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ON CLIFFORD'S THEOREM AND RAMIFICATION INDICES FOR SYMPLECTIC MODULES OVER A FINITE FIELD

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Introduction

Let K be a field, G a finite group. Let V be an (irreducible) KG-module, where KG is the group algebra consisting of all formal sums $\sum_{g\in G} a_g g$, $a_g \in K$, $g \in G$. The action of $\alpha = \sum a_g g$ on an element $v \in V$ obeys the rule $v(\sum_{g\in G} a_g g) = \sum_{g\in G} (a_g v)g$. If H is a subgroup of G, then, restricting the action of G on V to H, V is also a KH-module. Notation: V_H.

Let now N be a normal subgroup of G. The KN-module V_N is not irreducible in general, even when V is irreducible as KG-module. The well-known theorem of A. H. Clifford ([3], V.17.3) tells us precisely what is going on here.

Theorem (A. H. Clifford, 1938). *Let V be an irreducible KG-module. Let $N \lhd G$. Then the following properties hold.*

(a) *If W is an irreducible KN-submodule of V, then $V = \sum_{g\in G} Wg$. Every Wg is an irreducible KN-module and V is a completely reducible KN-module.*

(b) *Let $W_1, \ldots, W_n$ be representatives of the isomorphism classes of the irreducible KN-submodules of V. Write*

$$V_i = \sum_{\substack{W \subseteq V \\ W \cong W_i}} W \qquad (i = 1, \ldots, n).$$

Then V_i is homogeneous, i.e. it is a direct sum of KN-submodules of V, all being isomorphic to W_i, as KN-modules. Moreover $V = \bigoplus_{i=1}^{n} V_i$.

(c) *Let F_i be the irreducible representation of N on W_i. Then F_i^g, defined by $(w_i g)(F_i^g(n)) = (w_i F_i(n))g$, $w_i \in W_i$, $g \in G$ is the irreducible representation of N on $W_i g$.*

(d) *The homogeneous components V_i of the KN-module V are permuted transitively by elements of G by multiplication on the right.*

(e) *For every j the equality*

$$\{g \mid g \in G, V_j g = V_j\} = \{g \mid g \in G, F_j^g \text{ equivalent to } F_j\}$$

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holds. These elements g constitute the subgroup A_j (say) of G. Then V_j is an irreducible KA_j-module. We have $V \cong V_j \otimes_{KA_j} KG = V_j^G$ ("V is induced by V_j").

(f) *Let D be the representation of G on V. The irreducible constituents of D_N are precisely all the G-conjugates F^g of a single irreducible representation F of N. They occur all with the same multiplicity e.*

(g) *If χ is the trace function of D and if ϕ is the trace function of an irreducible constituent F of D_N, then $\chi_N = e(\sum_{i=1}^{n} \phi^{g_i})$, where the g_i are representatives of the right cosets of the subgroup $A = \{g \mid g \in G, F^g$ equivalent to $F\}$ in G. Notice that $A \supseteq N$. The positive integer e is called the inertia index (or ramification index) of D (or V) over N.*

Let G, N and A be the groups just mentioned in Clifford's Theorem. Sometimes we would like to know whether e divides $|A/N|$. This happens certainly in two well known cases:

1. K algebraically closed of charactertistic zero or of positive characteristic not dividing the order of G; see **[13]**, page 35.
2. K a finite field of odd characteristic not dividing the order of G and containing the primitive mth-roots of unity, where $m = |G|_{2'}$, G/N an elementary abelian p-group; see **[10]**, Theorem 13, due to W. Willems.

It is not true that the divisibility property of the inertia index always holds. As an example, take R cyclic of order 3, $K = \mathbb{F}_2$, $\{1\} = N \lhd R$. Then there exists an irreducible two-dimensional $\mathbb{F}_2$-representation of R with inertia index 2 over N. One of the purposes of this paper is to show that the behaviour of e can be described if G/N has prime order, G arbitrary, K a finite field. It is done in Theorem E.

In this paper we also study the situation in which $\mathbb{F}$ is a finite field, V a $\mathbb{F}G$-module, such that the vector space V carries a non-singular alternating bilinear form with values in $\mathbb{F}$, which is left invariant by G. Such a $\mathbb{F}G$-module is called *symplectic*. If L is a $\mathbb{F}G$-module, then L^* will denote the *dual module*. Thus $L^* = \mathrm{Hom}_{\mathbb{F}}(L, \mathbb{F})$ and the action of G on L^* is defined by $v(\alpha g) = (vg^{-1})\alpha$ for $\alpha \in L^*$, $g \in G$, $v \in L$. If $L \cong L^*$ as $\mathbb{F}G$-modules then L is called *self-dual*. It is well known that L is self-dual if and only if L carries a non-singular, G-invariant, bilinear form.

The following situations will be studied.

I. Let $\mathbb{F}$ be a finite field and let V be a faithful irreducible symplectic $\mathbb{F}G$-module. Let $N \lhd G$, $|G/N| =$ odd prime number. What does the decomposition of V_N look like? Or, what happens with $(V \otimes_{\mathbb{F}} K)_N$ for a suitable field extension K of finite degree over $\mathbb{F}$? Does an irreducible constituent of $V \otimes_{\mathbb{F}} K$ decompose as a direct sum of irreducible KN-modules, each being symplectic and standing perpendicular to each other with respect to the (tensored) symplectic K-form? What about the ramification index e? Is it equal to 1, to $|G/N|$, or to something else? An answer to these questions will be given in Theorem A. In a Corollary to Theorem A somewhat more can be said when $\mathbb{F}$ has characteristic 2.

II. If we impose more conditions on the group G, then we can sharpen Theorem A. The result is Theorem B. The proof of Theorem B is a corollary to Theorem A.

III. Suppose that the symplectic $\mathbb{F}G$-module V with $\mathbb{F}$ a finite field, is a direct sum of pairwise non-isomorphic, self-dual, irreducible $\mathbb{F}G$-modules. We say that such a $\mathbb{F}G$-module is *monoprimary*. Let $N \lhd G$. Suppose that the order of G/N is odd and assume that every prime divisor of $|G/N|$ divides $|\mathbb{F}|-1$. Then V_N is monoprimary (Theorem C). In order to prove that theorem we first consider the special case where V is an irreducible symplectic $\mathbb{F}G$-module, $|G/N|$ = odd prime number q, q divides $|\mathbb{F}|-1$. It turns out that V_N is monoprimary and so the inertia index e is equal to 1 (Theorem D). The statement of Theorem D resembles that of the analogous statement made in the proof of Theorem (3.1) of [**8**]. The method of the proof of Theorem D given here, can be regarded as a specialization of the proof of Theorem A. For an application of Theorem D we refer to Theorem (2.3) of [**12**]. It shows that in Theorem C the word "monoprimary" can be replaced by the word "anisotropic". As such, (2.3) of [**12**] is a generalization of (3.1) of [**8**]. It then yields one of the main results of [**12**] stated as follows.

Theorem ([**12**], R. W. van der Waall and N. S. Hekster). *Suppose that p is an odd prime, that G is a finite p-solvable group, that N is a normal subgroup of G, and that χ is a monomial irreducible character of N whose degree $\chi(1)$ is a power of p. Let η be an irreducible constituent of the induced character χ^G. Assume that every prime divisor of $|G/N|$ divides $p(p-1)$ and that G/N is supersolvable of odd order. Then η is a monomial character.*

The above theorem should be compared with Dade's Theorem (0) in [**2**]:

Theorem ([**2**], E. C. Dade). *Suppose that p is an odd prime, that G is a finite p-solvable group, that ψ is a monomial irreducible character of G whose degree $\psi(1)$ is a power of p, that N is a subnormal subgroup of G, and that an irreducible character χ of N is a constituent of the restriction ψ_N of ψ. Then χ is monomial.*

To conclude this Introduction, a few remarks are in order.

All the questions mentioned above about the inertia index e and on the symplectic Schur–Clifford theory play an essential role in the (complex) representation theory of finite groups today. The reader is referred to papers of Isaacs, Berger, Dade, Parks and van der Waall; see notably [**1, 2, 6, 7, 8, 9, 10, 11, 12**]. In all these papers monomial characters are focussed as a central theme.

Notations and conventions

Most of the notations are standard and can be found in [**3, 4, 5**] or are otherwise clear or self-explanatory. We recall some notions.

(1) Consider a type of operation on isomorphism classes of FG-modules (though apparently not in any natural way on the modules, themselves). We have in mind the following. Let α be an automorphism of F. If V is an FG-module, then by a choice of basis, V determines an F-representation X of G. Application of α to the entries of the matrices $X(G)$ yields a new F-representation X^{α}. This corresponds to some FG-module

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whose isomorphism class is uniquely determined by V and α. We shall write V^α to denote any module in this class. If F is a finite field with $b=p^n$ elements, with $p=\operatorname{char} F$, then $\operatorname{Gal}(F/\mathbb{F}_p)=\langle\beta\rangle$, where $\mathbb{F}_p$ is the prime field of F, and where β is the Frobenius automorphism $x\mapsto x^p$, $x\in F$. We then denote V^{β^i} sometimes by V^{p^i}.

(2) Definition (3.6) of [**8**]. Let $F\subseteq E$ be fields and let V be an EG-module. Then V is weakly self-dual over F if $V^*\cong V^\alpha$ for some $\alpha\in\operatorname{Gal}(E/F)$.

(3) Lemma (3.4) of [**8**]. Let $N\lhd G$ with G/N abelian and suppose that F is a splitting field for G/N with $\operatorname{char} F$ not dividing $|G/N|$. If V and W are irreducible FG-modules such that V_N and W_N have a common irreducible constituent, then $W\cong V\mu$ for some linear F-character μ of G/N.

(4) Proposition (3.7) of [**8**]. Let $E\supseteq F$ be fields with $\operatorname{Gal}(E/F)$ abelian, and let V be an EG-module which is weakly self-dual over F. If λ is an F-character of G of odd multiplicative order and $V\lambda$ is also weakly self-dual over F, then $V\cong V\lambda$.

(5) $0_{2'}(G)=$ product of all normal subgroups M of G with $2\nmid |M|$.
$F(G)=$ Fitting subgroup of G.
$\Omega_1(G)=\langle g\,|\,g\in G, g^p=1\rangle$; here G is a p-group for some prime p.
$0_p(G)=$ the maximal normal p-subgroup of G.
$\mathbb{F}_t=$ finite field consisting of t elements.
$\bar{E}=$ an algebraic closure of the field E.
$\mathbb{F}(\chi)$: see the definition given in the last lines of page 151 of [**5**].

The theorems and their proofs

Theorem A. *Let G be a finite group. Suppose V is a faithful irreducible non-singular symplectic $\mathbb{F}G$-module for a certain finite field $\mathbb{F}$. Let $N\lhd G$, $|G/N|=q$, where q is an odd prime number. Then there exists a finite field $\mathbb{K}$ containing $\mathbb{F}$ such that at least one of the following properties holds.*

(1) *The $\mathbb{K}G$-module $V\bigotimes_{\mathbb{F}}\mathbb{K}$ contains a faithful irreducible non-singular symplectic $\mathbb{K}G$-module W such that $W_N=U_1\perp\cdots\perp U_q$, where $U_i\not\cong U_j$ as $\mathbb{K}N$-modules if $i\neq j$, the U_i are irreducible non-singular symplectic $\mathbb{K}N$-submodules of W_N for the symplectic form on W restricted to U_i.*

(2) *The $\mathbb{K}G$-module $V\bigotimes_{\mathbb{F}}\mathbb{K}$ contains a faithful irreducible non-singular symplectic $\mathbb{K}G$-module W such that W is also irreducible when considered as $\mathbb{K}N$-module.*

(3) *There exists a self-dual absolutely irreducible $\mathbb{K}G$-module T which is also absolutely irreducible as $\mathbb{K}N$-module and there exists a 2-dimensional irreducible $\mathbb{K}G$-module S such that N acts trivially on S in such a way that $T\bigotimes_{\mathbb{K}}S$ is isomorphic to a faithful irreducible non-singular symplectic $\mathbb{K}G$-submodule of $V\bigotimes_{\mathbb{F}}\mathbb{K}$.*

Proof. There are two cases to be considered. Namely, (A) V_N is not homogeneous, (B) V_N is homogeneous.

(A) *Let V_N be not homogeneous.* Then it follows from Clifford's theorem ([**3**], V.17.3) that V_N is a direct sum of q pairwise non-isomorphic $\mathbb{F}N$-submodules. Call them $U_1,\ldots,U_q$. Hence

$$V_N=U_1\dotplus\cdots\dotplus U_q. \tag{1}$$

In fact we see that here any irreducible $\mathbb{F}N$-submodule T of V_N is equal to precisely one of the U_i. With respect to the symplectic form it follows from a well known folklore theorem that the completely reducible $\mathbb{F}N$-module V_N admits an orthogonal direct sum decomposition

$$V_N = M_1 \perp \cdots \perp M_s \perp (M_{s+1} \dotplus M_{s+1}^*) \perp \cdots \perp (M_{s+t} \dotplus M_{s+t}^*) \tag{2}$$

where the $M_1, \ldots, M_s$ are irreducible non-singular symplectic $\mathbb{F}N$-modules with the form on V restricted to M_i, and where all the $M_{s+1}, \ldots, M_{s+t}^*$ are irreducible totally isotropic $\mathbb{F}N$-modules; the matrix representation afforded by M_{s+i}^* is the inverse-transpose to that afforded by M_{s+i}. Following the Krull–Schmidt Theorem applied on (1) and (2) there is at least one U_1 (say) exactly equal to some M_i belonging to the set $\{M_1, \ldots, M_s\}$ as this set is not empty; namely $q = s + 2t$, again by the Krull–Schmidt Theorem as each of the $M_1, \ldots, M_{s+t}^*$ is its own homogeneous component in V_N. Write $M_1 = U_1$. The $M_1 g, g \in G$, are irreducible $\mathbb{F}N$-modules and they are all self-dual by construction of the action of g on V. Thus $M_1 g$ is precisely equal to one of the $M_1, \ldots, M_s$. Now, if t would be an integer larger than zero, then we would conclude that G does not act transitively on all the homogeneous components of V_N by multiplication on the right. Clifford's Theorem, however, implies that $\{M_1 g \mid g \in G\}$ is the set of the homogeneous components of V_N. Therefore $V_N = U_1 \perp \cdots \perp U_q$, $U_i \not\cong U_j$ if $\neq j$. Hence V_N is anisotropic in this case, i.e. V_N does not contain isotropic $\mathbb{F}N$-submodules other than (0).

(B) Let $b = p^t$ be the number of elements of $\mathbb{F}$, where $p = \operatorname{char} \mathbb{F}$. *We now assume that V_N is a direct sum of e isomorphic irreducible $\mathbb{F}N$-submodules. Let U be one of them. Set $V_N = eU$.*

(B.1) Let $q = p$. Then Green's Theorem ([4], VII.9.19) yields $e = 1$, $V_N = U$. Hence case (2) applies here with $\mathbb{K} = \mathbb{F}$.

(B.2) Let $q \neq p$. Then [5, 9.21] implies that

$$V \underset{\mathbb{F}}{\otimes} \bar{\mathbb{F}} = (V_1 \underset{K}{\otimes} \bar{\mathbb{F}}) \dotplus (V_1^b \underset{K}{\otimes} \bar{\mathbb{F}}) \dotplus \cdots \dotplus (V_1^{b^{\alpha-1}} \underset{K}{\otimes} \bar{\mathbb{F}}),$$

where $\alpha = |\mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F})|$ and $K = \mathbb{F}(\chi)$, and where

$$V \underset{\mathbb{F}}{\otimes} K = V_1 \dotplus V_1^b \dotplus \cdots \dotplus v_1^{b^{\alpha-1}}. \tag{3}$$

(Notice that $K = \mathbb{F}(\chi^{b^i})$, any $i = 0, \ldots, \alpha - 1$, by Theorem 9.21.c of [5].) Observe that $V_1^{b^i} \not\cong V_1^{b^j}$ if $i \neq j$ and that the $V_1^{b^i}$ are absolutely irreducible KG-modules for any i and that also

$$V_1^{b^i} \underset{K}{\otimes} \bar{\mathbb{F}} \not\cong V_1^{b^j} \underset{K}{\otimes} \bar{\mathbb{F}}$$

if $i \neq j$. Now, if $S(\cdot,\cdot)$ is the symplectic form governing the $\mathbb{F}G$-module V, with values in $\mathbb{F}$, then $S_1(\cdot,\cdot)$ defined by

$$S_1\left(\sum_i (x_i \otimes a_i), \sum_j (y_j \otimes b_j)\right) = \sum_{i,j} S(x_i, y_j)\, a_i b_j$$

for all $\sum_i(x_i \otimes a_i), \sum_j(x_j \otimes b_j)$ in $V \bigotimes_{\mathbb{F}} K$, makes $V \bigotimes_{\mathbb{F}} K$ into a non-singular symplectic KG-module. As $V \bigotimes_{\mathbb{F}} K$ is completely reducible as KG-module, it follows again that an orthogonal direct sum decomposition holds as indicated,

$$V \bigotimes_{\mathbb{F}} K = M_1 \perp \cdots \perp M_a \perp (M_{a+1} \dot{+} M^*_{a+1}) \perp \cdots \perp (M_{a+u} \dot{+} M^*_{a+u}). \tag{4}$$

Apply the Krull–Schmidt Theorem on (3) and (4). Then it follows that in (4) all the written M's are pairwise non-isomorphic and galois conjugated to each other.

(B.2.α) Assume $u=0$, i.e. $V \bigotimes_{\mathbb{F}} K = M_1 \perp \cdots \perp M_a$. Here M_1 is a faithful non-singular symplectic absolutely irreducible KG-submodule of $V \bigotimes_{\mathbb{F}} K$. If $a \geqq 2$, then we apply induction to the dimension of the given irreducible module as vector space over its ground field and we conclude that the theorem holds. More precisely, replace V by M_1 and $\mathbb{F}$ by K in the statement of the theorem and observe that $M_1 \bigotimes_K \mathbb{K}$ can be considered as $\mathbb{K}G$-submodule of $(V \bigotimes_{\mathbb{F}} K) \bigotimes_K \mathbb{K} \cong V \bigotimes_K \mathbb{K}$. Hence assume $a=1$. Then $V \bigotimes_{\mathbb{F}} \bar{\mathbb{F}} \cong M_1 \bigotimes_K \bar{\mathbb{F}}$ is irreducible and so V is an absolutely irreducible $\mathbb{F}G$-module. Hence

$$e\left(U \bigotimes_{\mathbb{F}} \bar{\mathbb{F}}\right) \cong (eU) \bigotimes_{\mathbb{F}} \bar{\mathbb{F}} \cong \left(V \bigotimes_{\mathbb{F}} \bar{\mathbb{F}}\right)_N = \begin{cases} L_1 \dot{+} \cdots \dot{+} L_q, L_i \ncong L_j \text{ if } i \neq j, \text{ or} \\ L_1, \end{cases}$$

where the L_j are the irreducible constituents of $(V \bigotimes_{\mathbb{F}} \bar{\mathbb{F}})_N$; here we made use of Theorem VII.9.18 of [4], applied to the cyclic p'-group G/N of order q. Therefore certainly $e=1$ and we are in case (2) with $\mathbb{K} = \mathbb{F}$.

(B.2.β) Let $u \geqq 1$. Since $(V_1^*)^{b^i} \cong (V_1^{b^i})^*$ for any i and since $M_t \cong M_t^*$ if $t \in \{1, \ldots, a\}$, it cannot happen that $a \geqq 1$. Indeed, let $V_1 = M_1$. Then for some j, $V_1^{b^j} = M_{a+1} \ncong M^*_{a+1} = (V_1^{b^j})^* \cong (V_1^*)^{b^j} \cong V_1^{b^j}$, with contradiction. Therefore we have

$$V \bigotimes_{\mathbb{F}} K = (M_1 \dot{+} M_1^*) \perp \cdots \perp (M_u \dot{+} M_u^*).$$

Now $M_1^* \cong M_1^r$ for some $r = b^f$ with $f \in \{1, \ldots, 2u-1\}$. Consider a matrix representation corresponding to the action of G on M_1. Let $\omega_1, \ldots, \omega_s$ be the eigenvalues (counted with multiplicities, i.e. the representation is s-dimensional) of a matrix corresponding to a particular element $g \neq 1$ of G. Then $\omega_1^{-1}, \ldots, \omega_s^{-1}$ are the eigenvalues for the inverse-transpose matrix corresponding to the element g. Therefore

$$\sum_{i=1}^{s} \omega_i^r = \sum_{i=1}^{s} \omega_i^{-1}$$

and also $\omega_{\sigma(i)}^{-1} = \omega_i^r$, $i = 1, \ldots, s$ for some σ contained in the symmetric group $\sum_s$. This leads to

$$\left(\sum_{i=1}^{s} \omega_i\right)^{r^2} = \left(\sum_{i=1}^{s} \omega_i^r\right)^r = \left(\sum_{i=1}^{s} \omega_i^{-1}\right)^r = \sum_{i=1}^{s} \omega_i^{-r} = \sum_{i=1}^{s} (\omega_i^r)^{-1}$$

$$= \sum_{i=1}^{s} (\omega_{\sigma(i)}^{-1})^{-1} = \sum_{i=1}^{s} \omega_{\sigma(i)} = \sum_{i=1}^{s} \omega_i.$$

Since $K=\mathbb{F}(\chi)$, it follows that $\sum_{i=1}^{s}\omega_i\in K\cap\mathbb{F}_{r^2}\subset\bar{\mathbb{F}}$. This holds for all such traces and so $K\subseteq\mathbb{F}_{r^2}$. Moreover $M_1^{r^2}\cong M_1$, but $M_1^r\cong M_1^*\not\cong M_1$. Certainly $r^2\in\{b^{2u},b^{4u},b^{6u},\dots\}$. As now $r=b^f\leq b^{2u-1}<b^{2u}$, we see that $r=b^f=b^u$ and so $K=\mathbb{F}_{r^2}$.

Thus we have $V\otimes_{\mathbb{F}}\mathbb{F}_r=L_1\dot{+}\dots\dot{+}L_u$, $L_i\not\cong L_j$ if $i\neq j$, and the L_i are irreducible $\mathbb{F}_rG$-modules. It is clear that a numbering of the $L_1,\dots,L_u$ can be chosen such that $L_i\otimes_{\mathbb{F}_r}K\cong M_i\dot{+}M_i^r\cong M_i\dot{+}M_i^*$, $i=1,\dots,u$. Because of $(L_i\otimes_{\mathbb{F}_r}K)^*\cong(M_i\dot{+}M_i^*)^*\cong M_i^*\dot{+}M_i\cong L_i\otimes_{\mathbb{F}_r}K$, [4, VII.8.4] and [5, 9.7] imply that any L_i is self-dual. By [4, VII.8.10.b] and the theorem of Krull–Schmidt we conclude that any L_i is a non-singular faithful irreducible symplectic $\mathbb{F}_rG$-submodule of $V\otimes_{\mathbb{F}}\mathbb{F}_r$ for the symplectic form on $V\otimes_{\mathbb{F}}\mathbb{F}_r$. Hence $V\otimes_{\mathbb{F}}\mathbb{F}_r=L_1\perp\cdots\perp L_u$; here it is also used that $V\otimes_{\mathbb{F}}\mathbb{F}_r$ is completely reducible as $\mathbb{F}_rG$-module.

Now, if $u>1$, then we can apply induction just as we did it in the case (B.2.α). Therefore, assume from now on that $u=1$. Hence $V\otimes_{\mathbb{F}}K=M_1\dot{+}M_1^*$. Thus $K=\mathbb{F}_{r^2}=\mathbb{F}_{b^2}$ and the M_1 and M_1^* are non-isomorphic absolutely irreducible $\mathbb{F}_{b^2}G$-modules. It follows from Corollary 9.7 of [5] that the irreducible $\bar{\mathbb{F}}G$-modules $M_1\otimes_K\bar{\mathbb{F}}$ and $M_1^*\otimes_K\bar{\mathbb{F}}$ are not isomorphic. As G/N is cyclic of prime order q not equal to p, we see that either $(M_1\otimes_K\bar{\mathbb{F}})_N$ is an irreducible $\bar{\mathbb{F}}N$-module (whence $(M_1^*\otimes_K\bar{\mathbb{F}})_N$ is irreducible as well), or

$$(M_1\otimes_K\bar{\mathbb{F}})_N=T_1\dot{+}\cdots\dot{+}T_q,\ T_j\not\cong T_m\quad\text{if}\quad j\neq m,$$

where the T_i are irreducible $\bar{\mathbb{F}}N$-modules (whence $(M_1^*\otimes_K\bar{\mathbb{F}})_N$ decomposes in an analogous way), see Theorem VII.9.18 of [4]. In the very last case it follows that $T_i^G\cong M_1\otimes_K\bar{\mathbb{F}}\not\cong M_1^*\otimes_K\bar{\mathbb{F}}\cong(M_1\otimes_K\bar{\mathbb{F}})^*\cong(T_i^G)^*\cong(T_i^*)^G$, whence all irreducible $\bar{\mathbb{F}}N$-modules contained in both $(M_1\otimes_K\bar{\mathbb{F}})_N$ and $(M_1^*\otimes_K\bar{\mathbb{F}})_N$ are pairwise non-isomorphic by the theorem of Frobenius–Nakayama. In that case we find

$$\left(V\bigotimes_{\mathbb{F}}\bar{\mathbb{F}}\right)_N\cong\left((M_1\dot{+}M_1^*)\bigotimes_K\bar{\mathbb{F}}\right)_N\cong\left(\sum_{i=1}^{q}T_i\dot{+}\sum_{i=1}^{q}T_i^*\right)$$

$$\cong(eU)\bigotimes_{\mathbb{F}}\bar{\mathbb{F}}\cong e\left(U\bigotimes_{\mathbb{F}}\bar{\mathbb{F}}\right).$$

The Krull–Schmidt Theorem implies now that $e=1$, and so case (2) has been arrived at. Therefore we can assume that $(M_1\otimes_K\bar{\mathbb{F}})_N$ and $(M_1^*\otimes_K\bar{\mathbb{F}})_N$ remain irreducible as $\bar{\mathbb{F}}N$-modules. This leads to

$$\left(V\bigotimes_{\mathbb{F}}\bar{\mathbb{F}}\right)_N=\left((M_1\dot{+}M_1^*)\bigotimes_K\bar{\mathbb{F}}\right)_N\cong\left(M_1\bigotimes_K\bar{\mathbb{F}}\right)_N\dot{+}\left(M_1^*\bigotimes_K\bar{\mathbb{F}}\right)_N\cong eU\bigotimes_{\mathbb{F}}\bar{\mathbb{F}}\cong e\left(U\bigotimes_{\mathbb{F}}\bar{\mathbb{F}}\right).$$

Applying the Krull–Schmidt Theorem we conclude that $e=1$ or $e=2$. Henceforth we are in case (2), or, as we will assume from now on, $e=2$. Write M instead of M_1.

Under that assumption it is clear from the above, that U is an absolutely irreducible $\mathbb{F}N$-module. Hence $U\otimes_{\mathbb{F}}K$ is an absolutely irreducible KN-module. We have also $M_N\cong M^*|_N\cong U\otimes_{\mathbb{F}}K$. We will show now that there exists an absolutely irreducible $\mathbb{F}G$-module T such that $T_N\cong U$. Namely, if follows from Theorem VII.9.13 of [4] that any irreducible KG-module L having $U \bigotimes_{\mathbb{F}} K$ in its restriction to N (i.e. $L_N = U_1 \dot{+} \cdots$ for a certain KN-submodule U_1 of L with $U_1 \cong U \bigotimes_{\mathbb{F}} K$) is of the form $M \bigotimes_K \Lambda$, where Λ is a one-dimensional KG-module such that N acts trivially on Λ. Call λ the corresponding one-dimensional representation of G. Let $\langle gN \rangle = G/N$. As $M_N \cong M^*|_N \cong U \bigotimes_{\mathbb{F}} K$, it therefore holds that $M^* \cong M \bigotimes_K \Lambda$, where $\lambda(g^i n) = \omega^i$, any $n \in N$, with ω a certain primitive qth-root of unity of K. Notice that $q | r^2 - 1$ but $q \not| r - 1$, whence $q | r + 1$. (Indeed, as $M \not\cong M^*$, some element $a = g^j n \in G \backslash N$ has $\operatorname{Tr} D(a) \neq 0$, where Tr means the trace function of the (matrix) representation D which corresponds to the KG-module M; likewise we denote D^* with respect to M^*. The fact that there must be such an element a in $G \backslash N$ is just forced by $M^* \cong M \bigotimes_K \Lambda$ and $M_N \cong M^*|_N$. So $\operatorname{Tr} D^*(a) = (\operatorname{Tr} D(a))^r = (\operatorname{Tr} D(a))\omega^j$, whence $\operatorname{Tr} D(a) = (\operatorname{Tr} D(a))^{r^2} = (\operatorname{Tr} D(a))^r \omega^{jr} = (\operatorname{Tr} D(a))\omega^{j(1+r)}$, so that $\omega^{1+r} = 1$. Thus if $q | r - 1$, then $\omega^2 = 1 = \omega^q$, whence $\omega = 1$, a contradiction.)

Thus we have $\operatorname{Tr} D^*(g^i n) = (\operatorname{Tr} D(g^i n))^r = \omega^i (\operatorname{Tr} D(g^i n))$. Let Λ^h be the one-dimensional KG-module corresponding to the representation λ^h defined by $\lambda^h(g^i n) = \omega^{ih}$ for all $n \in N$. Hence $\lambda^h(g^i n) = (\lambda(g^i n))^h$. Consider the irreducible KG-module $M \bigotimes_K \Lambda^{(q+1)/2}$. Then $M \bigotimes_K \Lambda^{(q+1)/2}$ is a self-dual KG-module, as we will show using the trace function. Indeed,

$$\begin{aligned}\operatorname{Tr}((D \otimes \lambda^{(q+1)/2})^*(g^i n)) &= \omega^{-i(q+1)/2}(\operatorname{Tr} D^*(g^i n)) = \omega^{-i(q+1)/2}\omega^i(\operatorname{Tr} D(g^i n)) \\ &= \omega^{i(q+1)/2}(\operatorname{Tr} D(g^i n)) = \operatorname{Tr}((D \otimes \lambda^{(q+1)/2})(g^i n)).\end{aligned}$$

Even more, as $\omega^r = \omega^{-1}$ by $q | r + 1$,

$$\begin{aligned}(\operatorname{Tr}((D \otimes \lambda^{(q+1)/2})(g^i n)))^r &= \omega^{ir(q+1)/2}(\operatorname{Tr} D(g^i n))^r \\ &= \omega^{-i(q+1)/2}(\operatorname{Tr} D^*(g^i n)) = \omega^{-i(q+1)/2}\omega^i(\operatorname{Tr} D(g^i n)) \\ &= \omega^{-i(q-1)/2}(\operatorname{Tr} D(g^i n)) \\ &= \omega^{i(q+1)/2}(\operatorname{Tr} D(g^i n)) = \operatorname{Tr}((D \otimes \lambda^{q+1)/2})(g^i n)).\end{aligned}$$

Therefore, Theorem VII.1.17 of [4] yields that $M \bigotimes_K \Lambda^{(q+1)/2}$ can be realized over $\mathbb{F}$. This $M \bigotimes_K \Lambda^{(q+1)/2}$ is now the desired $\mathbb{F}G$-module T in case (3) as we will see.

The map f, defined by

$$g^i n \overset{f}{\mapsto} \begin{pmatrix} 0 & -1 \\ 1 & \omega^{-(q-1)/2} + \omega^{(q-1)/2} \end{pmatrix}^i, \quad \text{for all } n \in N,$$

is a representation of G to $SL(2, \mathbb{F})$ with $\operatorname{Ker} f = N$. The representation f is irreducible as $\mathbb{F}$-representation; namely the eigenvalues of

$$\begin{pmatrix} 0 & -1 \\ 1 & \omega^{-(q-1)/2} + \omega^{(q-1)/2} \end{pmatrix}$$

are $\omega^{-(q-1)/2}$ and $\omega^{(q-1)/2}$, both contained in K, but not in $\mathbb{F}$.

Let S be the $\mathbb{F}G$-module corresponding to f. Consider the $\mathbb{F}G$-module $T \bigotimes_{\mathbb{F}} S$. Then

$$\begin{aligned}\operatorname{Tr}((D \otimes \lambda^{(q+1)/2} \otimes f)(g^i n)) &= \operatorname{Tr}(D(g^i n) \otimes \lambda^{(q+1)/2}(g^i n) \otimes f(g^i n)) \\ &= (\operatorname{Tr} D(g^i n))\omega^{i(q+1)/2}(\omega^{-i(q-1)/2} + \omega^{i(q-1)/2}) \\ &= (\operatorname{Tr} D(g^i n))\omega^{i(1+q)} = (\operatorname{Tr} D(g^i n))(\omega^i + 1) \\ &= \operatorname{Tr} D^*(g^i n) + \operatorname{Tr} D(g^i n).\end{aligned}$$

Hence we see that the irreducible $\mathbb{F}G$-module V (or rather the KG-module $V \bigotimes_{\mathbb{F}} K = M + M^*$) and the $\mathbb{F}G$-module $T \bigotimes_{\mathbb{F}} S$ afford the same trace function and that they have the same $\mathbb{F}$-dimension. Then Corollary 9.22 of [5] gives the result that V and $T \bigotimes_{\mathbb{F}} S$ are isomorphic as $\mathbb{F}G$-modules. Now, as

$$T^* \bigotimes_{\mathbb{F}} K \cong \left(T \bigotimes_{\mathbb{F}} K\right)^* \cong \left(M \bigotimes_{K} \Lambda^{(q+1)/2}\right)^* \cong M \bigotimes_{K} \Lambda^{(q+1)/2} \cong T \bigotimes_{\mathbb{F}} K$$

as KG-modules, it follows from the Deuring–Noether Theorem 9.7 of [5], that $T^* \cong T$ as $\mathbb{F}G$-modules. Hence we are in case (3). □

In the characteristic 2 case of Theorem A, we can say a bit more.

Corollary to Theorem A. *Let G be a finite group. Assume that $N \lhd G$ with $|G/N| = $ odd prime q, and there is no $B \trianglelefteq G$ with $BN = G$ and $B \cap N = \{1\}$. Suppose there exists a faithful irreducible non-singular symplectic $\mathbb{F}G$-module V where $\mathbb{F}$ is a finite field of characteristic 2. Then there exists a finite field $L \supseteq \mathbb{F}$ and a faithful irreducible non-singular symplectic LG-module M such that*

either

$M_N = U_1 \perp \cdots \perp U_q$, where $U_i \ncong U_j$ as LN-modules if $i \neq j$, the U_i are irreducible non-singular symplectic LN-submodules of M_N,

or

M_N is a faithful irreducible non-singular symplectic LN-module.

Proof. By assumption, $N \neq \{1\}$. Without loss of generality we may assume that we are in case (3) of Theorem A. Using the notation of that theorem, it follows that T_N is not an irreducible $\mathbb{K}N$-module for the trivial representation of N. Hence T is not the trivial $\mathbb{K}G$-module. Then, using char $\mathbb{K} = 2$, a theorem of Fong ([4], VII.8.13) implies that there exists a non-singular G-invariant symplectic form on T. As N is trivially represented on S and as $T \bigotimes_{\mathbb{K}} S$ is a faithful $\mathbb{K}G$-module, it follows from case (3) of Theorem A that T_N is faithful. Now, if T would not be faithful as a $\mathbb{K}G$-module, we should have the existence of $\{1\} \neq B \lhd G$ with $B \cap N = \{1\}$, whence $BN = G$. This is contrary to our assumption. Hence T is a faithful $\mathbb{K}G$-module. Certainly $\dim_{\mathbb{K}} T \leqq \frac{1}{2} \dim_{\mathbb{F}} V$. So we have an induction machine with respect to the dimensions of the appropriate modules over their ground fields. The corollary now follows. □

Theorem B. *Let G and V satisfy the hypotheses of Theorem A. Assume that* $O_{2'}(F(N)) \neq \{1\}$ *and that* $N/F(N)$ *is of odd order. Then case* (3) *of Theorem A never occurs.*

Proof. In the course of the proof of Theorem A we used an induction argument without specifying, at that time, what in fact the induction step was! Therefore it is enough to show that we have a contradiction as soon as we have reached the point in the proof of Theorem A, where we made the assumption that $e=2$. We proceed then as follows.

Hence it is clear that U is an absolutely irreducible $\mathbb{F}N$-module. Moreover, as $U \otimes_{\mathbb{F}} \bar{\mathbb{F}} \cong U^* \otimes_{\mathbb{F}} \bar{\mathbb{F}}$, see above, it follows that the inverse-transpose representation A^* of N corresponding to $(U \otimes_{\mathbb{F}} \bar{\mathbb{F}})^*$ is $\bar{\mathbb{F}}$-equivalent to the representation A of N on $U \otimes_{\mathbb{F}} \bar{\mathbb{F}}$. Consider a representing matrix $A(n)$ with $n \in N$. Then, if $\omega \in \bar{\mathbb{F}}$ is an eigenvalue of $A(n)$, the above conclusion implies that ω^{-1} is also an eigenvalue of $A(n)$. As G is represented irreducibly and faithfully on V, a module of characteristic p, it follows that $O_p(G)$ is contained in the (trivial) kernel of the representation of G on V, whence $O_p(G)=\{1\}$, see [3, V.5.17]. Therefore $\{1\} \neq B := \Omega_1(O_t(Z(O_{2'}(F(N)))))$ for a certain odd prime t unequal to p, by the hypothesis $O_{2'}(F(N)) \neq \{1\}$. Hence B is a non-trivial elementary abelian t-group with $B \lhd G$, and B is not contained in the trivial kernel of the representation of G on V. Using an obvious notation, we have $A_B = d(\zeta_1 + \cdots + \zeta_x)$; where $d \in \mathbb{N}$ and the ζ_j are pairwise non-isomorphic one-dimensional representations of B over $\bar{\mathbb{F}}$. Therefore, if ω is an eigenvalue of $A(g)$, $g \in B$, with $\omega \neq 1$, then ω^{-1} occurs with multiplicity d in $A(g)$ as well. Let $\zeta_1(g) = \omega \neq 1$. 1. Define ζ^- via $\zeta^-(b) = (\zeta_1(b))^{-1}$, any $b \in B$. Since B is abelian of odd order, ζ^- is a one-dimensional representation of B over $\bar{\mathbb{F}}$ with $\zeta^- \neq \zeta_1$. Now $A_B(b) = A_B(b^{-1})$ for all $b \in B$. Thus by applying an orthogonality relation it follows that ζ^- occurs in A_B, say $\zeta^- = \zeta_2$. Now observe that $x = |N\text{: (inertia group of } \zeta_i \text{ in } N)|$ divides $|N/F(N)|$, as (inertia group of ζ_i in $N) \supseteq F(N)$. As $|N/F(N)|$ is odd by assumption, this means that at least one ζ_i is the trivial character of B over $\bar{\mathbb{F}}$, say $\zeta_y = 1_B$. Then immediately it holds that B is trivially represented on V for we know from Clifford's Theorem that all the ζ's are N-conjugated to each other. However, as $V_N = 2U$, U is faithful as $\mathbb{F}N$-module and we have a contradiction. □

For the convenience of the reader we repeat the definition of a monoprimary module.

Definition. Let K be a finite field. Let V be a non-singular symplectic KG-module for the finite group G. Then V is called *monoprimary* if it is a direct sum of pairwise non-isomorphic, self-dual, irreducible KG-modules.

There are places in the literature, such as [**2**, **7**, **8**, **9**, **11**], where the property of being a monoprimary module yields results in the theory of M-groups. As a tool for applications one would like to know a theorem like "If $N \trianglelefteq G$, V a monoprimary KG-module, then V_N is a monoprimary KN-module". This is certainly not true in its full generality. In this respect we can prove such a theorem in a particular case.

Theorem C. *Let G be a finite group,* $N \ntrianglelefteq G$, G/N *solvable of odd order. Suppose that every prime divisor of* $|G/N|$ *divides* $|\mathbb{F}|-1$ *with* $\mathbb{F}$ *a finite field. Let V be a monoprimary* $\mathbb{F}G$*-module. Then V is also monoprimary as* $\mathbb{F}N$*-module.*

Proof. In order to prove the theorem, we can clearly restrict outselves to the case $|G/N|=q$, q odd prime. Next we argue that it suffices to assume that V is an irreducible $\mathbb{F}G$-module. Namely, let $V=A_1 \perp \cdots \perp A_t$, $A_i \not\cong A_j$ as $\mathbb{F}G$-modules when $i \neq j$, the A's non-singular symplectic irreducible $\mathbb{F}G$-modules. Since $\mathbb{F}$ is a splitting field for G/N with $(\mathrm{char}\,\mathbb{F}) \not| \, |G/N|$, it is possible to apply Lemma (3.4) of [**8**]. In that lemma it is proved that if $A_i|_N$ and $A_j|_N$ have a common irreducible constituent in the Clifford sense, $A_i \cong A_j\lambda$ for some one-dimensional $\mathbb{F}$-character λ of G/N. Now A_i and A_j are both self-dual. Then, since λ is a $\mathbb{F}$-character of G of odd multiplicative order, Proposition (3.7) of [**8**] implies that $A_j\lambda \cong A_j$, whence $i=j$. Thus from now on, we assume that V is an irreducible non-singular symplectic $\mathbb{F}G$-module. The proof of the theorem follows now from a variation of Theorem A, to be called Theorem D. □

The proof of the following Theorem D can be regarded as a specialization of the proof of Theorem A, but there are some subtleties in it. As mentioned in the Introduction, the statement of Theorem D resembles that of the analogous statement made in the proof of Theorem (3.1) of [**8**].

Theorem D. *Let G be a finite group. Suppose G admits an irreducible non-singular symplectic $\mathbb{F}G$-module V for a certain finite field $\mathbb{F}$. Let $N \lhd G$, $|G/N|=q$, q odd prime. Assume q divides $|\mathbb{F}|-1$. Then precisely one of the following statements holds.*

(1) *$V_N=U_1 \perp \cdots \perp U_q$, $U_i \not\cong U_j$ if $i \neq j$, the U_t are irreducible non-singular symplectic $\mathbb{F}N$-submodules of V_N.*

(2) *V_N is an irreducible (whence non-singular symplectic) $\mathbb{F}N$-module.*

Proof. It is clear that we can assume that

$$V_N \text{ is homogeneous, say } V_N=eU; \tag{α}$$

just follow part (A) of the proof of Theorem A. Let K be the field defined in the beginning of part (B) of the proof of Theorem A. Again we have

$$V \bigotimes_{\mathbb{F}} K = M_1 \perp \cdots \perp M_a \perp (M_{a+1} \dotplus M^*_{a+1}) \perp \cdots \perp (M_{a+u} \dotplus M^*_{a+u}). \tag{β}$$

In this equality (β) all the written M's and M^*'s are all pairwise non-isomorphic and they are all galois conjugated to each other. Next we split up.

Assume $u=0$, i.e. $V \otimes_{\mathbb{F}} K = M_1 \perp \cdots \perp M_a$. Hence $V \otimes_{\mathbb{F}} K$ is monoprimary. Now $(V \otimes_{\mathbb{F}} K)_N$ is monoprimary as soon as we have proved that each $M_i|_N$ is monoprimary. Indeed, $|\mathbb{F}|-1$ divides $|K|-1$, so q divides $|K|-1$ and we can use Lemma (3.4) and Proposition (3.7) of [**8**] again. Observe however that $M_i|_N$ satisfies either statement of Theorem D. It holds because M_i is an absolutely irreducible KG-module, being also non-singular symplectic, following Theorem VII.9.18 of [**4**]. Hence, as $(V \otimes_{\mathbb{F}} K)_N \cong V_N \otimes_{\mathbb{F}} K \cong (eU) \otimes_{\mathbb{F}} K \cong e(U \otimes_{\mathbb{F}} K)$, the Krull–Schmidt Theorem immediately gives $e=1$.

Let $u \geqq 1$. Just as it is done in part (B.2.β) of the proof of Theorem A, we have $V \otimes_{\mathbb{F}} K = (M_1 \dotplus M_1^*) \perp \cdots \perp (M_u \dotplus M_u^*)$. Again there is here a field tower $\mathbb{F} \subseteq \mathbb{F}_r \subset \mathbb{F}_{r^2} = K$

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such that $V \otimes_{\mathbb{F}} \mathbb{F}_r = L_1 \perp \cdots \perp L_u$, that is, $V \otimes_{\mathbb{F}} \mathbb{F}_r$ is monoprimary. As $|\mathbb{F}|-1$ divides $|\mathbb{F}_r|-1$, it holds that $q \mid |\mathbb{F}_r|-1$. By Lemma (3.4) and Proposition (3.7) of [8], $(V \otimes_{\mathbb{F}} \mathbb{F}_r)_N$ is monoprimary as soon as each $L_i|_N$ is monoprimary. Having achieved that result, the Krull–Schmidt Theorem gives $e=1$ in the relation.

$$\left(V \underset{\mathbb{F}}{\otimes} \mathbb{F}_r\right)_N \cong (eU) \underset{\mathbb{F}}{\otimes} \mathbb{F}_r \cong e\left(U \underset{\mathbb{F}}{\otimes} \mathbb{F}_r\right) \cong (L_1 \perp \cdots \perp L_u)_N.$$

We now pick such a $\mathbb{F}_r G$-module L_i, we call it L. Thus L is a non-singular symplectic irreducible $\mathbb{F}_r G$-module with $L \otimes_{\mathbb{F}_r} K = M \dotplus M^*$ and these KG-modules M and M^* are dual to each other. Besides that, they are absolutely irreducible non-isomorphic isotropic KG-modules.

Next assume that M_N and $M^*|_N$ have a common irreducible constituent in the Clifford sense. Then $M^* \cong M\mu$ for some one-dimensional K-character μ of G/N. As $q \mid |\mathbb{F}_r|-1$ and $(|\mathbb{F}_r|-1) \mid (|K|-1)$, we see that μ is in fact a one-dimensional $\mathbb{F}_r$-character of G/N of odd order. Since $M^* \cong M^\sigma$ for some $\sigma \in \mathrm{Gal}(K/\mathbb{F}_r)$ with $\sigma^2=1$, M is a so-called weakly self-dual module over $\mathbb{F}_r$, see Definition (3.6) of [8]. As both M and $M\mu \cong M^*$ are weakly self-dual over $\mathbb{F}_r$, Proposition (3.7) of [8] yields $M\mu \cong M$. Thus we have a contradiction and so M_N and $M^*|_N$ do not have common irreducible constituents.

Hence, applying the Krull–Schmidt Theorem and the fact that M and M^* are absolutely irreducible KG-modules, $(L \otimes_{\mathbb{F}_r} K)_N$ decomposes into a direct sum of pairwise non-isomorphic irreducible KN-modules. Then L_N must also decompose in a direct sum of pairwise non-isomorphic irreducible $\mathbb{F}_r N$-modules. Now, go to the written text in the proof of Theorem A in case (A) for the non-singular symplectic $\mathbb{F}_r G$-module L instead of the $\mathbb{F}G$-module V written there. It follows then, that L_N is monoprimary.

Therefore $(V \otimes_{\mathbb{F}} \mathbb{F}_r)_N$ is monoprimary as we have seen. Hence the Krull–Schmidt Theorem applied to $(V \otimes_{\mathbb{F}} \mathbb{F}_r)_N \cong e(U \otimes_{\mathbb{F}} \mathbb{F}_r)$ yields $e=1$. □

In the next theorem we show that the value of the ramification index e is restricted in the case that we work with modules over a finite field.

Theorem E. *Let G be a finite group, $N \lhd G$, $|G/N|=q$, q some prime integer. Assume V is an irreducible $\mathbb{F}G$-module for a certain finite field $\mathbb{F}$. Suppose that $V_N = eU$, that is, if V is considered as $\mathbb{F}N$-module, it is a direct sum of e isomorphic copies of the irreducible $\mathbb{F}N$-submodule U of V_N. Then $e=1$ or $e=q$ or e divides $q-1$.*

Proof. Let $\operatorname{char} \mathbb{F} = p$. We can assume that $q \neq p$ for otherwise Green's Theorem VII.9.19 of [4] gives $e=1$. Hence let $q \neq p$. By Theorem VII.2.6 of [4] there exists a finite field K containing $\mathbb{F}$ such that K is a splitting field for G, for N and for G/N all together. Consider $V \otimes_{\mathbb{F}} K$. Then, for suitable integers u and s, we have the following decompositions into irreducible KG-modules R_j and irreducible KN-modules T_j:

$$V \underset{\mathbb{F}}{\otimes} K = R_1 \dotplus \cdots \dotplus R_u, \quad (eU) \underset{\mathbb{F}}{\otimes} K \cong e\left(U \underset{\mathbb{F}}{\otimes} K\right) = e(T_1 \dotplus \cdots \dotplus T_s).$$

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Since Schur indices for modules over finite fields are all equal to one ([4], VII.1.16.e), it follows that the R_i are pairwise non-isomorphic absolutely irreducible KG-modules affording characters which are galois conjugated to each other, see [5, 9.21]. The same statement holds for the KN-modules T_i.

(1) *Let $R_1|_N$ be not homogeneous.* Let W be an irreducible constituent of the KN-module $R_1|_N$. Then Clifford's Theorem yields $R_1 \cong W \otimes_{KN} KG$, that is, R_1 is induced by W. Moreover, $R_1|_N$ is the direct sum of q pairwise non-isomorphic G-conjugated KN-submodules. All these KN-modules are absolutely irreducible. From the Krull–Schmidt Theorem we see that some T_j is isomorphic to W as KN-modules. Since all the T_i have the same K-dimension, it follows that, after an eventual renumbering, $qu=s$, $e=1$, $R_i \cong T_i \otimes_{KN} KG$. Notice that it is implicitly used here that if some $R_i|_N$ happens to be homogeneous that $R_i|_N$ is irreducible as KN-module, by [4, VII.9.19 and VII.9.18], just by the splitting field property of K. Thus in fact all $R_i|_N$ are here not homogeneous.

(2) *Suppose now that all $R_i|_N$ are homogeneous.* Then [4, VII.9.18] implies that all the $R_i|_N$ are absolutely irreducible KN-modules. Let $D=\{R_1,\ldots,R_u\}$. Let Y be an $\mathbb{F}(\chi)$-submodule of $V\otimes_{\mathbb{F}}\mathbb{F}(\chi)$, where χ is the trace function of R_i (the field $\mathbb{F}(\chi)$ does not depend on the index i, by [5, 9.21.c]). Then $Y\otimes_{\mathbb{F}(\chi)}K$ is an absolutely irreducible KG-module isomorphic as KG-module to a member of D. See [5, 9.21.e]. So we have

$$V \underset{\mathbb{F}}{\otimes} \mathbb{F}(\chi) = S_1 \dotplus \cdots \dotplus S_u,$$

where, say, $R_i \cong S_i \otimes_{\mathbb{F}(\chi)} K$, and where, by the Deuring–Noether Theorem [5, 9.7], $S_i \not\cong S_j$ if $i \neq j$, as $\mathbb{F}(\chi)G$-modules. Observe that any S_i is an absolutely irreducible $\mathbb{F}(\chi)G$-module. Consider an irreducible constituent Z of $S_i|_N$. By the Krull–Schmidt Theorem it is isomorphic as $\mathbb{F}(\chi)N$-module to some irreducible constituent of $U\otimes_{\mathbb{F}}\mathbb{F}(\chi)$. Then some irreducible constituent of $Z\otimes_{\mathbb{F}(\chi)}K$ is, by Krull–Schmidt again, isomorphic to some irreducible constituent of $U\otimes_{\mathbb{F}}K \cong (U\otimes_{\mathbb{F}}\mathbb{F}(\chi))\otimes_{\mathbb{F}(\chi)}K$. That last constituent must be isomorphic to one of the $R_i|_N$. By comparison of dimensions it now holds that $S_i|_N$ is an absolutely irreducible $\mathbb{F}(\chi)N$-module for any i. Notice now that $u=[\mathbb{F}(\chi):\mathbb{F}]=|\mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F})|$ and that $\mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F})$ is cyclic, generated by the Frobenius automorphism $x\mapsto x^b$, where $b=|\mathbb{F}|$, any $x\in\mathbb{F}(\chi)$. We have $et=u$, where t is just the number of all the isomorphy types of the irreducible $\mathbb{F}(\chi)N$-submodules of $U\otimes_{\mathbb{F}}\mathbb{F}(\chi)$. Such a module is isomorphic to some $S_i|_N$. Suppose from now on that $e \geqq 2$ and let $\bar{D}=\{S_1,\ldots,S_u\}$. Hence there are $A,B\in\bar{D}$ with $A\neq B$ with $A_N\cong B_N=S_1|_N$, say. Observe that $\bar{D}=\{A^\tau \mid \tau\in\mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F})\}$. Let $\Pi=\{\sigma\in\mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F}) \mid A^\sigma|_N\cong S_1|_N\}$. So $\Pi\neq\{1\}$. Hence, if $\sigma\in\Pi$, there exists by [4, VII.9.13], a unique one-dimensional $\mathbb{F}(\chi)G$-representation Λ_σ, depending on $\sigma\in\Pi$ and with N acting trivially on Λ_σ, such that $A^\sigma\cong A\otimes_{\mathbb{F}(\chi)}\Lambda_\sigma$. Therefore if $\alpha,\beta\in\Pi$,

$$(A^\alpha)^\beta \cong \left(A \underset{\mathbb{F}(\chi)}{\otimes} \Lambda_\alpha\right)^\beta \cong A^\beta \underset{\mathbb{F}(\chi)}{\otimes} (\Lambda_\alpha)^\beta$$

$$\cong \left(A \underset{\mathbb{F}(\chi)}{\otimes} \Lambda_\beta\right) \underset{\mathbb{F}(\chi)}{\otimes} (\Lambda_\alpha)^\beta.$$

So, if $\alpha,\beta\in\Pi$ then $\alpha\beta\in\Pi$. Hence Π is a subgroup of the cyclic group $\mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F})$.

Let $\Pi = \langle\gamma\rangle$ and let $\mathbb{F}_{b^j}$ be the invariant field of $\langle\gamma\rangle$. Notice that $1=(b,q)$. It follows that $E:=\{A, A^{\gamma}, \ldots A^{\gamma^{|\gamma|-1}}]$ is precisely the subset of D consisting of those members which are isomorphic to $S_1|_N$ when they are realized as $\mathbb{F}(\chi)N$-modules. Notice $|E|=|\gamma|$. Let $\phi \in \mathrm{Gal}(\mathbb{F}(\chi)/\mathbb{F})$ be arbitrary. Then $A^{\phi} \in \bar{D}$ and each member of $\bar{D}$ is of this form. It follows that (with $\Lambda = \Lambda_{\gamma}$), $(A^{\phi})^{\gamma^i} = (A^{\gamma^i})^{\phi} \cong (A \bigotimes_{\mathbb{F}(\chi)} \Lambda^f)^{\phi} \cong A^{\phi} \bigotimes_{\mathbb{F}(\chi)} (\Lambda^f)^{\phi}$, where

$$\Lambda^f \cong \underbrace{\Lambda \bigotimes_{\mathbb{F}(\chi)} \Lambda \bigotimes_{\mathbb{F}(\chi)} \cdots \bigotimes_{\mathbb{F}(\chi)} \Lambda}_{f\text{-times}}$$

with

$$f = \frac{(b^j)^i - 1}{b^j - 1}.$$

Hence $(A^{\phi})^{\gamma^i}|_N \cong A^{\phi}|_N$, $i = 1, \ldots, |\gamma|-1$. Therefore we see that $|\gamma| t = u$. So $e = |\gamma|$. It follows that $A \cong A^{\gamma^e} \cong A \bigotimes_{\mathbb{F}(\chi)} \Lambda^h$ with

$$h = \frac{(b^j)^e - 1}{b^j - 1}.$$

Now, by [**4**, VII.9.12.c], $\Lambda^h = I$, the trivial one-dimensional $\mathbb{F}(\chi)G$-module. As Λ has order q, we have $q\,|\,h$. If now q divides

$$\frac{(b^j)^a - 1}{b^j - 1}$$

for some $a \in \{1, \ldots, e-1\}$, then $A^{\gamma^a} \cong A$ and so $|E| \leqq a < e = |E|$, a contradiction. Hence q does not divide

$$\frac{(b^j)^a - 1}{b^j - 1}$$

if $a \in \{1, \ldots, e-1\}$. Next, if q does not divide $b^j - 1$, then it follows that e is precisely equal to the order of b^j modulo q. By Fermat's Theorem $(b^j)^{q-1} \equiv 1 \pmod{q}$, whence $e\,|\,q-1$. Therefore assume now that $q\,|\,b^j - 1$. This means that

$$A^{\gamma^2} \cong \left(A \underset{\mathbb{F}(\chi)}{\bigotimes} \Lambda\right)^{\gamma} \cong A^{\gamma} \underset{\mathbb{F}(\chi)}{\bigotimes} \Lambda^{b^j} \cong A^{\gamma} \underset{\mathbb{F}(\chi)}{\bigotimes} \Lambda \cong A \underset{\mathbb{F}(\chi)}{\bigotimes} \Lambda \underset{\mathbb{F}(\chi)}{\bigotimes} \Lambda.$$

Hence

$$A^{\gamma^i} \cong A \underset{\mathbb{F}(\chi)}{\bigotimes} \underbrace{(\Lambda \otimes \cdots \otimes \Lambda)}_{i\text{-times}}$$

Therefore $|\gamma| = q$, and then $e = |\gamma| = q$. This finishes the proof of the theorem. $\square$

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