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ON THE FUNCTIONAL CENTRAL LIMIT THEOREM FOR REVERSIBLE MARKOV CHAINS WITH NONLINEAR GROWTH OF THE VARIANCE

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Abstract

In this paper we study the functional central limit theorem (CLT) for stationary Markov chains with a self-adjoint operator and general state space. We investigate the case when the variance of the partial sum is not asymptotically linear in n, and establish that conditional convergence in distribution of partial sums implies the functional CLT. The main tools are maximal inequalities that are further exploited to derive conditions for tightness and convergence to the Brownian motion.

Keywords: Maximal inequality; reversible process; martingale approximation; Markov chain; tightness; functional central limit theorem

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1. Introduction

Kipnis and Varadhan (1986) showed that, for an additive functional zero mean S_n of a stationary reversible Markov chain, the condition $\mathrm{var}(S_n)/n \to \sigma^2$ implies convergence of $S_{[nt]}/\sqrt{n}$ to the Brownian motion (here $[nt]$ denotes the integer part of nt). A considerable number of papers further extend and apply this result to infinite particle systems, random walks, processes in random media, and Metropolis–Hastings algorithms. Among others, Kipnis and Landim (1999) considered interacting particle systems, and Tierney (1994) discussed the applications to Markov chain Monte Carlo methods. Wu (1999) and Zhao and Woodroofe (2008) studied the law of the iterated logarithm, and Derriennic and Lin (2001) and Cuny and Peligrad (2012) investigated the central limit theorem (CLT) started at a point.

Recently, Zhao *et al.* (2010) addressed the conditional CLT question under the weaker condition $\mathrm{var}(S_n) = nh(n)$, where h is a slowly varying function (i.e. $\lim_{n\to\infty} h(nt)/h(n) = 1$ for all $t > 0$). They showed by example the surprising result that the distribution of $S_{[nt]}/\sqrt{\mathrm{var}(S_n)}$ needs not converge to the standard normal distribution in this case. They developed sufficient conditions for convergence to a (possibly nonstandard) normal distribution imposed to an approximating martingale.

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In this paper we address the issue of the functional CLT for the case considered in Zhao *et al.* (2010). Our goal is to establish sufficient conditions imposed on the original sequence. We also show that, for reversible Markov chains, conditional convergence in distribution of partial sums properly normalized implies the functional CLT. The main tools to prove this result are new maximal inequalities based on a triangular forward–backward martingale decomposition and tightness results.

Our paper is organized as follows. Section 2 contains the definitions, a short background of the problem, and the results. Section 3 is devoted to the proofs. Section 4 contains a functional CLT for an additive functional associated to a Metropolis–Hastings algorithm, with the variance of partial sums behaving asymptotically like $nh(n)$ (where h is a slowly varying function). Throughout the paper, '$\Rightarrow$' denotes weak convergence, $[x]$ denotes the integer part of x, and '$\xrightarrow{\mathrm{P}}$' denotes convergence in probability. The notation $a_n \sim b_n$ means that $a_n/b_n \to 1$ as $n \to \infty$, and $a_n = o(b_n)$ means that $a_n/b_n \to 0$ as $n \to \infty$.

2. Definitions, background, and results

We assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary Markov chain defined on a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ with values in a general state space $(S, \mathcal{A})$. The marginal distribution is denoted by $\pi(A) = \mathrm{P}(\xi_0 \in A)$. Assume that there is a regular conditional distribution for ξ_1 given ξ_0 denoted by $Q(x, A) = \mathrm{P}(\xi_1 \in A \mid \xi_0 = x)$. Let Q also denote the Markov operator acting via $(Qf)(x) = \int_S f(s)Q(x, \mathrm{d}s)$. Next, let $\mathbb{L}_0^2(\pi)$ be the set of measurable functions on S such that $\int f^2\,\mathrm{d}\pi < \infty$ and $\int f\,\mathrm{d}\pi = 0$. For some function $f \in \mathbb{L}_0^2(\pi)$, let

$$X_i = f(\xi_i), \qquad S_n = \sum_{i=1}^{n} X_i, \qquad \sigma_n = (\mathrm{E}\,S_n^2)^{1/2}. \tag{1}$$

Denote by $\mathcal{F}_k$ the σ-field generated by ξ_i with $i \le k$.

For any integrable random variable X, we define $\mathrm{E}_k(X) = \mathrm{E}(X \mid \mathcal{F}_k)$. Under this notation, $\mathrm{E}_0(X_1) = (Qf)(\xi_0) = \mathrm{E}(X_1 \mid \xi_0)$. We denote by $\|X\|_p$ the norm in $\mathbb{L}_p(\Omega, \mathcal{F}, \mathrm{P})$.

The Markov chain is called reversible if $Q = Q^*$, where Q^* is the adjoint operator of Q. The condition of reversibility is equivalent to requiring that (ξ_0, ξ_1) and (ξ_1, ξ_0) have the same distribution or

$$\int_A Q(\omega, B)\pi(\mathrm{d}\omega) = \int_B Q(\omega, A)\pi(\mathrm{d}\omega)$$

for all Borel sets $A, B \in \mathcal{A}$.

Kipnis and Varadhan (1986) assumed that

$$\lim_{n\to\infty} \frac{\sigma_n^2}{n} = \sigma_f^2, \tag{2}$$

and proved that, for any reversible Markov chain defined by (1), this condition implies that

$$\frac{S_{[nt]}}{\sqrt{n}} \Rightarrow |\sigma_f| W(t),$$

where $W(t)$ is the standard Brownian motion.

Recently, Zhao *et al.* (2010) analyzed the case in which

$$\sigma_n^2 = nh(n) \quad \text{with } h \text{ a slowly varying function.} \tag{3}$$

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In their Proposition 1, they showed that, without loss of generality, we can assume that $h(n) \to \infty$, since otherwise either (2) holds (and this case is already known) or $2S_n = (1+(-1)^{n-1})X_1$ almost surely (a.s.). Then, in their Proposition 2 they showed that representation (3) implies that

$$\|\mathrm{E}_0(S_n)\|_2 = o(\sigma_n). \tag{4}$$

On the other hand, it is well known that (4) implies (3); see, for instance, Lemma 1 of Wu and Woodroofe (2004). Therefore, we can state Proposition 2 of Zhao *et al.* (2010) as follows.

Proposition 1. *For a stationary reversible Markov chain $(X_n)_{n\in\mathbb{Z}}$ defined by (1), relations (3) and (4) are equivalent.*

In their Corollary 2, Zhao *et al.* (2010) gave sufficient conditions for the validity of the conditional CLT in terms of conditions imposed on the differences of an approximating martingale. In addition, they provided an example of a reversible Markov chain satisfying (3), for which the CLT holds with a different normalization.

Throughout this paper, we will assume that $\sigma_n^2 \to \infty$.

By conditional convergence in distribution, denoted by $Y_n \mid \mathcal{F}_0 \Rightarrow Y$, we understand that, for any function g which is continuous and bounded,

$$\mathrm{E}_0(g(Y_n)) \xrightarrow{\mathrm{P}} \mathrm{E}\, g(Y) \quad \text{as } n \to \infty.$$

In other words, let P^x be the probability associated with the Markov chain started from x and let E^x be the corresponding expectation. Then, for any $\varepsilon > 0$,

$$\pi\{x : |\mathrm{E}^x g(Y_n) - \mathrm{E}\, g(Y)| > \varepsilon\} \to 0.$$

One of our results is the following invariance principle for functionals of stationary reversible Markov chains. Define

$$W_n(t) = \frac{S_{[nt]}}{\sigma_n}.$$

Theorem 1. *Assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary reversible Markov chain as defined above. Define $(X_i)_{i\in\mathbb{Z}}$ by (1), and assume that (3) is satisfied and S_n/σ_n is conditionally convergent in distribution to L. Then,*

$$W_n(t) \Rightarrow cW(t), \tag{5}$$

where $W(t)$ is a standard Brownian motion and c is the standard deviation of L.

Theorem 1 does not require special properties of the Markov chain, such as irreducibility and aperiodicity. However, if these properties are satisfied, we have the following simplification.

Corollary 1. *Assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary, reversible, irreducible, and aperiodic Markov chain such that (3) is satisfied. Then $S_n/\sigma_n \Rightarrow L$ implies (5).*

The proof of Theorem 1 requires the development of several tools. First, we will establish maximal inequalities that are of interest in themselves. As in the Doob maximal inequalities for martingales case, we will compare moments and tail distributions of the maximum of partial sums with those of the corresponding partial sums.

Proposition 2. *Let $(X_i)_{i\in\mathbb{Z}}$ be defined by (1), and let $Q = Q^*$. Let $p > 1$ and $q > 1$ such that $1/p + 1/q = 1$. Then, for all $n \geq 1$,*

$$\Big\| \max_{1\le i\le n} |S_i| \Big\|_p \le \Big\| \max_{1\le i\le n} |X_i| \Big\|_p + (4q+3) \max_{1\le i\le n} \|S_i\|_p.$$

Remark 1. Let $p = 2$. Since $(X_i)_{i\in\mathbb{Z}}$ is stationary, it is well known that

$$\Big\| \max_{1\le i\le n} |X_i| \Big\|_2 = o(n^{1/2}) \quad \text{as } n \to \infty.$$

If we assume in addition that $\liminf_n \sigma_n^2/n > 0$, we deduce that there exists $C > 0$ such that

$$\Big\| \max_{1\le i\le n} |S_i| \Big\|_2 \le C \max_{1\le i\le n} \|S_i\|_2.$$

For a proof of tightness, it is also convenient to have inequalities for the tail probabilities of partial sums. We will also establish the following.

Proposition 3. *Let $(X_i)_{i\in\mathbb{Z}}$ be defined by (1), and let $Q = Q^*$. Then, for every $x > 0$ and $n \ge 1$,*

$$\mathrm{P}\Big(\max_{1\le i\le n} |S_i| > x\Big) \le \frac{2}{x}\bigg[18\,\mathrm{E}\,|S_n|\,\mathbf{1}\Big(|S_n| > \frac{x}{12}\Big) + 55 \max_{1\le i\le n} \|\mathrm{E}_0(S_i)\|_1 + \Big\| \max_{1\le i\le n} |X_i| \Big\|_1\bigg].$$

An important step in the proof of Theorem 1 is the use of tightness conditions. We will give two necessary conditions for tightness that will ensure continuity of every limiting process.

Proposition 4. *Assume that X_i is defined by (1), that condition (3) is satisfied, and that one of the following two conditions holds:*

1. *$(S_n^2/\sigma_n^2)_{n\ge1}$ is uniformly integrable;*
2. *S_n/σ_n is convergent in distribution.*

Then $W_n(t)$ is tight in $D(0, 1)$ endowed with uniform topology and any limiting process is continuous.

Finally, we give sufficient conditions for convergence to the standard Brownian motion.

Proposition 5. *Assume that $(\xi_n)_{n\in\mathbb{Z}}$ is a stationary reversible Markov chain. Define $(X_i)_{i\in\mathbb{Z}}$ by (1), and assume that (4) is satisfied. Assume that $(S_n^2/\sigma_n^2)_{n\ge1}$ is uniformly integrable and that*

$$\lim_{n\to\infty} \frac{\|\mathrm{E}_0(S_n^2) - \sigma_n^2\|_1}{\sigma_n^2} = 0.$$

Then

$$W_n(t) \Rightarrow W(t).$$

3. Proofs

We start with a preliminary martingale decomposition that combines ideas from Wu and Woodroofe (2004) with the forward–backward martingale approximation of Meyer and Zheng (1984) and Lyons and Zheng (1988).

3.1. Forward–backward martingale decomposition

As in Wu and Woodroofe (2004) for fixed $n \ge 1$, define the stationary sequences

$$\theta_k^n = \frac{1}{n}\sum_{i=0}^{n-1} \mathrm{E}_k(X_k + \cdots + X_{k+i}) \quad \text{and} \quad D_k^n = \theta_k^n - \mathrm{E}_{k-1}(\theta_k^n).$$

Then, $(D_k^n)_{k\in Z}$ is a triangular array of martingale differences adapted to the filtration $\mathcal{F}_n = \sigma(\xi_i,\ i \le n)$. Note that

$$\begin{aligned}\theta_k^n &= X_k + \frac{1}{n}\sum_{i=1}^{n-1} \mathrm{E}_k(S_{k+i} - S_k)\\ &= X_k + \mathrm{E}_k(\theta_{k+1}^n) - \frac{1}{n}\mathrm{E}_k(S_{k+n} - S_k)\\ &= X_k + \theta_{k+1}^n - D_{k+1}^n - \frac{1}{n}\mathrm{E}_k(S_{k+n} - S_k).\end{aligned}$$

Therefore,

$$X_k = D_{k+1}^n + \theta_k^n - \theta_{k+1}^n + \frac{1}{n}\mathrm{E}_k(S_{k+n} - S_k). \tag{6}$$

We now construct a martingale approximation for the reversed process adapted to the filtration $\mathcal{G}_n = \sigma(\xi_i,\ i \ge n)$. We introduce the notation $\tilde{\mathrm{E}}_1(X_0) = \mathrm{E}(X_0 \mid \mathcal{G}_1) = \mathrm{E}(X_0 \mid \xi_1) = (Q^* f)(\xi_1)$.

Let

$$\tilde{\theta}_k^n = \frac{1}{n}\sum_{i=0}^{n-1} \tilde{\mathrm{E}}_k(X_{k-i} + \cdots + X_k).$$

With this notation,

$$X_{k+1} = \tilde{D}_k^n + \tilde{\theta}_{k+1}^n - \tilde{\theta}_k^n + \frac{1}{n}\tilde{\mathrm{E}}_{k+1}(X_{-n+k+1} + \cdots + X_k), \tag{7}$$

where the $\tilde{D}_k^n$ are martingale differences with respect to the filtration $\mathcal{G}_k = \sigma(\xi_i,\ i \ge k)$, $\tilde{D}_k^n = \tilde{\theta}_k^n - \mathrm{E}_{k+1}\,\tilde{\theta}_k^n$.

If we assume that $Q = Q^*$, we have $\tilde{\mathrm{E}}_1(X_0) = \mathrm{E}(X_2 \mid \xi_1) = (Qf)(\xi_1)$. Therefore, $\tilde{\theta}_k^n = \theta_k^n$, $\tilde{\theta}_{k+1}^n = \theta_{k+1}^n$, and $\tilde{\mathrm{E}}_{k+1}(X_{-n+k+1} + \cdots + X_k) = \mathrm{E}_{k+1}(X_{k+2} + \cdots + X_{k+n+1})$. Adding relations (6) and (7) leads to

$$X_k + X_{k+1} = D_{k+1}^n + \tilde{D}_k^n + \frac{1}{n}\mathrm{E}_k(S_n - S_k) + \frac{1}{n}\mathrm{E}_{k+1}(S_{k+n+1} - S_{k+1}).$$

Summing these relations we obtain the representation

$$\sum_{i=0}^{k-1}(X_i + X_{i+1}) = \sum_{i=1}^{k}\biggl[(D_i^n + \tilde{D}_{i-1}^n) + \frac{1}{n}\mathrm{E}_{i-1}(S_{n+i-1} - S_{i-1}) + \frac{1}{n}\mathrm{E}_i(S_{n+i} - S_i)\biggr].$$

So,

$$2S_k + (X_0 - X_k) = \sum_{i=1}^{k}(D_i^n + \tilde{D}_{i-1}^n) + \bar{R}_k^n,$$

where

$$\bar{R}_k^n = \frac{1}{n}\sum_{i=1}^{k}[\mathrm{E}_{i-1}(S_{n+i-1} - S_{i-1}) + \mathrm{E}_i(S_{n+i} - S_i)].$$

Therefore, in the reversible case, we obtain the forward–backward martingale representation

$$S_k = \tfrac{1}{2}[(X_k - X_0) + (M_k^n + \tilde{M}_k^n) + \bar{R}_k^n], \tag{8}$$

where $M_k^n = \sum_{i=1}^k D_i^n$ is a forward martingale adapted to the filtration $\mathcal{F}_k$ and $\tilde{M}_k^n = \sum_{i=0}^{k-1}\tilde{D}_i^n$ is a backward martingale adapted to the filtration $\mathcal{G}_k$.

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Also, it is convenient to point out a related martingale approximation which helps us relate the partial sums with a martingale adapted to the same filtration. Note that

$$\theta_k^n = X_k + \frac{1}{n}\sum_{i=1}^{n-1} \mathrm{E}_k(S_{k+i} - S_k) = X_k + \bar{\theta}_k^n, \quad \text{where} \quad \bar{\theta}_k^n = \frac{1}{n}\sum_{i=1}^{n-1} \mathrm{E}_k(S_{k+i} - S_k).$$

Starting from (6) and using this notation, we obtain

$$X_{k+1} = D_{k+1}^n + \bar{\theta}_k^n - \bar{\theta}_{k+1}^n + \frac{1}{n}\mathrm{E}_k(S_{k+n} - S_k).$$

So, summing these relations, defining, as above, $M_k^n = \sum_{i=1}^k D_i^n$, we obtain, for every stationary sequence, not necessarily reversible, and any n and m,

$$S_m = M_m^n + R_m^n, \quad \text{where} \quad R_m^n = \bar{\theta}_0^n - \bar{\theta}_m^n + \frac{1}{n}\sum_{k=0}^{m-1} \mathrm{E}_k(S_{k+n} - S_k). \tag{9}$$

3.2. Proof of Proposition 2

We start from (8) and take the maximum on both sides. We easily obtain

$$\max_{1\le i\le n} |S_i| \le \frac{1}{2}\Big(|X_0| + \max_{1\le i\le n}|X_i| + \max_{1\le i\le n}|M_i^n + \tilde{M}_i^n| + \max_{1\le i\le n}|\bar{R}_i^n|\Big). \tag{10}$$

Note that

$$\max_{1\le i\le n}|\bar{R}_i^n| \le \frac{1}{n}\sum_{i=1}^n (|\mathrm{E}_{i-1}(S_{n+i-1} - S_{i-1})| + |\mathrm{E}_i(S_{n+i} - S_i)|),$$

whence, by Minkowski's inequality and stationarity, for any $p \ge 1$,

$$\Big\|\max_{1\le i\le n}|\bar{R}_i^n|\Big\|_p \le \frac{2}{n}\sum_{i=1}^n \|\mathrm{E}_i(S_{n+i} - S_i)\|_p = 2\|\mathrm{E}_0(S_n)\|_p. \tag{11}$$

Taking into account the fact that $\max_{1\le k\le n}|\tilde{M}_k^n| \le |\tilde{M}_n^n| + \max_{1\le k\le n}|\tilde{M}_n^n - \tilde{M}_k^n|$, we easily deduce that

$$\Big\|\max_{1\le k\le n}|M_k^n + \tilde{M}_k^n|\Big\|_p \le \Big\|\max_{1\le k\le n}|M_k^n|\Big\|_p + \Big\|\max_{1\le k\le n}|\tilde{M}_n^n - \tilde{M}_k^n|\Big\|_p + \|\tilde{M}_n^n\|_p,$$

whence, by applying Doob's maximal inequality twice, stationarity, and reversibility,

$$\Big\|\max_{1\le k\le n}|M_k^n + \tilde{M}_k^n|\Big\|_p \le q\|M_n^n\|_p + (q+1)\|\tilde{M}_n^n\|_p = (2q+1)\|M_n^n\|_p$$

(where q is the conjugate of p).

From (9) we have $M_n^n = S_n - R_n^n$, and from Minkowski's inequality we deduce that

$$\|M_n^n\|_p \le \|S_n\|_p + \frac{2}{n}\sum_{i=0}^{n-1}\|\mathrm{E}_0(S_i)\|_p + \|\mathrm{E}_0(S_n)\|_p,$$

whence

$$\|M_n^n\|_p \le \|S_n\|_p + 3\max_{1\le i\le n}\|\mathrm{E}_0(S_i)\|_p. \tag{12}$$

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From (10), (11), and (12), we deduce the following extension of Doob's maximal inequality for reversible processes:

$$\Big\| \max_{1\le i\le n} |S_i| \Big\|_p \le \frac{1}{2}\Big(\|X_0\|_p + \Big\| \max_{1\le i\le n} |X_i| \Big\|_p + (2q+1)\Big[\|S_n\|_p + 3 \max_{1\le i\le n} \|\mathrm{E}_0(S_i)\|_p \Big] + 2\|\mathrm{E}_0(S_n)\|_p \Big).$$

Taking into account the fact that $\|\mathrm{E}_0(S_i)\|_p \le \|S_i\|_p$ completes the proof of Proposition 2.

3.3. Proof of Proposition 3

For the proof of this proposition, we will use the following claim that can be easily obtained by truncation.

Claim 1. *Let X and Y be two positive random variables. Then, for all $x \ge 0$,*

$$\mathrm{E}\, X\, \mathbf{1}(Y > x) \le \mathrm{E}\, X\, \mathbf{1}\Big(X > \frac{x}{2} \Big) + \frac{x}{2} \mathrm{P}(Y > x).$$

For every $x \ge 0$, using (10), we obtain

$$\mathrm{P}\Big(\max_{1\le i\le n} |S_i| > x \Big) \le \mathrm{P}\Big(\max_{1\le i\le n} |M_i^n + \tilde{M}_i^n| > x \Big) + \mathrm{P}\Big(|X_0| + \max_{1\le i\le n} |X_i| + \max_{1\le i\le n} |\bar{R}_i^n| > x \Big). \tag{13}$$

Applying the Markov inequality, then the triangle inequality followed by (11) with $p = 1$, we obtain

$$\mathrm{P}\Big(|X_0| + \max_{1\le i\le n} |X_i| + \max_{1\le i\le n} |\bar{R}_i^n| > x \Big) \le \frac{2}{x}\Big(\Big\| \max_{1\le i\le n} |X_i| \Big\|_1 + \|\mathrm{E}_0(S_n)\|_1 \Big). \tag{14}$$

By the triangle inequality and reversibility,

$$\begin{aligned}\mathrm{P}\Big(\max_{1\le i\le n} |M_i^n + \tilde{M}_i^n| > x \Big) &\le \mathrm{P}\Big(\max_{1\le i\le n} |M_i^n| > \frac{x}{3} \Big) + \mathrm{P}\Big(\max_{1\le i\le n} \Big| \sum_{k=i}^{n} \tilde{D}_i^n \Big| > \frac{x}{3} \Big) \\ &\quad + \mathrm{P}\Big(|\tilde{M}_n^n| > \frac{x}{3} \Big) \\ &\le 3\, \mathrm{P}\Big(\max_{1\le i\le n} |M_i^n| > \frac{x}{3} \Big).\end{aligned}$$

Then, by Doob's maximal inequality and Claim 1 applied to $X = |M_n^n|$ and $Y = \max_{1\le i\le n} |M_i^n|$, we obtain

$$\begin{aligned}\mathrm{P}\Big(\max_{1\le i\le n} |M_i^n| > \frac{x}{3} \Big) &\le \frac{3}{x} \mathrm{E}\, |M_n^n|\, \mathbf{1}\Big(\max_{1\le i\le n} |M_i^n| > \frac{x}{3} \Big) \\ &\le \frac{3}{x} \mathrm{E}\, |M_n^n|\, \mathbf{1}\Big(|M_n^n| > \frac{x}{6} \Big) + \frac{1}{2} \mathrm{P}\Big(\max_{1\le i\le n} |M_i^n| > \frac{x}{3} \Big),\end{aligned}$$

implying that

$$\mathrm{P}\Big(\max_{1\le i\le n} |M_i^n| > \frac{x}{3} \Big) \le \frac{6}{x} \mathrm{E}\, |M_n^n|\, \mathbf{1}\Big(|M_n^n| > \frac{x}{6} \Big).$$

We now express the right-hand side in terms of S_n. By (9) we have $M_n^n = S_n - R_n^n$, and using the fact that, for all positive real numbers x, y, and a, we have $(x+y)\,\mathbf{1}(x+y>a) \leq 2x\,\mathbf{1}(x>a/2) + 2y\,\mathbf{1}(y>a/2) \leq 2x\,\mathbf{1}(x>a/2) + 2y$, we obtain

$$\begin{aligned}\mathrm{E}\,|M_n^n|\,\mathbf{1}\Big(|M_n^n| > \frac{x}{6}\Big) &\leq 2\,\mathrm{E}\,|S_n|\,\mathbf{1}\Big(|S_n| > \frac{x}{12}\Big) + 2\|R_n^n\|_1 \\ &\leq 2\,\mathrm{E}\,|S_n|\,\mathbf{1}\Big(|S_n| > \frac{x}{12}\Big) + 6\max_{1\leq i\leq n}\|\mathrm{E}_0(S_i)\|_1.\end{aligned}$$

Therefore,

$$\mathrm{P}\Big(\max_{1\leq i\leq n}|M_i^n| > \frac{x}{3}\Big) \leq \frac{6}{x}\Big[2\,\mathrm{E}\,|S_n|\,\mathbf{1}\Big(|S_n| > \frac{x}{12}\Big) + 6\max_{1\leq i\leq n}\|\mathrm{E}_0(S_i)\|_1\Big],$$

and so

$$\mathrm{P}\Big(\max_{1\leq k\leq n}|M_k^n + \tilde{M}_k^n| > x\Big) \leq \frac{18}{x}\Big[2\,\mathrm{E}\,|S_n|\,\mathbf{1}\Big(|S_n| > \frac{x}{12}\Big) + 6\max_{1\leq i\leq n}\|\mathrm{E}_0(S_i)\|_1\Big]. \tag{15}$$

Thus, (13), (14), and (15) lead to

$$\mathrm{P}\Big(\max_{1\leq i\leq n}|S_i| > x\Big) \leq \frac{2}{x}\Big[18\,\mathrm{E}\,|S_n|\,\mathbf{1}\Big(|S_n| > \frac{x}{12}\Big) + 55\max_{1\leq i\leq n}\|\mathrm{E}_0(S_i)\|_1 + \Big\|\max_{1\leq i\leq n}|X_i|\Big\|_1\Big].$$

3.4. Proof of Proposition 4

We first prove the conclusion of the proposition under the assumption that $(S_n^2/\sigma_n^2)_{n\geq 1}$ is uniformly integrable.

By stationarity and Theorem 8.3 of Billingsley (1968, p. 137) formulated for random elements of D, we have to show that, for all $\varepsilon > 0$,

$$\lim_{\delta\to 0^+}\limsup_{n\to\infty}\frac{1}{\delta}\mathrm{P}\Big(\max_{1\leq k\leq [n\delta]}|S_k| > \varepsilon\sigma_n\Big) = 0. \tag{16}$$

By Proposition 3,

$$\begin{aligned}&\mathrm{P}\Big(\max_{1\leq k\leq [n\delta]}|S_k| > \varepsilon\sigma_n\Big) \\ &\quad\leq \frac{2}{\varepsilon\sigma_n}\Big[18\,\mathrm{E}\,|S_{[n\delta]}|\,\mathbf{1}\Big(|S_{[n\delta]}| > \frac{\varepsilon\sigma_n}{12}\Big) + 55\max_{1\leq i\leq [n\delta]}\mathrm{E}\,|\mathrm{E}_0(S_i)| + \mathrm{E}\max_{1\leq i\leq n}|X_i|\Big].\end{aligned} \tag{17}$$

We will analyze each term on the right-hand side of inequality (17) separately.

By the fact that $\lim_{n\to\infty}\sigma_{[n\delta]}^2/\delta\sigma_n^2 = 1$, taking into account the uniform integrability of $(S_n^2/\sigma_n^2)_{n\geq 1}$ leads to

$$\begin{aligned}&\lim_{\delta\to 0^+}\limsup_{n\to\infty}\frac{1}{\delta\sigma_n}\mathrm{E}\,|S_{[n\delta]}|\,\mathbf{1}\Big(|S_{[n\delta]}| > \frac{\varepsilon\sigma_n}{12}\Big) \\ &\quad\leq \lim_{\delta\to 0^+}\limsup_{n\to\infty}\frac{24}{\varepsilon\sigma_n^2}\mathrm{E}\,S_n^2\,\mathbf{1}\Big(\frac{|S_n|}{\sigma_n} > \frac{\varepsilon}{24\delta^{1/2}}\Big) \\ &\quad= 0.\end{aligned}$$

By stationarity and the fact that $\liminf_n \sigma_n^2/n > 0$, we have

$$\frac{1}{\sigma_n^2}\Big(\mathrm{E}\max_{1\le i\le n}|X_i|\Big)^2 \le \frac{1}{\sigma_n^2}\mathrm{E}\max_{1\le i\le n}|X_i|^2 \to 0 \quad \text{as } n\to\infty. \tag{18}$$

Then, by condition (4) and Proposition 1,

$$\frac{1}{\sigma_n^2}\max_{1\le i\le [n\delta]}(\mathrm{E}\,|\mathrm{E}_0(S_i)|)^2 \le \frac{1}{\sigma_n^2}\max_{1\le i\le [n\delta]}\mathrm{E}[\mathrm{E}_0(S_i)]^2 \to 0 \quad \text{as } n\to\infty. \tag{19}$$

Combining the last three convergence results with inequality (17) leads to (16).

To prove the second part of this proposition, assume now that $S_n/\sigma_n \Rightarrow L$. By Theorem 5.3 of Billingsley (1968), we note that the limit has finite second moment, namely,

$$\mathrm{E}\,L^2 \le \liminf_{n\to\infty}\frac{\|S_n\|_2^2}{\sigma_n^2} = 1. \tag{20}$$

Furthermore, since $(|S_n|/\sigma_n)_{n\ge 1}$ is uniformly integrable (because $\mathrm{E}\,S_n^2/\sigma_n^2 = 1$), by (4) and Theorem 5.4 of Billingsley (1968), it follows that

$$\begin{aligned}\limsup_{n\to\infty}\frac{1}{\sigma_n}\mathrm{E}\,|S_{[n\delta]}|\,\mathbf{1}\Big(|S_{[n\delta]}| > \frac{\varepsilon\sigma_n}{12}\Big) &\le \frac{1}{\sqrt{\delta}}\lim_{n\to\infty}\frac{1}{\sigma_{[n\delta]}}\mathrm{E}\,|S_{[n\delta]}|\,\mathbf{1}\Big(\frac{|S_{[n\delta]}|}{\sigma_{[n\delta]}} > \frac{\varepsilon}{24\sqrt{\delta}}\Big)\\ &= \frac{1}{\sqrt{\delta}}\mathrm{E}\,|L|\,\mathbf{1}\Big(|L| > \frac{\varepsilon}{24\sqrt{\delta}}\Big).\end{aligned} \tag{21}$$

By passing to the limit in relation (17) and using (18), (19), and (21), we obtain

$$\limsup_{n\to\infty}\frac{1}{\delta}\mathrm{P}\Big(\max_{1\le k\le [n\delta]}|S_k| > \varepsilon\sigma_n\Big) \le \frac{36}{\varepsilon\delta^{1/2}}\mathrm{E}\,|L|\,\mathbf{1}\Big(|L| > \frac{\varepsilon}{24\sqrt{\delta}}\Big).$$

Then, clearly,

$$\limsup_{n\to\infty}\frac{1}{\delta}\mathrm{P}\Big(\max_{1\le k\le [n\delta]}|S_k| > \varepsilon\sigma_n\Big) \le \frac{36\times 24}{\varepsilon}\mathrm{E}\,L^2\,\mathbf{1}\Big(|L| > \frac{\varepsilon}{24\sqrt{\delta}}\Big).$$

Finally, taking into account (20), the conclusion follows by letting $\delta \to 0^+$.

3.5. Proof of Theorem 1

Because conditional convergence in distribution implies weak convergence, it follows that $S_n/\sigma_n \Rightarrow L$. Then, by the second part of Proposition 4, $W_n(t)$ is tight in $C(0,1)$ endowed with the uniform topology with all possible limits in $C(0,1)$. Now, let us consider a convergent subsequence, say $W_{n'}(t) \Rightarrow X(t)$. Then $X(t)$ is continuous and since S_n/σ_n is conditionally convergent in distribution, $X(t)$ has independent increments (by Lemma 1 below applied on subsequences). It is well known (see, for instance, Doob (1953, Chapter VIII)) that the process $X(t)$ has the representation $X(t) = at + bW(t)$ for some constants a and b, where $W(t)$ is the standard Brownian motion. Without restricting the generality, by symmetry we can assume that $b > 0$. To identify the constants, we use the convergence of moments in the limit theorem, namely Theorem 5.4 of Billingsley (1968). Note that $(S_n/\sigma_n)_{n\ge 1}$ is uniformly integrable in $\mathbb{L}_1$ since it is bounded in $\mathbb{L}_2$. We use this remark to obtain

$$\mathrm{E}\,L = \lim_{n\to\infty}\frac{\mathrm{E}\,S_n}{\sigma_n} = 0 = \lim_{n'\to\infty}\mathrm{E}\,W_{n'}(1) = \mathrm{E}\,X(1) = a + b\,\mathrm{E}\,W(1) = a,$$

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so $a = 0$. Finally, by the same argument, it follows that

$$\lim_{n\to\infty} \frac{\mathrm{E}\,|S_n|}{\sigma_n} = \mathrm{E}\,|L| = \lim_{n'\to\infty} \mathrm{E}\,|W_{n'}(1)| = \mathrm{E}\,|X(1)| = b\,\mathrm{E}\,|W(1)| = b\sqrt{\frac{2}{\pi}},$$

and so $b = \mathrm{E}\,|L|\sqrt{\pi/2}$. It follows that $X(t) = (\mathrm{E}\,|L|\sqrt{\pi/2})W(t)$. In particular, it follows that L has normal distribution and, therefore, $\mathrm{E}\,|L|\sqrt{\pi/2}$ is the standard deviation of L.

Lemma 1. *Under the assumptions of Theorem 1, if $W_n(t) \Rightarrow X(t)$ then $X(t)$ has independent increments.*

Proof. Without loss of generality, for simplicity, we consider only two increments. For any $0 \le s < t \le 1$, we will show that

$$(W_n(s), W_n(t) - W_n(s)) \Rightarrow (X(s), X(t) - X(s)),$$

where $X(s)$ and $X(t) - X(s)$ are independent. By the Cramér–Wold device, it is enough to show that, for any two real numbers a and b,

$$A = \mathrm{E}\exp[\mathrm{i}aW_n(s) + \mathrm{i}b(W_n(t) - W_n(s))] - \mathrm{E}\exp[\mathrm{i}aX(s)]\,\mathrm{E}\exp[\mathrm{i}b(X(t) - X(s))] \to 0.$$

To see this, note that

$$\mathrm{E}\exp[\mathrm{i}aW_n(s) + \mathrm{i}b(W_n(t) - W_n(s))] = \mathrm{E}\exp[\mathrm{i}aW_n(s)]\,\mathrm{E}_{[ns]}\exp[\mathrm{i}b(W_n(t) - W_n(s))].$$

By adding and subtracting $\mathrm{E}\exp[\mathrm{i}aW_n(s)]\,\mathrm{E}\exp[\mathrm{i}b(X(t) - X(s))]$ to A, we easily obtain

$$\begin{aligned}|A| &\le \mathrm{E}\,|\mathrm{E}_{[ns]}\exp[\mathrm{i}b(W_n(t) - W_n(s)] - \mathrm{E}\exp[\mathrm{i}b(X(t) - X(s))]| \\ &\quad + |\mathrm{E}\exp[\mathrm{i}aW_n(s)] - \mathrm{E}\exp[\mathrm{i}aX(s)]| \\ &= I + II.\end{aligned}$$

Since we assume that $W_n(s) \Rightarrow X(s)$, it follows that $II \to 0$. Furthermore, by (3), $X(s)$ and $s^{1/2}L$ are identically distributed.

To treat the term I, note that, by stationarity and the definition of $W_n(t)$, we have

$$I = \mathrm{E}\left|\mathrm{E}_0\exp\left[\mathrm{i}b\left(\frac{S_{[nt]-[ns]}}{\sigma_n}\right)\right] - \mathrm{E}\exp[\mathrm{i}b(X(t) - X(s))]\right|. \tag{22}$$

Because we assume that $\sigma_n \to \infty$ we have

$$\frac{1}{\sigma_n}\,\mathrm{E}\,|S_{[nt]-[ns]} - S_{[n(t-s)]}| \to 0, \tag{23}$$

which easily implies that, for all b,

$$\mathrm{E}\left|\mathrm{E}_0\exp\left[\mathrm{i}b\left(\frac{S_{[nt]-[ns]}}{\sigma_n}\right)\right] - \mathrm{E}_0\exp\left[\mathrm{i}b\left(\frac{S_{[n(t-s)]}}{\sigma_n}\right)\right]\right| \to 0. \tag{24}$$

Now, since $S_{[n(t-s)]}/\sigma_n \Rightarrow X(t-s)$ and $S_{[nt]} - S_{[ns]}/\sigma_n \to X(t) - X(s)$, we deduce from (23) and stationarity that $X(t-s)$ and $X(t) - X(s)$ have the same distribution. Furthermore, by (3), we deduce that $S_{[n(t-s)]}/\sigma_n$ is also conditionally convergent in distribution; so, in addition, $X(t-s)$ is distributed as $(t-s)^{1/2}L$. By taking into account (22) and (24) as well, it follows that

$$\limsup_{n\to\infty} I = \limsup_{n\to\infty} \mathrm{E}\left|\mathrm{E}_0\exp\left[\mathrm{i}b\left(\frac{S_{[n(t-s)]}}{\sigma_n}\right)\right] - \mathrm{E}\exp[\mathrm{i}b(X(t-s))]\right| = 0,$$

leading to the conclusion.

3.6. Proof of Corollary 1

The proof of this corollary follows the lines of Theorem 1 with the exception that we replace Lemma 1 by the following lemma.

Lemma 2. *Under the assumptions of Corollary 1, if $W_n(t) \Rightarrow X(t)$ then $X(t)$ has independent increments.*

Proof. Note that, as the Markov chain is stationary, irreducible, and aperiodic, it follows that it is absolutely regular (see Theorems 21.5 and Corollary 21.7 of Bradley (2007b)). It is well known that an absolutely regular sequence is strong mixing (see the chart on page 186 of Bradley (2007a)). This means that $\alpha_n \searrow 0$, where

$$\alpha_n = \sup \mathrm{P}(A \cap B) - \mathrm{P}(A)\,\mathrm{P}(B);$$

here the supremum is taken over all $A \in \sigma(\xi_i,\ i \le 0)$ and $B \in \sigma(\xi_i,\ i \ge n)$. Because we know from the proof of Theorem 1 that the process $X(t)$ is continuous, it is enough to show that, for all k and $0 < s_1 < t_1 < s_2 < t_2 < \cdots < s_k < t_k < 1$, the increments $(X(t_i) - X(s_i))_{1 \le i \le k}$ are independent. Now, using the definitions of α_n and $W_n(t)$, we obtain, by recurrence,

$$\left| \mathrm{P}\left(\bigcap_{i=1}^{k} (W_n(t_i - s_i) \in A_i) \right) - \prod_{i=1}^{k} \mathrm{P}(W_n(t_i - s_i) \in A_i) \right| \le \min_{1 \le i \le k-1} \alpha_{[n(s_{i+1} - t_i)]}$$
$$\to 0 \quad \text{as } n \to \infty$$

for any Borelians $A_1, \ldots, A_k$. The conclusion follows by passing to the limit with n.

3.7. Proof of Proposition 5

By Proposition (1) we know that $\sigma_n^2 = nh(n)$ with h a function slowly varying at ∞. Then, by the first part of Proposition 4, $W_n(t)$ is tight in $D(0, 1)$. It remains to apply Theorem 19.4 of Billingsley (1968).

4. Application to a Metropolis–Hastings algorithm

In this section we analyze a standardized example of a stationary irreducible and aperiodic Metropolis–Hastings algorithm with uniform marginal distribution. This type of Markov chain is interesting since it can easily be transformed into Markov chains with different marginal distributions. We point out a CLT under a normalization other than the variance of partial sums. Markov chains of this type are often studied in the literature from different points of view; see Doukhan *et al.* (1994), Rio (2000), (2009), and Merlevède and Peligrad (2012). The idea of considering the Metropolis–Hastings algorithm in this context comes from Zhao *et al.* (2010).

Let $E = [-1, 1]$. We now define the transition probabilities of a Markov chain by

$$Q(x, A) = (1 - |x|)\delta_x(A) + |x|\upsilon(A),$$

where δ_x denotes the Dirac measure and υ on $[-1, 1]$ satisfies

$$\upsilon(\mathrm{d}x) = |x|\,\mathrm{d}x.$$

Then there exists a unique invariant measure, with uniform distribution on $[-1, 1]$,

$$\pi(\mathrm{d}x) = \tfrac{1}{2}\mathrm{d}x,$$

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and the stationary Markov chain $(\xi_i)_i$ with values in E and transition probability $Q(x, A)$ is reversible and positively recurrent. Moreover, for any odd function f, we have

$$Q^k(f)(\xi_0) = \mathrm{E}(f(\xi_k) \mid \xi_0) = (1 - |\xi_0|)^k f(\xi_0) \quad \text{a.s.}$$

For the odd function $f(x) = \operatorname{sgn} x$, define $X_i = \operatorname{sgn} \xi_i$. In this context we will show the following result.

Result 1. *Let $(X_j)_{j\geq 1}$ be as defined above. Then $\sigma_n^2/(2n \log n) \to 1$ and*

$$\frac{1}{\sigma_n} \sum_{j=1}^{[nt]} X_j \Rightarrow \frac{1}{2^{1/2}} W(t),$$

where $W(t)$ is the standard Brownian motion.

Proof. For any $m \geq 0$, we have

$$\mathrm{E}(X_0 X_m) = \mathrm{E}(f(\xi_0) Q^m(f)(\xi_0)) = \int_E (1 - |x|)^m \pi(\mathrm{d}x) = \frac{1}{m+1}.$$

Therefore, by simple computations we obtain

$$\sigma_n^2 \sim 2n \log n \quad \text{as } n \to \infty.$$

To find the limiting distribution of S_n properly normalized, we study the regeneration process. Let

$$T_0 = \inf\{i > 0 \colon \xi_i \neq \xi_0\}$$

and

$$T_{k+1} = \inf\{i > T_k \colon \xi_i \neq \xi_{i-1}\}, \qquad \tau_k = T_{k+1} - T_k.$$

It is well known that $(\xi_{\tau_k}, \tau_k)_{k\geq 1}$ are independent and identically distributed (i.i.d.) random variables with ξ_{τ_k} having the distribution υ. Furthermore,

$$\mathrm{P}(\tau_1 > n \mid \xi_{\tau_1} = x) = (1 - |x|)^n.$$

Then it follows that

$$\mathrm{E}(\tau_1 \mid \xi_{\tau_1} = x) = \frac{1}{|x|} \quad \text{and} \quad \mathrm{E}(\tau_1) = 2.$$

So, by the law of large numbers, $T_n/n \to 2$ a.s.

Let us study the tail distribution of τ_1. Since

$$\mathrm{P}(\tau_1 |X_{\tau_1}| > y \mid \xi_{\tau_1} = x) = \mathrm{P}(\tau_1 > y \mid \xi_{\tau_1} = x) = (1 - |x|)^y,$$

by integration we obtain

$$\mathrm{P}(\tau_1 > y) = \int_{-1}^{1} (1 - |x|)^y |x| \, \mathrm{d}x = 2 \int_0^1 (1 - x)^y x \, \mathrm{d}x \sim 2y^{-2} \quad \text{as } y \to \infty.$$

Moreover, $\mathrm{E}(\tau_k X_{\tau_k}) = 0$ by symmetry. Also,

$$H(y) = \mathrm{E}(\tau_1^2 \mathbf{1}(\tau_1 \leq y)) \sim 4 \ln y.$$

Define a normalization satisfying $b_n^2 \sim nH(b_n)$. In our case, $b_n^2 \sim 4n \ln b_n$, implying that $b_n^2 \sim 2n \ln n$.

For each n, let m_n be such that $T_{m_n} \le n < T_{m_n+1}$.

We have the representation

$$\sum_{k=1}^{n} X_k - \sum_{k=1}^{[n/2]} Y_k = (T_0 - 1)X_0 + \left(\sum_{k=1}^{m_n} \tau_k X_{\tau_k} - \sum_{k=1}^{[n/2]} \tau_k X_{\tau_k}\right) + \sum_{k=T_{m_n+1}}^{n} X_k, \tag{25}$$

where $Y_k = \tau_k X_{\tau_k}$ is a centered i.i.d. sequence in the domain of attraction of a normal law. By the limit theorem for i.i.d. variables in the domain of attraction of a stable law (see Feller (1971)) we obtain

$$\frac{\sum_{k=1}^{[n/2]} Y_k}{b_{[n/2]}} \Rightarrow N(0, 1). \tag{26}$$

By Theorem 4.1 of Billingsley (1968), the CLT for $(\sum_{k=1}^{n} X_k)/b_{[n/2]}$ will follow from (25) and (26) provided we show that the normalized quantity on the right-hand side of (25) converges in probability to 0. Clearly, because $b_{[n/2]} \to \infty$ we have

$$\frac{(T_0 - 1)X_0}{b_{[n/2]}} \Rightarrow 0.$$

Also,

$$\mathrm{E}\, \frac{|\sum_{k=T_{m_n+1}}^{n} X_k|}{b_{[n/2]}} \le \frac{\mathrm{E}\,|\tau_{m_n+1}|}{b_{[n/2]}} = \frac{2}{b_{[n/2]}} \to 0.$$

Therefore, it remains to study the middle term. Let $\delta > 0$. Then

$$\begin{aligned}\mathrm{P}\left(\left|\sum_{k=1}^{m_n} Y_k - \sum_{k=1}^{[n/2]} Y_k\right| > \varepsilon b_{[n/2]}\right) &\le \mathrm{P}\left(\left|\frac{m_n}{n} - \frac{1}{2}\right| \ge \delta\right) \\ &\quad + \mathrm{P}\left(\max_{n/2-\delta n<l<n/2+\delta n}\left|\sum_{k=1}^{l} Y_k - \sum_{k=1}^{[n/2]} Y_k\right| > \varepsilon b_{[n/2]}\right) \\ &= I + II.\end{aligned}$$

By the definition of m_n and the law of large numbers for the i.i.d. sequence $(\tau_i)_{i\ge 1}$, we know that $m_n/n \to 1/\mathrm{E}(\tau_1) = \frac{1}{2}$ a.s. Therefore, the first term converges to 0 for every fixed δ as $n \to \infty$. As for the second term, by stationarity and the fact that the Y_k are i.i.d.,

$$II \le 2\,\mathrm{P}\left(\max_{1\le l\le[\delta n]+1}\left|\sum_{k=1}^{l} Y_k\right| > \frac{\varepsilon b_{[n/2]}}{2}\right),$$

and by Theorem 1.1.5 of De la Peña and Giné (1999),

$$II \le 2\,\mathrm{P}\left(\max_{1\le l\le[\delta n]+1}\left|\sum_{k=1}^{l} Y_k\right| > \frac{\varepsilon b_{[n/2]}}{2}\right) \le 18\,\mathrm{P}\left(\left|\sum_{k=1}^{[\delta n]+1} Y_k\right| > \frac{\varepsilon b_{[n/2]}}{60}\right).$$

Then, by the CLT in (26) and the fact that $b_n^2 \sim 2n \ln n$, we have

$$\limsup_{n\to\infty} \mathrm{P}\bigg(\bigg|\sum_{k=1}^{[\delta n]+1} Y_k\bigg| > \frac{\varepsilon b_{[n/2]}}{60}\bigg) = \limsup_{n\to\infty} \mathrm{P}\bigg(\frac{|\sum_{k=1}^{[\delta n]+1} Y_k|}{b_{[\delta n]}} > \frac{\varepsilon b_{[n/2]}}{60 b_{[\delta n]}}\bigg)$$
$$\leq \mathrm{P}\bigg(N(0,1) > \frac{\varepsilon \delta^{-1/2}}{120}\bigg),$$

which converges to 0 as $\delta \to 0$.

It follows that

$$\frac{S_n}{b_{[n/2]}} \Rightarrow N(0,1).$$

We recall that $\sigma_n^2 = 2n \log n = b_n^2$, implying that

$$\frac{S_n}{\sigma_n} \Rightarrow N\bigg(0, \frac{1}{2}\bigg).$$

Consequently, because the chain is irreducible and aperiodic, by Corollary 1,

$$W_n(t) \Rightarrow 2^{-1/2} W(t).$$

For a different example having this type of asymptotic behavior, we cite Zhao *et al.* (2010). Our Corollary 1 will also provide a functional CLT for their example.

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