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SOME INEQUALITIES FOR DIOPHANTINE APPROXIMATION BY CONTINUED FRACTIONS

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Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0; a_1, a_2, \ldots, a_n, \ldots]$, let p_n/q_n be its nth convergent and let $\theta_n = q_n|q_n\xi - p_n|$. In this paper a general method is introduced to deduce a series of inequalities involving the triple $(\theta_{n-1}, \theta_n, \theta_{n+1})$.

1. INTRODUCTION

Let ξ be an irrational number with simple continued fraction expansion $\xi = [a_0; a_1, a_2, \ldots, a_n, \ldots]$, where a_0 is an integer and the a_i $(i = 1, 2, \ldots)$ are positive integers. Let p_n/q_n denote the nth convergent. The sequence of approximation constants is defined as follows.

$$\theta_n = q_n|q_n\xi - p_n|. \tag{1}$$

About forty years ago, Brauer and Macon [**1**, **4**] proved the following inequalities involving the triple $(\theta_{n-1}, \theta_n, \theta_{n+1})$:

$$\theta_{n-1}\theta_n\theta_{n+1} < 4/27, \tag{2}$$

$$\theta_{n-1} + \theta_n + \theta_{n+1} < 2, \tag{3}$$

$$\frac{1}{\theta_{n-1}} + \frac{1}{\theta_n} + \frac{1}{\theta_{n+1}} > 6. \tag{4}$$

Recently, Jager and Kraaikamp [**2**] generalised the above results. They proved the following inequalities which include (2), (3) and (4) for $a_{n+1} = 1$.

$$\theta_{n-1}\theta_n\theta_{n+1} < (a_{n+1} + 1)^2/(a_{n+1} + 2)^3, \tag{5}$$

$$\theta_{n-1} + \theta_n + \theta_{n+1} < (2a_{n+1} + 3)/(a_{n+1} + 2), \tag{6}$$

$$\frac{1}{\theta_{n-1}} + \frac{1}{\theta_n} + \frac{1}{\theta_{n+1}} > 4 + a_{n+1} + \frac{2}{a_{n+1} + 1}. \tag{7}$$

In this paper, using the idea in [**7**, **8**], we introduce a general method for deducing a series of inequalities involving the triple $(\theta_{n-1}, \theta_n, \theta_{n+1})$. The results cited above are very special cases of the new inequalities we obtain.

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2. Preliminaries

Let $\xi = [a_0; a_1, a_2, \dots, a_n, \dots]$, $\alpha_{n+1} = [a_{n+1}; a_{n+2}, \dots]$. Then (see [**5**, **6**]) we have

$$\xi - \frac{p_n}{q_n} = \frac{\alpha_{n+1} p_n + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{(\alpha_{n+1} + q_{n-1}/q_n) q_n^2}.$$

Since $q_{n-1}/q_n = [0, a_n, a_{n-1}, \dots, a_1]$, we have that

$$\theta_n = q_n |q_n \xi - p_n| = 1/([a_{n+1}; a_{n+2}, \dots] + [0; a_n, a_{n-1}, \dots, a_1]).$$

Set $P = [a_{n+2}; a_{n+3}, \dots]$, $Q = [a_n; a_{n-1}, \dots, a_1]$ and $\Delta = a_{n+1}PQ + P + Q$. Then we have the following relations.

$$\theta_{n-1} = \left(Q + \frac{1}{a_{n+1} + P^{-1}}\right)^{-1} = \frac{a_{n+1}P + 1}{a_{n+1}PQ + P + Q} = \frac{a_{n+1}P + 1}{\Delta}, \tag{8}$$

$$\theta_n = \left(a_{n+1} + P^{-1} + Q^{-1}\right)^{-1} = \frac{PQ}{a_{n+1}PQ + P + Q} = \frac{PQ}{\Delta}, \tag{9}$$

$$\theta_{n+1} = \left(P + \frac{1}{a_{n+1} + Q^{-1}}\right)^{-1} = \frac{a_{n+1}Q + 1}{a_{n+1}PQ + P + Q} = \frac{a_{n+1}Q + 1}{\Delta}. \tag{10}$$

Viewing P and Q as variables, one easily finds the following formal partial derivatives with respect to P, Q.

$$\theta'_{n-1,P} = -1/\Delta^2; \qquad \theta'_{n-1,Q} = -(a_{n+1}P + 1)^2/\Delta^2, \tag{11}$$

$$\theta'_{n,P} = Q^2/\Delta^2; \qquad \theta'_{n,Q} = P^2/\Delta^2, \tag{12}$$

$$\theta'_{n+1,P} = -(a_{n+1}Q + 1)^2/\Delta^2; \qquad \theta'_{n+1,Q} = -1/\Delta^2. \tag{13}$$

3. Main results

We need a very simple lemma (one need only observe that $P > a_{n+2}$ and $Q > a_n$).

Lemma 1. *Let $f(\theta_{n-1}, \theta_n, \theta_{n+1})$ be a function defined for θ_{n-1}, θ_n, $\theta_{n+1} > 0$. Let $\Delta_0 = a_n a_{n+1} a_{n+2} + a_n + a_{n+2}$. Then*

(i) *if $f((a_{n+1}P + 1)/\Delta, (PQ)/\Delta, (a_{n+1}Q + 1)/\Delta)$ is a decreasing function in both P and Q, we have*

$$f(\theta_{n-1}, \theta_n, \theta_{n+1}) < f\left(\frac{a_{n+1}a_{n+2} + 1}{\Delta_0}, \frac{a_n a_{n+2}}{\Delta_0}, \frac{a_{n+1}a_{n+2} + 1}{\Delta_0}\right); \tag{14}$$

(ii) *if $f((a_{n+1}P + 1)\Delta, (PQ)/\Delta, (a_{n+1}Q + 1)/\Delta)$ is an increasing function in both P and Q, we have*

$$f(\theta_{n-1}, \theta_n, \theta_{n+1}) > f\left(\frac{a_{n+1}a_{n+2} + 1}{\Delta_0}, \frac{a_n a_{n+2}}{\Delta_0}, \frac{a_{n+1}a_{n+2} + 1}{\Delta_0}\right). \tag{15}$$

Since $a_n \geqslant 1$, $a_{n+2} \geqslant 1$, (14) and (15) can be replaced by

$$f(\theta_{n-1},\theta_n,\theta_{n+1}) < f\left(\frac{a_{n+1}+1}{a_{n+1}+2}, \frac{1}{a_{n+1}+2}, \frac{a_{n+1}+1}{a_{n+1}+2}\right), \tag{14'}$$

$$f(\theta_{n-1},\theta_n,\theta_{n+1}) > f\left(\frac{a_{n+1}+1}{a_{n+1}+2}, \frac{1}{a_{n+1}+2}, \frac{a_{n+1}+1}{a_{n+1}+2}\right). \tag{15'}$$

Theorem 1. *If $\alpha, \gamma \geqslant \beta > 0$, then*

$$\theta_{n-1}^{\alpha}\theta_n^{\beta}\theta_{n+1}^{\gamma} < \frac{(a_{n+1}+1)^{\alpha+\gamma}}{(a_{n+1}+2)^{\alpha+\beta+\gamma}}. \tag{16}$$

Proof: Let $f(\theta_{n-1},\theta_n,\theta_{n+1}) = \theta_{n-1}^{\alpha}\theta_n^{\beta}\theta_{n+1}^{\gamma}$. Then

$$\begin{aligned}(\log f)'_P &= \frac{\alpha}{\theta_{n-1}}\theta'_{n-1,P} + \frac{\beta}{\theta_n}\theta'_{n,P} + \frac{\gamma}{\theta_{n+1}}\theta'_{n+1,P} \\ &= \frac{-\alpha}{(a_{n+1}P+1)\Delta} + \frac{\beta Q}{P\Delta} + \frac{-\gamma(a_{n+1}Q+1)}{\Delta} \\ &< \frac{\beta Q}{\Delta} - \frac{\gamma Q}{\Delta} < 0.\end{aligned}$$

Similarly one can show that $(\log f)'_Q < 0$. Therefore $f(\theta_{n-1},\theta_n,\theta_{n+1})$ is a decreasing function in P and Q. By Lemma 1 and (14′) we have (16). ∎

Letting $\alpha = \beta = \gamma = 1$, we have inequality (5).

Theorem 2. *If $3 \geqslant \alpha, \gamma \geqslant \beta \geqslant 1$, A, B, $C > 0$, and αA, $\gamma C \geqslant \beta B$, then*

$$A\theta_{n-1}^{\alpha} + B\theta_n^{\beta} + C\theta_{n+1}^{\gamma} < A\left(\frac{a_{n+1}+1}{a_{n+1}+2}\right)^{\alpha} + B\left(\frac{1}{a_{n+1}+2}\right)^{\beta} + C\left(\frac{a_{n+1}+1}{a_{n+1}+2}\right)^{\gamma}. \tag{17}$$

Proof: Let $f(\theta_{n-1},\theta_n,\theta_{n+1}) = A\theta_{n-1}\alpha + B\theta_n^{\beta} + C\theta_{n+1}^{\gamma}$. Then

$$\begin{aligned} f'_P &= \alpha A\theta_{n-1}^{\alpha-1}\theta'_{n-1,P} + \beta B\theta_n^{\beta-1}\theta'_{n,P} + \gamma C\theta_{n+1}^{\gamma-1}\theta'_{n+1,P} \\ &= \frac{-\alpha A}{(a_{n+1}P+1)^{\alpha-1}\Delta^{3-\alpha}} + \frac{BQ^{3-\beta}}{P^{\beta-1}\Delta^{3-\beta}} + \frac{-\gamma C(a_{n+1}Q+1)^{3-\gamma}}{\Delta^{3-\gamma}} \\ &< \frac{\beta BQ^{3-\beta}}{\Delta^{3-\beta}} - \frac{\gamma CQ^{3-\gamma}}{\Delta^{3-\gamma}} \\ &= \Delta^{\beta-3}(\beta BQ^{3-\beta} - \Delta^{\gamma-\beta}\gamma CQ^{3-\gamma}).\end{aligned}$$

Since $\gamma \geqslant \beta$, we have $\Delta^{\gamma-\beta} = (a_{n+1}PQ + P + Q)^{\gamma-\beta} > Q^{\gamma-\beta}$, hence

$$f'_P < \frac{Q^{3-\beta}}{\Delta^{3-\beta}}(\beta B - \gamma C) < 0.$$

Similarly, one can show that $f'_Q < 0$. By Lemma 1 and (14′), we have inequality (17). ∎

Letting $A = B = C = 1$ and $\alpha = \beta = \gamma = 1$, we have inequality (6).

THEOREM 3. *Let $\alpha, \gamma \geqslant 1 \geqslant \beta > 0$, let $A, B, C > 0$ and $\alpha A, \gamma C \geqslant \beta B$. Then*

$$\frac{A}{\theta_{n-1}^{\alpha}} + \frac{B}{\theta_n^{\beta}} + \frac{C}{\theta_{n+2}^{\gamma}} > A\left(\frac{a_{n+1}+2}{a_{n+1}+1}\right)^{\alpha} + B(a_{n+2}+2)^{\beta} + C\left(\frac{a_{n+2}+2}{a_{n+1}+1}\right)^{\gamma}$$

PROOF: Let $f(\theta_{n-1}, \theta_n, \theta_{n+1}) = A\theta_{n-1}^{-\alpha} + B\theta_n^{-\beta} + C\theta_{n+1}^{-\gamma}$. Then

$$\begin{aligned} f'_P &= \frac{-\alpha A}{\theta_{n-1}^{\alpha+1}}\theta'_{n-1,P} + \frac{-\beta B}{\theta_n^{\beta+1}}\theta'_{n,P} + \frac{-\gamma C}{\theta_{n+1}^{\gamma+1}}\theta'_{n+1,P} \\ &= \frac{\alpha A \Delta^{\alpha-1}}{(a_{n+1}P+1)^{\alpha+1}} - \frac{\beta B \Delta^{\beta-1}}{P^{\beta+1}Q^{\beta-1}} + \frac{\gamma C \Delta^{\gamma-1}}{(a_{n+1}Q+1)^{\gamma-1}} \\ &> \frac{\gamma C \Delta^{\gamma-1}}{(a_{n+1}Q+1)^{\gamma-1}} - \frac{\beta B Q^{1-\beta}}{P^{\beta+1}\Delta^{1-\beta}} \end{aligned}$$

Since $\Delta = a_{n-1}PQ + P + Q > a_{n+1}Q + 1 > Q$, we have $f'_P > \gamma C - \beta B \geqslant 0$.

Similarly one can show that $f'_Q > 0$. By Lemma 1 and (15′), we have inequality (18). ∎

Letting $\alpha = \beta = \gamma = 1$ and $A = B = C = 1$ in Theorem 3, we have inequality (7).

THEOREM 4. *Let $A, C \geqslant B \geqslant 1$. Then*

$$A^{\theta_{n-1}} B^{\theta_n} C^{\theta_{n+1}} < A^{\frac{a_{n+1}+1}{a_{n+1}+2}} B^{\frac{1}{a_{n+1}+2}} C^{\frac{a_{n+1}+1}{a_{n+1}+2}} \tag{19}$$

PROOF: Let $f(\theta_{n-1}, \theta_n, \theta_{n+1}) = A^{\theta_{n-1}} B^{\theta_n} C^{\theta_{n+1}}$. Then

$$\begin{aligned} (\log f)'_P &= \theta'_{n-1,P}\log A + \theta'_{n,P}\log B + \theta'_{N+1,P}\log C \\ &= \frac{1}{\Delta^2}\left(-\log A + Q^2\log B - (a_{n+1}Q+1)^2\log C\right) \\ &< \frac{1}{\Delta^2}(Q^2\log B - Q^2\log C) \\ &< 0. \end{aligned}$$

Similarly one can show that $f'_Q < 0$. By Lemma 1 and (14′), we have inequality (19). ∎

Letting $A = B = C$, we again have inequality (6).

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The significance of Theorems 1,2,3, and 4 is not just that of their statements. They are examples of a general method. For a given function $f(\theta_{n-1}, \theta_n, \theta_{n+1})$ involving parameters, one first finds f'_P and f'_Q by using expressions (8)-(13) (this is the decisive step), then one considers proper conditions on the parameters to make f decreasing or increasing in P or Q. In this way one may deduce numerous other inequalities involving the triple $(\theta_{n-1}, \theta_n, \theta_{n+1})$.

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