. Hence

$$\int_0^R \varphi(t)\psi(t)\,dt \leq \int_0^R \varphi(t)X(t)\,dt \leq \int_0^\infty \varphi(t)X(t)\,dt \tag{2.7}$$

for each $R > 0$. The result follows now if $R \to \infty$. If $\varphi(0+) = \infty$, define for each $\delta > 0$

$$\varphi_\delta(t) = \begin{cases} \varphi(\delta) & t \in (0, \delta) \\ \varphi(t) & t \in (\delta, \infty)\,. \end{cases}$$

Hence from (2.7)

$$\int_0^\infty \varphi_\delta(t)\psi(t)\,dt \leq \int_0^\infty \varphi_\delta(t)\,X(t)\,dt \leq \int_0^\infty \varphi(t)\,X(t)\,dt\,,$$

and by Fatou's lemma

$$\int_0^\infty \varphi(t)\,\psi(t)\,dt \leq \lim_{\delta \to 0} \int_0^\infty \varphi_\delta(t)\psi(t)\,dt \leq \int_0^\infty \varphi(t)\,X(t)\,dt\,,$$

which is the result.

LEMMA 2.2. If

$$H(t) = \int_0^\infty A(u, t)\,h(u)\,du$$

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exists for almost all $t > 0$, where $A(u,t)$ satisfies

$$\int_0^\infty |A(u,t)|\,du \le K \quad \text{and} \quad \int_0^\infty |A(u,t)|\,dt \le K$$

for some constant K, then for each $R > 0$

$$\int_0^R H^*(t)dt \le K \int_0^R h^*(t)dt ,$$

where H^* and h^* are the equimeasurable rearrangements of decreasing order of $|H|$ and $|h|$, respectively.

Lemma 2.2 is proved in the same way as Theorem 3.8.1 of [1].

LEMMA 2.3. If $\varphi(t)$ is a non-increasing, positive function defined on $(0, \infty)$ and $\psi(t)$ non-negative on $(0,\infty)$ then

$$\int_0^\infty \varphi(t)\psi(t)\,dt \le \int_0^\infty \varphi(t)\psi^*(t)\,dt , \tag{2.8}$$

where ψ^* is the rearrangement of decreasing order of ψ.

<u>Proof.</u> If the right side of (2.8) is not finite, the result is obvious. Otherwise for each $R > 0$,

$$\int_0^R \psi^*(t)\,dt \le \frac{1}{\varphi(R)} \int_0^\infty \varphi(t)\psi^*(t)\,dt < \infty .$$

Since for each $R > 0$

$$\int_0^R \psi(t)\,dt \le \int_0^R \psi^*(t)\,dt ,$$

(2.8) is an immediate consequence of Lemma 2.1 with X replaced by ψ^*.

LEMMA 2.4. If

$$A_k(u,t) \equiv \frac{1}{k!}\, e^{-ku/t}\, u^k \left(\frac{k}{t}\right)^{k+1} , \quad u > 0,\ t > 0 \tag{2.9}$$

and $k = 1, 2, \ldots$, then

$$\int_0^\infty A_k(u, t)du = \int_0^\infty A_k(u, t)dt = 1 .$$

DEFINITION 2.1. A function $\varphi(t)$ defined for $t > 0$ belongs to the class A if $\varphi(t)$ is a non-increasing function for all $t > 0$ and if there exists a function $K(x) > 0$ non-decreasing for all $x > 0$ such that

$$\varphi(t) \geq e^{-tx} K(x) \tag{2.10}$$

for all $x > 0$.

3. $L_p(\varphi)$ - representation theory.

THEOREM 3.1. Let φ belong to class A. Necessary and sufficient conditions that a function $f(x)$ defined for $x > 0$ be the Laplace transform of a function F in $L_p(\varphi)$, $p > 1$, are that

(3.1) $f(x)$ has derivatives of all orders in $0 < x < \infty$

(3.2) $f(x) = o(1) \quad (x \to \infty)$

and that

$$\| L_{k,} [f] \|_{L_p(\varphi)} \leq M , \qquad k = 1, 2, \ldots . \tag{3.3}$$

Proof. Let

$$f(x) = \int_0^\infty e^{-xt} F(t)\, dt , \quad x > 0 ,$$

with $F \in L_p(\varphi)$. Then (3.1) holds. By Hőlder's inequality with

$\frac{1}{p} + \frac{1}{p'} = 1$ and (2.10)

$$\int_0^\infty e^{-xt} |F(t)|\, dt \leq \{\int_0^\infty e^{-xt} dt\}^{1/p'} \{\int_0^\infty e^{-xt} |F(t)|^p dt\}^{1/p} \tag{3.4}$$

$$\leq x^{-1/p'} [K(x)]^{-1/p} \{\int_0^\infty \varphi(t) |F(t)|^p\, dt\}^{1/p}$$

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so that (3.2) holds. Next we show that (3.3) is satisfied. Define f_η, $\eta > 0$ by

$$f_\eta(x) = \int_0^\infty e^{-xt} f_\eta(t)dt , \quad x > 0 ,$$

where

$$F_\eta(t) = \begin{cases} F(t) & 0 < t < \eta \\ 0 & \text{otherwise.} \end{cases}$$

Clearly $f_\eta(x)$ has derivatives of all orders in $0 < x < \infty$, so that by Hőlder's inequality

$$|L_{k,t}[f_\eta]|^p = \left|\frac{1}{k!} \int_0^\infty e^{-ku/t} u^k (\tfrac{k}{t})^{k+1} F_\eta(u)\, du\right|^p$$

$$\leq \frac{1}{k!} \int_0^\infty e^{-ku/t} u^k (\tfrac{k}{t})^{k+1} |F_\eta(u)|^p du \left\{\frac{1}{k!} \int_0^\infty e^{-ku/t} u^k (\tfrac{k}{t})^{k+1} du\right\}^{p/p'}$$

(3.5)

$$= \frac{1}{k!} \int_0^\infty e^{-ku/t} u^k (\tfrac{k}{t})^{k+1} |F_\eta(u)|^p du$$

$$= \frac{1}{k!} \int_0^\eta e^{-ku/t} u^k (\tfrac{k}{t})^{k+1} |F(u)|^p du$$

$$\leq \frac{e^{-k} k^{k+1}}{t\ k!} \int_0^\eta |F(u)|^p du \leq \frac{e^{-k} k^{k+1}}{t\ k!\, \varphi(\eta)} \int_0^\infty \varphi(u) |F(u)|^p du < \infty ,$$

for $t > 0$. Let

$$H_{k,\eta}(t) = \int_0^\infty A_k(u,t) |F_\eta(u)|^p du \quad t > 0 ,$$

where $A_k(u,t)$ is the function defined by (2.9). Then by (3.5), $H_{k,\eta}(t)$ exists for $t > 0$ and by Lemmas 2.2 and 2.4, for each $R > 0$,

$$\int_0^R H^*_{k,\eta}(t)\, dt \leq \int_0^R (|F_\eta(t)|^p)^* dt .$$

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In particular, for $R = \eta$,

$$\int_0^\eta H_{k,\eta}(t)\,dt \le \int_0^\eta H^*_{k,\eta}(t)\,dt \le \int_0^\eta (|F_\eta(t)|^p)^*\,dt$$

(3.6)
$$= \int_0^\eta |F_\eta(t)|^p\,dt = \int_0^\eta |F(t)|^p\,dt\,.$$

But $\eta > 0$ is arbitrary, so that by (3.5), (3.6) and Lemma 2.1

(3.7) $$\int_0^\infty \varphi(t)\,|L_{k,t}[f_\eta]|^p\,dt \le \int_0^\infty \varphi(t)\,H_{k,\eta}(t)\,dt \le \int_0^\infty \varphi(t)\,|F(t)|^p\,dt\,.$$

Now by definition of F_η, $\lim_{\eta\to\infty} F_\eta(t) = F(t)$ and $|F_\eta(t)| \le |F(t)|$, so that by (3.4), Lebesgue's theorem of dominated convergence yields

$$\lim_{\eta\to\infty} f_\eta(x) = \lim_{\eta\to\infty} \int_0^\infty e^{-xt}\,F_\eta(t)\,dt = \int_0^\infty e^{-xt}\,F(t)\,dt = f(x)\,.$$

Similarly, we obtain for $x > 0$,

$$\lim_{\eta\to\infty} f_\eta^{(k)}(x) = \lim_{\eta\to\infty} (-1)^k \int_0^\infty e^{-xt}\,t^k\,F_\eta(t)\,dt$$

$$= (-1)^k \int_0^\infty e^{-xt}\,t^k\,F(t)dt = f^k(x) \quad k = 1, 2, \ldots,$$

and hence by (1.1)

$$\lim_{\eta\to\infty} |L_{k,t}[f_\eta]|^p = |L_{k,t}[f]|^p\,.$$

Therefore, by Fatou's lemma and (3.7),

$$\int_0^\infty \varphi(t)\,|L_{k,t}[f]|^p dt \le \underline{\lim}_{\eta\to\infty} \int_0^\infty \varphi(t)\,|L_{k,t}[f_\eta]|^p\,dt$$

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$$\leq \int_0^\infty \varphi(t)\,|F(t)|^p\,dt\ ,$$

proving (3.3).

Sufficiency. By (3.1) $L_{k,t}[f]$ exists and by Hőlder's inequality, (2.10) and (3.3) for $x > 0$

$$\int_0^x |L_{k,t}[f]|\,dt = \int_0^x \varphi(t)^{1/p}\,\varphi(t)^{-1/p}\,|L_{k,t}[f]|\,dt$$

$$\leq \{\int_0^x \varphi(t)^{-p'/p}dt\}^{1/p'}\{\int_0^x \varphi(t)\,|L_{k,t}[f]|^p\,dt\}^{1/p}$$

$$\leq \frac{M}{K(y)^{1/p}}\{\int_0^x e^{typ'/p}\,dt\}^{1/p'} = \frac{Mp^{1/p'}}{K(y)^{1/p}}[\frac{e^{xyp'/p}-1}{yp'}]^{1/p'}$$

Thus, for some $c > 0$ and $k = 1, 2, \ldots$,

$$\int_0^x L_{k,t}[f]\,dt = 0(e^{cx})\ , \quad (x \to \infty)\ ,$$

and Theorem 2.1 is applicable, so that with $f(\infty) = 0$

$$(3.8) \qquad \lim_{k\to\infty}\int_0^\infty e^{-xt}\,L_{k,t}[f]\,dt = f(x)\ .$$

Next, define γ_k , $k = 1, 2, \ldots$, by $\gamma_k(t) = \varphi(t)^{1/p}\,L_{k,t}[f]$. Then by (3.3)

$$\int_0^\infty |\gamma_k(t)|^p\,dt \leq M^p,\ k = 1, 2, \ldots\ .$$

By the weak compactness argument [4, Theorem 17a, Chapter 1, § 17] there is an increasing unbounded subsequence $\{k_i\}_{i=1}^\infty$ and a function $\gamma(t) = \varphi(t)^{1/p}\,F(t)\ \varepsilon\ L_p\,(0, \infty)$, such that for every $\beta(t)\ \varepsilon\ L_{p'}\ (0, \infty)$

$$(3.9) \qquad \lim_{i\to\infty} \int_0^\infty \beta(t)\, \gamma_{k_i}(t)\, dt = \int_0^\infty \beta(t)\, \gamma(t)\, dt \ .$$

Let in particular $\beta(t) = \varphi(t)^{-1/p} e^{-tx}$, $x > 0$, $p > 1$; then $\beta(t) \in L_{p'}(0, \infty)$, for by (2.10)

$$\int_0^\infty |\beta(t)|^{p'} dt = \int_0^\infty \varphi(t)^{-p'/p} e^{-txp'} dt \leq K(x)^{-p'/p} \int_0^\infty e^{xtp'/p} e^{-xtp'} dt < \infty .$$

Hence by (3.9)

$$\lim_{i\to\infty} \int_0^\infty e^{-xt} L_{k_i, t}[f]\, dt = \int_0^\infty e^{-xt} F(t)\, dt ,$$

so that by (3.8)

$$f(x) = \int_0^\infty e^{-xt} F(t)\, dt , \quad x > 0 ,$$

where $F \in L_p(\varphi)$, $p > 1$.

4. $\Lambda(\varphi, p)$ - representation theory. We note that if φ belongs to class A, then the Laplace transform of a function $F \in \Lambda(\varphi, p)$, $p > 1$ exists. For by [1, p. 60],

$$(4.1) \qquad F^{*p} = (|F|^p)^* \quad p \geq 1 ,$$

Hőlder's inequality, (2.10) and Lemma 2.3 with ψ replaced by $|F(\cdot)|^p$ yields

$$\int_0^\infty e^{-xt} F(t)\, dt \leq \{\int_0^\infty e^{-xt} dt\}^{1/p'} \{\int_0^\infty e^{-xt} |F(t)|^p dt\}^{1/p}$$

$$\leq x^{-1/p'} K(x)^{-1/p} \{\int_0^\infty \varphi(t) |F(t)|^p dt\}^{1/p}$$

$$\leq x^{-1/p'} K(x)^{-1/p} \{\int_0^\infty \varphi(t)\, F^*(t)^p dt\}^{1/p} < \infty .$$

THEOREM 4.1. Let φ belong to class A. Necessary and sufficient conditions that a function $f(x)$ defined for $x > 0$ be the Laplace transform of a function $F \in \Lambda(\varphi, p)$, $p > 1$ are that (3.1), (3.2) and

$$\| L_{k,} : [f] \|_{\Lambda(\varphi, p)} \leq M , \qquad k = 1, 2, \ldots, \tag{4.2}$$

hold.

Proof. By Lemma 2.3 and (4.1)

$$\int_0^\infty \varphi(t) |F(t)|^p \, dt \leq \int_0^\infty \varphi(t) F^*(t)^p \, dt ,$$

so that the proof of the necessity part follows as in Theorem 3.1.

Sufficiency. Using Lemma 2.3 and (4.1) we see as in the proof of Theorem 3.1 that

$$\lim_{k \to \infty} \int_0^\infty e^{-xt} L_{k,t}[f] \, dt = f(x) . \tag{4.3}$$

By [1, Theorem 3.7.3, §3.7, p.74] the spaces $\Lambda(\varphi, p)$, $p > 1$ are reflexive and hence [3, Theorem 4.61 c, §4.61] the unit sphere in $\Lambda(\varphi, p)$ is weakly compact. We next define the functional G_x, $x > 0$, on $\Lambda(\varphi, p)$, $p > 1$, by

$$G_x(u) = \int_0^\infty e^{-xt} u(t) \, dt .$$

Clearly G_x is linear and also bounded, since Hőlder's inequality (2.10), Lemma 2.3 and (4.1) yield

$$|G_x(u)| \leq \int_0^\infty e^{-xt} |u(t)| \, dt \leq \{ \int_0^\infty e^{-xt} dt \}^{1/p'} \{ \int_0^\infty e^{-xt} |u(t)|^p dt \}^{1/p}$$

$$\leq x^{-1/p'} K(x)^{-1/p} \{ \int_0^\infty \varphi(t) |u(t)|^p dt \}^{1/p}$$

$$\leq x^{-1/p'} K(x)^{-1/p} \{ \int_0^\infty \varphi(t) u^*(t)^p dt \}^{1/p}$$

$$= x^{-1/p'} K(x)^{-1/p} \|u\|_{\Lambda(\varphi, p)} .$$

By [3, Theorem 4.41 B, § 4.41], for any bounded linear functional G on $\Lambda(\varphi, p)$, $p > 1$ and for each bounded sequence $\{u_k\}_{k=1}^{\infty}$ in $\Lambda(\varphi, p)$ there is a subsequence $\{k_i\}_{i=1}^{\infty}$ and a function $u \in \Lambda(\varphi, p)$, such that

$$\lim_{i \to \infty} G(u_{k_i}) = G(u) .$$

This holds in particular for the functional G_x and the sequence $L_{k,t}[f] \in \Lambda(\varphi, p)$. Hence there is a sequence $\{k_i\}_{i=1}^{\infty}$ and a function $F \in \Lambda(\varphi, p)$, such that

$$(4.4) \qquad \lim_{i \to \infty} \int_0^{\infty} e^{-xt} L_{k_i, t}[f]\, dt = \int_0^{\infty} e^{-xt} F(t)\, dt .$$

From (4.3) and (4.4) we obtain

$$f(x) = \int_0^{\infty} e^{-xt} F(t)\, dt , \quad x > 0 ,$$

which is the result.

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