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THE CENTRALIZER OF A SUBGROUP IN A GROUP ALGEBRA

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Abstract Let F be an algebraically closed field, G be a finite group and H be a subgroup of G. We answer several questions about the centralizer algebra FG^H. Among these, we provide examples to show that

- the centre $\mathrm{Z}(FG^H)$ can be larger than the F-algebra generated by $\mathrm{Z}(FG)$ and $\mathrm{Z}(FH)$,
- FG^H can have primitive central idempotents that are not of the form ef, where e and f are primitive central idempotents of FG and FH respectively,
- it is not always true that the simple FG^H-modules are the same as the non-zero FG^H-modules $\mathrm{Hom}_{FH}(S, T\downarrow H)$, where S and T are simple FH and FG-modules, respectively.

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If R is a commutative ring, G is a finite group and H is a subgroup of G, then the *centralizer algebra* RG^H is the set of all elements of RG that commute with all elements of H. The algebra RG^H is a Hecke algebra in the sense that it is isomorphic to $\mathrm{End}_{RH\times G}(RG) = \mathrm{End}_{RH\times G}(1_{\Delta H}\uparrow^{H\times G})$, where $\Delta H = \{(h,h)\colon h \in H\}$. Thus Robinson's methods from [**12**] and Alperin's methods form [**2**] can be applied, replacing their G by $H\times G$ and their H by ΔH. We have studied the representation theory of centralizer algebras in several papers [**4**–**7**], mainly in cases where G is p-solvable and H is normal, or when $G = S_n$ and $H = S_m$ for $n-3 \leqslant m \leqslant n$. Part of our original motivation was to see whether there might be a 'weight conjecture' for these algebras—one that would simultaneously generalize Alperin's weight conjecture and Brauer's First Main Theorem on Blocks. This idea is explained in more detail in [**5**], [**4**] and [**6**]. Also, when H is a p-subgroup these algebras play an important role in Green's approach to modular representation theory and in Puig's theory of points. Along the way, several fairly basic and general questions have come up. This paper mainly consists of counter-examples to conjectures that one might be led to make based on the evidence in our earlier papers.

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When coefficients belong to an algebraically closed field F of characteristic 0, or of characteristic p where $p \nmid |G|$, the representation theory of a centralizer algebra FG^H is easy to understand. (See [**5**, Lemma 2.1] and [**10**, Lemma 1.0.1] for proofs.)

(i) The algebra FG^H is semisimple.

(ii) If S is a simple FH-module and T is a simple FG-module such that

$$\mathrm{Hom}_{FH}(S, T\downarrow_H) \neq 0,$$

then the FG^H-module $\mathrm{Hom}_{FH}(S, T\downarrow_H)$ is simple. (The space of homomorphisms is an FG^H-module via the multiplication $(a\varphi)(v) = a(\varphi(v))$ for all $a \in FG^H$, $\varphi \in \mathrm{Hom}_{FH}(S, T\downarrow_H)$ and $v \in S$.)

(iii) Every simple FG^H-module arises in this way, and appears just once as S and T run through all possibilities.

(iv) The centre of FG^H is generated as an F-algebra by the centres of FG and FH.

(v) Every primitive central idempotent of FG^H has the form ef, where e is a primitive central idempotent of FG and f is a primitive central idempotent of FH.

If the characteristic of the field does divide $|G|$, then FG^H is not semisimple, because the one-dimensional space spanned by $\sum_{g \in G} g$ is a nilpotent two-sided ideal. It is natural to ask whether items (ii)–(v) are still true in the non-semisimple case. As we will see, none of them is true in general. However, some of the counter-examples were not easy to find. Asking whether they are true or close to true in particular cases has been a useful approach. For example, we show in [**5**] that (ii)–(v) are all true when $G = S_n$ and $H = S_{n-1}$. In the preprint [**7**], we show that (v) is true when $G = S_n$ and $H = S_{n-2}$ or S_{n-3}.

From now on, F is a field of characteristic p. For any subset A of G, we let $A^+ = \sum_{g \in A} g \in FG$. There is a basis for FG^H consisting of all elements of the form C^+, where C is an orbit for the conjugation action of H on G.

Question 1. *Is FG^H a symmetric algebra?*

In general, the answer is no. Take $H = G$, so that $FG^H = \mathrm{Z}(FG)$. If G has more p-regular classes than blocks, $\mathrm{Z}(FG)$ is not symmetric. The Reynolds ideal, $\mathrm{Soc}(FG) \cap \mathrm{Z}(FG)$, has as a basis the p-regular section sums (see [**11**, (39)]). So $\mathrm{Soc}(\mathrm{Z}(FG))$ has dimension greater than or equal to $l(FG)$, the number of simple modules. On the other hand, the dimension of $\mathrm{Hd}(\mathrm{Z}(FG))$ equals the number of p-blocks.

Question 2. *If S is a simple FH-module and T is a simple FG-module such that $\mathrm{Hom}_{FH}(S, T\downarrow_H) \neq 0$, is $\mathrm{Hom}_{FH}(S, T\downarrow_H)$ a simple FG^H-module?*

In general, the answer is no. To construct a counter-example, we will use the following proposition.

Proposition 1. *Let P be a normal p-subgroup of G such that $C_G(P) \subseteq P$. Let V be the simple FP-module. Let U be a simple FG-module. If $\mathrm{Hom}_{FP}(V, U\downarrow_P)$ is a simple FG^P-module, then U has dimension 1 as a vector space over F.*

Proof. Since V is the unique simple FP-module, it follows from Clifford's Theorem that P acts trivially on U. Pick a non-zero element v of V. It is easily checked that the map $\phi \mapsto \phi(v)$ gives an isomorphism $\mathrm{Hom}_{FP}(V, U) \cong U\downarrow_{FG^P}$ as modules over FG^P. Hence $U\downarrow_{FG^P}$ is simple.

Next, we show that $U\downarrow_{FC_G(P)}$ is simple. Let W be an $FC_G(P)$-submodule of U. We claim that W is also an FG^P-submodule of $U\downarrow_{FG^P}$. To see this, let C be an orbit of the conjugation action of P on G. Since P acts trivially on U, every element of C acts the same way on U. Because $|C|$ is a power of p, it follows that C^+ acts as 0 on U unless $C = \{x\}$ for some $x \in C_G(P)$.

Since $C_G(P) \subseteq P$, and $U\downarrow_{FC_G(P)}$ is simple, it follows that U has dimension 1 as a vector space over F. □

It is now easy to construct a counter-example. Let P be an elementary abelian p-group of order p^2. Let $K = \mathrm{SL}(2, p)$, acting on P by ordinary matrix multiplication. Let G be the semidirect product of K and P. We have $C_G(P) \subseteq P$; it therefore follows from Proposition 1 that for any simple FG-module U with $\dim_F(U) > 1$, $\mathrm{Hom}_{FP}(V, U\downarrow_P)$ is not simple as an FG^P-module. In [**1**], Alperin lists all simple FK-modules. They have dimensions $1, 2, \dots, p$. Any one of these inflated by the projection $G \to K$ is a simple FG-module.

It can, however, easily happen that for a particular pair G, H, the answer to Question 2 is affirmative, even when FG^H is not semisimple. For example, Kleshchev's branching rule shows that if $G = S_n$ and $H = S_{n-1}$, then the answer to Question 2 is affirmative.

Question 3. *Is every simple FG^H-module isomorphic to $\mathrm{Hom}_{FH}(S, T\downarrow_H)$ for some simple FH-module S and simple FG-module T?*

In general, the answer is no. The following counter-example was communicated to us by Burkhard Külshammer.

An affirmative answer would imply that $l(FG^H)$, the number of simple FG^H-modules, is less than or equal to the product $l(G)l(H)$. In order to construct a counter-example to this inequality, let us take H to be a p-subgroup of G, so that $l(H)$ equals 1. Then $l(FG^H)$ is at least as big as $l(\mathrm{C}_G(H))$; this follows from the fact that the Brauer homomorphism with respect to H maps FG^H onto $F\mathrm{C}_G(H)$. So a positive answer to Question 3 would imply that $l(G) \geqslant l(\mathrm{C}_G(H))$.

As a counter-example, take G to be the dihedral group of order 80, H to be a subgroup of G of order 5, and F to have characteristic 5. The subgroup H is a normal 5-subgroup of G, and G/H is a dihedral group of order 16. Hence FG has seven simple modules in characteristic 5, four of dimension 1 and three of dimension 2. On the other hand, $C_G(H)$ is a cyclic group of order 40 and so $FC_G(H)$ has eight simple modules in characteristic 5.

As a weak form of Question 3, we ask the following.

Question 4. *Is every simple FG^H-module a composition factor of a module of the form $\mathrm{Hom}_{FH}(S, T\downarrow_H)$, where S is a simple FH-module and T is a simple FG-module?*

The answer is affirmative, as we now show. This line of argument was suggested by Boltje, Külshammer, Linckelmann and Scott (personal communication).

First, we point out that the map $\varphi \mapsto \varphi(1)$ gives an isomorphism of FG^H-modules

$$\mathrm{Hom}_{FH}(FH, FG) \cong FG.$$

The affirmative answer to Question 4 now follows from the next proposition.

Proposition 2. *Let M be an FG-module, and let N be an FH-module. If D is a composition factor of the FG^H-module $\mathrm{Hom}_{FH}(N, M)$, then there are a composition factor S of N and a composition factor T of M such that D is a composition factor of $\mathrm{Hom}_{FH}(S, T)$.*

Proof. Let

$$0 \to M_1 \to M \to M_2 \to 0$$

be a short exact sequence of FG-modules. Left exactness of the functor $\mathrm{Hom}(N, \cdot)$ gives an exact sequence

$$0 \to \mathrm{Hom}_{FH}(N, M_1) \to \mathrm{Hom}_{FH}(N, M) \to \mathrm{Hom}_{FH}(N, M_2),$$

where the last map is not necessarily surjective. It is easily checked that the maps are FG^H-module homomorphisms. Hence, each composition factor D of $\mathrm{Hom}_{FH}(N, M)$ is also a composition factor of $\mathrm{Hom}_{FH}(N, T)$ for some composition factor T of M.

Similarly, left exactness of the functor $\mathrm{Hom}(\cdot, T)$ tells us that if

$$0 \to N_1 \to N \to N_2 \to 0$$

is a short exact sequence of FH-modules, then there is an exact sequence

$$0 \to \mathrm{Hom}_{FH}(N_2, T) \to \mathrm{Hom}_{FH}(N, T) \to \mathrm{Hom}_{FH}(N_1, T).$$

The maps are easily checked to be FG^H-homomorphisms.

Thus, if D is a composition factor of $\mathrm{Hom}_{FH}(N, T)$, then there exists a composition factor S of N such that D is a composition factor of $\mathrm{Hom}_{FH}(S, T)$. □

Question 5. *Is the centre of FG^H generated as an algebra by $\mathrm{Z}(FG)$ and $\mathrm{Z}(FH)$?*

The answer is no, in general. We will exhibit a counter-example.

If R is a suitable local ring of characteristic 0 with residue field F of characteristic p, we get an algebra epimorphism $RG^H \to FG^H$ that maps $\mathrm{Z}(RG^H)$ into $\mathrm{Z}(FG^H)$. Let $\overline{\mathrm{Z}(RG^H)}$ be the image of this map. Since $\langle \mathrm{Z}(RG), \mathrm{Z}(RH) \rangle \subseteq \mathrm{Z}(RG^H)$, and $\langle \mathrm{Z}(RG), \mathrm{Z}(RH) \rangle$ maps onto $\langle \mathrm{Z}(FG), \mathrm{Z}(FH) \rangle$, we have

$$\langle \mathrm{Z}(FG), \mathrm{Z}(FH) \rangle \subseteq \overline{\mathrm{Z}(RG^H)} \subseteq \mathrm{Z}(FG^H).$$

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If it were true that $\langle \mathrm{Z}(FG), \mathrm{Z}(FH)\rangle = \mathrm{Z}(FG^H)$, then it would also be true that $\overline{\mathrm{Z}(RG^H)} = \mathrm{Z}(FG^H)$. Thus, in order to produce a counter-example to the conjecture $Z(FG^H) = \langle Z(FG), Z(FH)\rangle$, it is enough to exhibit an element of $Z(FG^H)$ that is not in the image of the map coming from $Z(RG^H)$.

For a counter-example, let $G = S_4$, let H be the normal Klein 4-group containing the products of disjoint 2-cycles, and let F have characteristic 2.

The conjugation action of H on G has 12 orbits. The four elements of H lie in singleton orbits. The two containing $(1,2,3)$ and $(1,3,2)$ have size 4. In addition, there are six orbits of size 2, with representatives $(1,2)$, $(1,3)$, $(1,4)$, $(1,2,3,4)$, $(1,3,4,2)$, $(1,4,2,3)$. Denote the orbit of g by O_g, and the orbit sum in RG^H by O_g^+, for each representative g.

We claim that $O^+_{(1,2,3)}$ is not in $\mathrm{Z}(RG^H)$ but its image is in $Z(FG^H)$.

To see this, note first that $O^+_{(1,2,3)}$ commutes with $O^+_{(1,2,3)}$ and $O^+_{(1,3,2)}$ and all elements of H, as it is a class sum of A_4. Next,

$$O^+_{(1,2,3)}O^+_{(1,2)} = 2(O^+_{(1,4)} + O^+_{(1,3,4,2)}),$$

but

$$O^+_{(1,2)}O^+_{(1,2,3)} = 2(O^+_{(1,3)} + O^+_{(1,2,3,4)}).$$

Conjugating by $(1,2,3)$ and $(1,3,2)$, we get similar equations for the terms

$$O^+_{(1,2,3)}O^+_{(1,3)}, \quad O^+_{(1,3)}O^+_{(1,2,3)}, \quad O^+_{(1,2,3)}O^+_{(1,4)} \quad \text{and} \quad O^+_{(1,4)}O^+_{(1,2,3)}.$$

Nevertheless, there are also quite a few examples for which the answer to Question 5 is affirmative. The paper [**6**] shows that the answer is affirmative when $G = S_n$ and $H = S_{n-1}$. Computer calculations done by the first author using Magma [**3**] and GAP [**8**] have shown that the answer is affirmative when $G = S_n$ and $H = S_m$ for all cases with $m \leqslant n \leqslant 8$. Some of these calculations used a fairly recent theorem of Alperin [**2**]. Alperin's theorem provides us with a way to compute the order of the finite abelian group $\mathrm{Z}(\mathbb{Z}G^H)/\langle \mathrm{Z}(\mathbb{Z}G), \mathrm{Z}(\mathbb{Z}H)\rangle$, where as usual $\mathbb{Z}$ denotes the integers; this group has order equal to the product of the elementary divisors of a certain matrix called the reduced class–coset table for the groups ΔH and $H \times G$. (See [**2**] for the definition.)

As a weak form of Question 5, one can ask the following.

Question 6. *Is* $\mathrm{Z}(FG^H) = \langle \mathrm{Z}(FG), \mathrm{Z}(FH)\rangle + \mathrm{J}(\mathrm{Z}(FG^H))$*?*

Or equivalently, *is every block idempotent of FG^H of the form ef, where e is a block idempotent of FG and f is a block idempotent of FH?*

The answer is no, in general, although there are many examples in which the answer is affirmative. The following is a counter-example, originally found by the first author using Magma [**3**]. The counter-example for Question 5 is not a counter-example for Question 6, as Proposition 3 below shows.

Example. If $G = S_6$, $H = A_4$ and F is a splitting field of characteristic 5, then $\langle Z(FG), Z(FH)\rangle$ does not contain all primitive idempotents of $Z(FG^H)$. In particular, there are block idempotents e for FG and f for FH such that ef is not central primitive in FG^H. The algebra $efFG^H$ is not indecomposable as an F-algebra.

Proof. We take e to be the principal 5-block idempotent of FG and f to be the principal 5-block idempotent of FH. Since FH is semisimple, $fFH \cong F$. Note that f is a primitive idempotent in fFH (and not just a centrally primitive idempotent).

Think of fFG and $fFGe$ as right FG-modules. The first two paragraphs of the proof of Proposition 2.6 of [**5**] show that there is a natural injective F-algebra map $efFG^H \to \mathrm{End}_{FG}(fFGe)$; the map sends $efx \in efFG^H$ to multiplication on the left by efx. (Proposition 2.6 in [**5**] applies just to $G = S_n$ and $H = S_{n-1}$, but the relevant part of the proof only uses the fact that f has p-defect 0 and f is primitive in fFH.) In our case, the map happens to be an isomorphism. We can see this by comparing dimensions of domain and range—looking at ordinary character multiplicities we see that both are four dimensional. Thus $efFG^H \cong \mathrm{End}_{FG}(fFGe)$.

Now we analyse $fFGe$ as a right FG-module. The FG-module fFG is isomorphic to the induced module $(fFH)\uparrow^G \cong F_H \uparrow^G$. However, consider the chain of groups $H \leqslant N \leqslant G$, where $N = S_4$. Then

$$F_H \uparrow^N = F_N \oplus \mathrm{sgn}_N \quad \text{and} \quad F_H \uparrow^G = F_N \uparrow^G \oplus \mathrm{sgn}_N \uparrow^G,$$

where F_N is the trivial module and sgn_N is the sign module.

We need some information from [**9**] about the principal 5-block eFG of S_6. There are four simple eFG-modules, labelled by the 5-regular partitions $[6]$, $[4,2]$, $[3,2,1]$ and $[2^2,1^2]$. Denote the corresponding simple modules by $D([6])$, $D([4,2])$, etc., and their projective covers by $P([6])$, $P([4,2])$, etc. With respect to these labellings, the Cartan matrix is

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}.$$

In particular, the principal indecomposable modules $P([6])$ and $P([2^2,1^2])$ have no composition factor in common.

Now we return to the isomorphism

$$fFG \cong F_H \uparrow^G = F_N \uparrow^G \oplus \mathrm{sgn}_N \uparrow^G .$$

It is easy to see that

$$F_N \uparrow^G e = P([6]) \quad \text{and} \quad \mathrm{sgn}_N \uparrow^G e = \mathrm{sgn}_G \otimes F_N \uparrow^G e = P([2^2,1^2]).$$

Since $P([6])$ and $P([2^2,1^2])$ have no composition factor in common, it follows that

$$\begin{aligned} \mathrm{End}_{FG}(fFGe) &= \mathrm{End}_{FG}(P([6]) \oplus P([2^2,1^2])) \\ &= \mathrm{End}_{FG}(P([6])) \oplus \mathrm{End}_{FG}(P([2^2,1^2])). \end{aligned}$$

Thus $efFG^H$, which is isomorphic to $\mathrm{End}_{FG}(fFGe)$, decomposes into a direct product of two non-zero F-algebras (each two dimensional and commutative with a one-dimensional Jacobson radical). □

Note. From the ordinary character multiplicities we see that $efFG^H$ is commutative (even though FG^H is certainly not commutative). Now, with $N = S_4$, it is clear that $Z(FN) \subseteq FG^H$ (for very general reasons). Thus $ef\mathrm{Z}(FN) \subseteq \mathrm{Z}(efFG^H)$, because $efFG^H$ is commutative. What is happening then is that $ef = aef + bef$ (a non-trivial orthogonal decomposition in the centre of the algebra), where a is the block idempotent of the 5-block of N containing F_N and b is the block idempotent of the 5-block of N containing sgn_N.

Some positive results along the lines of Question 6 are possible. For example, the answer is affirmative when $G = S_n$ and $H = S_m$ for $m = n-1$, $n-2$ or $n-3$. The following proposition gives another situation in which the answer is affirmative.

Proposition 3. *Assume that P is a normal p-subgroup of G. Then every central idempotent of FG^P is in $\mathrm{Z}(FG)$.*

Proof. Let e be a primitive central idempotent of FG^P. The Brauer map $\mathrm{Br}_P\colon FG^P \to FC_G(P)$ is a surjective homomorphism. Its kernel is a nilpotent ideal. (To see this, note that if C is an orbit of the conjugation action of P on $G \backslash C_G(P)$, then C^+ acts as 0 on each simple FG-module, so C^+ is in $\mathrm{J}(FG) \cap FG^P$.) Let $f = \mathrm{Br}_P(e)$. Then $f = e + j$, with $j \in \mathrm{J}(FG^P)$. Pick an n such that $j^{p^n} = 0$. Then

$$f = f^{p^n} = (e+j)^{p^n} = e^{p^n} + j^{p^n} = e.$$

Thus $e \in FC_G(P)$. Since e must be central in $FC_G(P)$, it follows that e is a linear combination of elements of $C_G(P)$ of order prime to p.

Assume, for a contradiction, that e is not in the centre of FG. Let $g \in G$ such that $g^{-1}eg \neq e$. Then $g^{-1}ege = 0$ since $g^{-1}eg$ is another primitive central idempotent. Let C be the orbit of g under the conjugation action of P. Since P is normal, $C \subseteq gP$. Let $x_1, x_2, \dots, x_s$ be elements of P such that $C = \{gx_1, gx_2, \dots, gx_s\}$. Let $a = x_1 + x_2 + \dots + x_s$. Then $C^+ = ga$. Since e is a central idempotent of FG^P, it follows that $gae = ega$. Since $a \in FP$, $ae = ea$, so $gea = ega$, and hence $ea = g^{-1}ega$. Multiplying from the left by e, we obtain $ea = eea = eg^{-1}ega = 0a = 0$.

However, e is a linear combination of p'-elements of $C_G(P)$, so for each i, ex_i is a linear combination of elements with p-part x_i. Therefore, the elements of the set $\{ex_1, \dots, ex_s\}$ have disjoint support. It follows that the sum $ea = ex_1 + ex_2 + \dots + ex_s$ cannot be 0. This contradiction completes the proof. $\square$

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