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THE EMPTY SPHERE
Part II

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Blow up a sphere in one of the interstices of a lattice until it is held rigidly. There will be no lattice points in the interior and sufficiently many on the boundary so that their convex hull is a solid figure. Such a sphere was called an *empty sphere* by B. N. Delone in 1924 when he introduced his method for lattice coverings [**3**, **4**]. The circumscribed polytope is called an *L-polytope*. Our interest in such matters stems from the following result [**6**, Theorems 2.1 and 2.3]: With a list of the L-polytopes for lattices of dimension $\leqq n$ one can give a geometrical description of the possible sets of integer solutions of

$$f(x) = a_0 + \sum_{i=1}^{n} a_i x_i + \sum_{i,j=1}^{n} a_{ij} x_i x_j = 0$$

$$(a_{ij} = a_{ji};\ a_{ij},\ a_i,\ a_0 \in \mathbf{R})$$

where f satisfies the following condition (in which $\mathbf{Z}$ denotes the integers):

$$f(z) \geqq 0, \quad z \in \mathbf{Z}^n.$$

In [**6**] we also described the 19 possible 4-dimensional L-polytopes as well as all those appearing in lattices of dimension 1, 2, 3. Our purpose here is to provide the remaining details necessary to verify that our list of 4-dimensional L-polytopes is complete [**6**, Theorem 6.2].

1. Introduction. The collection of all L-polytopes of a given n-dimensional lattice Γ forms a decomposition of space, an *L-decomposition* (described in [**6**, Section 3]). To enumerate the possible L-polytopes we first classify the L-decompositions up to affine equivalence and then search these for the various species of L-polytope.

In his second memoir [**12**], published posthumously in 1909, Voronoi introduced L-decompositions and showed how their affine structure could be studied. There he made extensive use of the correspondence between positive quadratic forms and lattices; the method of continuous parame-

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ters of Hermite: form f corresponds to lattice Γ if Γ has a fundamental basis $E = \{e_1, e_2, \ldots, e_n\}$ such that

$$f(x) = |e_1x_1 + e_2x_2 + \ldots + e_nx_n|^2.$$

This correspondence is many to one. The action of an element Q of $GL(n, \mathbf{Z})$ on E produces an equally valid basis E' for Γ but the coefficient matrices for the corresponding quadratic forms are related by the formula $A' = Q^{\mathrm{tr}}AQ$ and are generally distinct. If P^N is the cone of positive forms then integrally equivalent elements of P^N correspond to the same lattice. Here, $N = \binom{n+1}{2}$ is the dimension of this cone of positive forms.

Voronoi showed that the action of $GL(n, \mathbf{Z})$ induces a partition of P^N into disjoint relatively open convex subcones of dimension $1, 2, \ldots, N$ such that [**12**, Part IV]:

(i) On each of these subcones the affine structure of the L-decompositions of corresponding lattices is constant. Subcones are integrally equivalent if and only if the affine structure of corresponding L-decompositions is identical.

(ii) Subcones of dimension N correspond to general lattices having simplicial L-decompositions. These L-type domains are polyhedral – they are described by a finite system of strict linear inequalities.

(iii) A subcone of dimension less than N is a relatively open proper face of two or more L-type domains. If such a cone makes contact with the boundary of an L-type domain it necessarily is a face of that domain. The *special lattice* corresponding to a form on such a face has among its L-polytopes some which are not simplexes.

For dimensions $n = 1, 2, 3$ the L-type domains of P^N are all integrally equivalent and were called by Voronoi type I. There is a common analysis for these, described in [**6**, Section 4], which extends to any dimension. In four dimensions his investigations revealed exactly three inequivalent domains; types I, II and III [**12**, Part V, pp. 164-178]. Each corresponds to lattices with simplicial L-decompositions which for the three types are affinely inequivalent.

It will be convenient to refer to an L-polytope as type I if it belongs to a lattice corresponding to a form in or on the boundary of a type I domain. Similarly we will refer to type II and type III L-polytopes. This assignment of a type is of course not unique. As it turns out (see the proof of Theorem 4.3) there is only one species of 4-dimensional simplicial L-polytope and it is type I, II and III.

By examining the facial structure of a type I domain, we found in [**6**] all of the type I L-polytopes for $n \leqq 4$. Here we complete our study by enumerating the 4-dimensional L-polytopes which are not type I. This involves a similar examination of the type II and III domains.

The material of Section 2 is not new; it has been used for different purposes than ours on the references cited. In Section 3 the L-decomposition of the red triangular lattice is a new result; the central lattice has been discussed elsewhere but our treatment is new. In Section 4 we give the concluding arguments of our treatment of the 4-dimensional L-polytopes and this material is completely new.

2. The geometry of Voronoi's second perfect domain. In Voronoi's theory of integral reduction via perfect forms, an arbitrary quadratic form in $n = 4$ variables is integrally equivalent to one lying in either the first or second perfect reduction domain [**11**, p. 172]. The first coincides with a type I domain. Here we examine the geometry of the second perfect domain, more completely discussed in [**8**, Section 11-15], [**9**, Section 9].

The second perfect domain consists of all forms

$$\begin{aligned} f(x_1, x_2, x_3, x_4) &= \beta_{12}x_1^2 + \rho_{12}x_2^2 + \rho_{13}x_3^2 + \rho_{14}x_4^2 \\ &+ \beta_{23}(x_1 - x_3)^2 + \beta_{24}(x_1 - x_4)^2 + \rho_{23}(x_2 - x_3)^2 \\ &+ \rho_{24}(x_2 - x_4)^2 + \rho_{34}(x_3 - x_4)^2 + \beta_{13}(x_1 + x_2 - x_3)^2 \\ &+ \beta_{14}(x_1 + x_2 - x_4)^2 + \beta_{34}(x_1 + x_2 - x_3 - x_4)^2, \end{aligned}$$

where $\rho_{ij}, \beta_{ij} \geqq 0$. Using Vononoi's transformation

$$(x_1, x_2, x_3, x_4) \rightarrow (x_1 + x_2, x_1 - x_2, x_1 - x_3, x_1 - x_4)$$

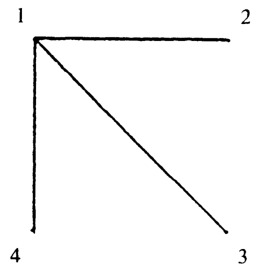
this expression can be rewritten as

$$f(x_1, x_2, x_3, x_4) = \sum_{i<j}^{4} [\beta_{ij}(x_i + x_j)^2 + \rho_{ij}(x_i - x_j)^2].$$

The collection of all such forms is a polyhedral cone, say K.

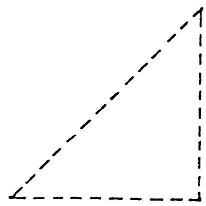
Since the Voronoi transformation has determinant 2, the correspondence with lattices has been altered. The coefficient matrix of a form in K is the Gram matrix for a lattice basis which is not fundamental. As is easy to compute, the coordinate vector of a lattice point referred to such a basis is either integer or the sum of an integer vector with $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$.

The relatively open faces of the second perfect domain correspond to collections of forms with some of the parameters β_{ij}, ρ_{ij} set equal to zero; the others strictly positive. A convenient representation of any such face is a graph on four vertices with edges of two colors. If $\beta_{ij} = 0$ add a black edge to the graph connecting vertices i and j; if $\rho_{ij} = 0$ add a red edge (dotted line). It is easy to verify that any graph with three edges which is either forked or triangular corresponds to a 9-dimensional face (a facet) of the second perfect domain (note that dim $K = 10$):



$\beta_{12} = \beta_{13} = \beta_{14} = 0$

3 black edges –
a black forked facet

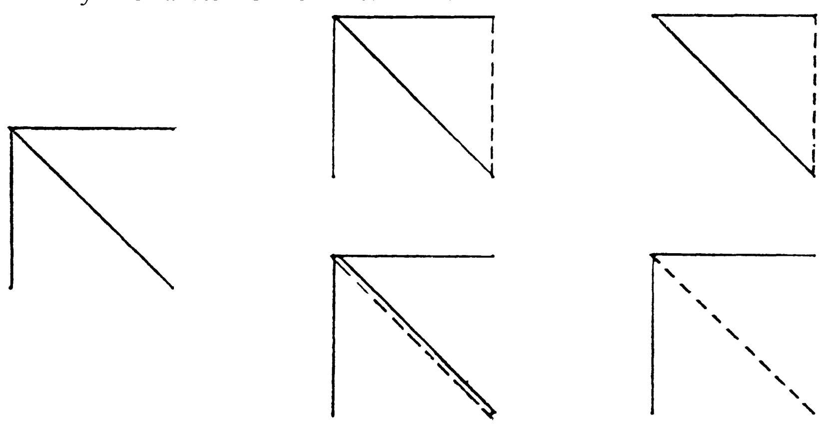


$\rho_{23} = \rho_{24} = \rho_{34} = 0$

3 red edges –
a red triangular facet

Taking into account the two colors there are 32 forked and 32 triangular graphs and corresponding facets.

All of these facets are simplicial; the number of extreme rays in their closure equals their dimension, which is nine. Thus each of these facets has nine relatively open 8-dimensional faces, their graphs generated by adding a single edge in all possible ways to the graph of the facet. By experimentation with graphs one quickly learns that any of these are faces of precisely two facets from our list of 64.



9-dimensional facet | 8-dimensional facet | neighboring facet

It is a simple consequence of this fact that:

2.1. Proposition. *The second perfect reduction domain K has exactly* 64 *facets* (*all simplicial*), 32 *of which are triangular and* 32 *forked.*

The cone K is invariant under the linear maps induced by:

$$k_i : x_i \to -x_i, \quad x_j = x_j, \quad j \neq i,$$

$$s_{ij} : x_i \leftrightarrow x_j, \quad x_k \to x_k, \quad i \neq j \neq k.$$

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In fact, the second perfect domain appears in Voronoi's theory whenever $n \geqq 4$ (replace 4 by n in the above expression for $f \in K$) and these transformations generate the full symmetry group when $n \geqq 5$. This is the group of the n-cube, B_n. It is a peculiarity of 4-space that the additional transformation

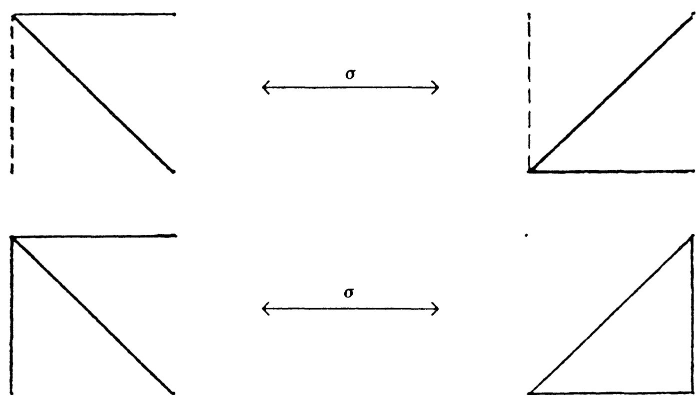
$$\sigma : x_i \rightarrow -x_i + \frac{1}{2} \sum_{k=1}^{4} x_k$$

leaves K invariant. The group generated by all these actions, say G, can be identified as F_4 with order 1152. This follows since $s_{14}s_{23}\sigma$ is an orthogonal reflection and the only finite reflection group on $\mathbf{R}^4$ that properly contains B_4 is F_4 [**2**, p. 578].

2.2. PROPOSITION. $|G| = 1152$.

The action of G on the facets of K can easily be described using the graphs. The element s_{ij} permutes vertices i and j. The element k_i converts all red edges incident at vertex i to black and all black to red; the edges incident at i are recolored. All of these actions map triangular facets into triangular and forked into forked.

The effect of σ is to fix red edges and replace black edges by their complements (the complement of the black edge between 2 and 3 is the black edge between 1 and 4).



the action of σ

Black forked facets are exchanged with black triangular under this action.

2.3. PROPOSITION. (i) *The* 64 *facets of K fall into two G-equivalence classes. That containing the black forked facets has* 48 *elements*; *that containing the red triangular facets* 16. *Accordingly, we will refer to facets as either* BF *or* RT *depending upon their equivalence class.*

(ii) *An arbitrary* BF *facet shares six of its nine* 8*-dimensional faces with* BF *facets; the remaining three neighbors are* RT *facets.*

(iii) *All nine neighbors of an* RT *facet are* BF *facets.*

Proof. Part (i) of the proposition follows by considering the action of G on the graphs of the facets. Parts (ii) and (iii) by taking a typical facet of each type, enumerating the nine neighbors and classifying these by inspection.

For the following result see [**11**, p. 172]; a discussion and proof can be found in [**8**, Section 15].

2.4. THEOREM. *The second perfect reduction domain K shares* BF *facets with domains which are integrally equivalent to the first perfect reduction domain. The* RT *facets are shared by K with domains which are integrally equivalent to it.*

There is a one-dimensional ray, the *central axis* of K, which is invariant under G. The elements of this ray are given by the formula

$$f(x_1, x_2, x_3, x_4) = \rho \sum_{i<j}^{4} (x_i + x_j)^2 + (x_i - x_j)^2$$
$$= 6\rho(x_1^2 + x_2^2 + x_3^2 + x_4^2).$$

The interior of the convex hull of the central axis with any of the 64 relatively open 9-dimensional facets of K forms an open 10-dimensional simplicial cone. By supplementing these with all of their relatively open faces of varying dimension we have a complete partition of K. The connection with L-type domains is supplied by the following result (see [**12**, Part V, pp. 164-178]; a modern proof can be found in [**9**, Section 9]).

2.5. THEOREM. *The* 48 *open simplicial cones built with* BF *facets are equivalent L-type domains, Voronoi's type II. The remaining* 16 *cones built with* RT *facets are type III domains.*

We are particularly interested in the relatively open faces of varying dimension of type II and III domains which are not type I. By searching the special lattices corresponding to these, new species of L-polytopes may be found which are neither simplicial nor type I.

2.6. PROPOSITION. (i) *All type III faces, with the exception of the* RT *facets*, are type II. *The* RT *facets bound only type III domains.*

(ii) *Of the type II faces, the* BF *facets and faces lying on the boundaries of these are type I. The remaining type II faces are interior to K and are not type I.*

Proof. Both parts follow directly from 2.3, 2.4, and 2.5.

We now know where to look for new species of L-polytope which are not type I:

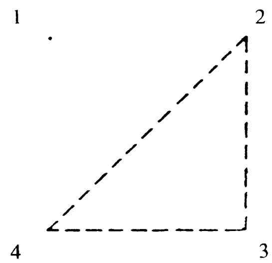
2.7. PROPOSITION. *Any non-simplicial L-polytope which is not type I must belong to a lattice with form lying on either a* RT *facet or a type II face interior to K. To determine whether there are species of simplicial L-polytopes which are not type I an additional search of a type II and III domain must be made.*

3. The L-decomposition of two symmetrical lattices. The *red triangular* and the *central* lattice correspond to forms on a red triangular facet and the central axis of the second perfect domain, K. See [**9**, Section 9] for some material relating to our discussion of the red triangular lattice and [**10**] for a treatment of the central lattice in a more general context.

As metrical form for the red triangular lattice we take

$$\begin{aligned} f(x) &= \frac{1}{2}\{(x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_1 + x_4)^2 + (x_2 + x_4)^2 \\ &\qquad + (x_3 + x_4)^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 + (x_1 - x_4)^2\} \\ &= 3x_1^2 + 2(x_2^2 + x_3^2 + x_4^2) + x_2x_3 + x_2x_4 + x_3x_4. \end{aligned}$$

This form lies on the central ray of the red triangular facet with graph



red triangular facet

and is invariant under the stability group of the graph.

As is easy to compute, there are (up to sign) nine non-zero lattice vectors having the minimal length $\sqrt{2}$; the *short edges* of the red triangular lattice. Together with six additional with length $\sqrt{3}$ (the *long edges*) the fifteen shortest lattice vectors have the coordinates tabled below.

short edges ($\sqrt{2}$)		long edges ($\sqrt{3}$)	
$[0, 1, 0, 0]$	$\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$	$[1, 0, 0, 0]$	$\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$
$[0, 0, 1, 0]$	$\left[\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]$	$[0, 1, -1, 0]$	$\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$
$[0, 0, 0, 1]$	$\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$	$[0, 1, 0, -1]$	
$\left[\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$	$\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right]$	$[0, 0, 1, -1]$	
$\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right]$			

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Consider polytope A:

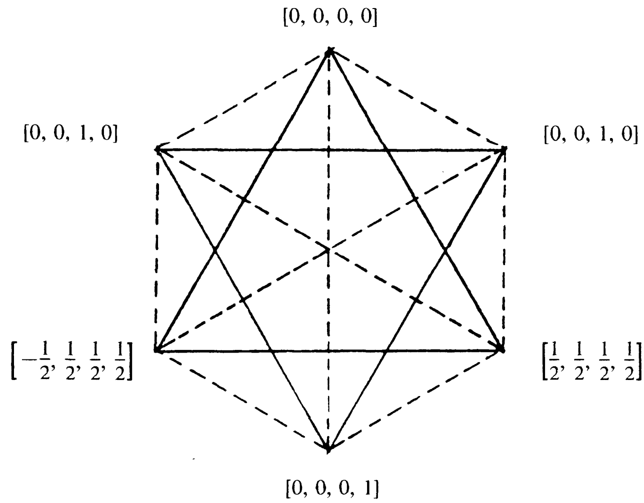

Figure 1. Cyclic Polytope A

Some important properties of A can be established by easy computations:

(i) The edges of the two triangles drawn with solid lines are long with length $\sqrt{3}$, the others are short (dotted lines) with length $\sqrt{2}$.

(ii) The solid equilateral triangles lie in mutually perpendicular planes. Their common centroid, $\left[0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$, is the center of a unit sphere circumscribing A.

(iii) There are nine facets, formed by choosing pairs of vertices from each of the solid triangles in all possible ways, then taking convex hulls. All are equivalent 3-simplexes under the group of isometries of A. Each has a pair of perpendicular opposite edges which are long; the other four edges are short.

(iv) Since each pair of vertices of A is joined by an edge, A is a cyclic polytope.

3.1. PROPOSITION. *A is an L-polytope.*

Proof. By re-expressing f as a sum of squares, the distance from $[x_1, x_2, x_3, x_4]$ to the centroid can be written as

$$3x_1^2 + \left(x_2 - \frac{1}{3}\right)^2 + \left(x_3 - \frac{1}{3}\right)^2 + \left(x_4 - \frac{1}{3}\right)^2$$
$$+ \frac{1}{2}\left\{\left(x_2 + x_3 - \frac{2}{3}\right)^2 + \left(x_2 + x_4 - \frac{2}{3}\right)^2 + \left(x_3 + x_4 - \frac{2}{3}\right)^2\right\}.$$

In order that this length not exceed 1, an integer coordinate vector must have $x_1 = 0$; at most one of x_2, x_3, x_4 may be non-zero with value 1. We already know (property (ii) above) that the four possible integer vectors satisfying these conditions lie on the unit sphere centered at $\left[0, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]$. Similarly all half-integer coordinate vectors except $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$, $\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$ lie outside this sphere. This sphere is therefore empty and has the six vertices of A lying on it. Thus A is an L-polytope.

3.2. PROPOSITION. *The simplex*

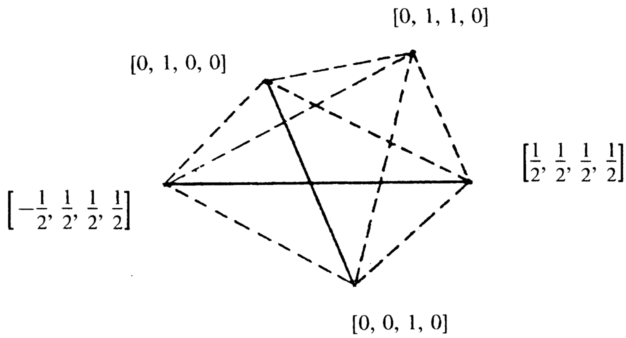

Figure 2. Simplex S

with two long edges (solid lines) of length $\sqrt{3}$ *and eight short edges (dotted lines) of length* $\sqrt{2}$ *is an L-polytope. The facet containing the long edges is shared with A.*

Proof. After checking edge lengths note that the shared facet is the convex hull of the vertices with coordinates [0, 1, 0, 0], [0, 0, 1, 0], $\left[-\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$. Since the center of the circumscribed 3-sphere is also the barycenter of this 3-simplex, $\left[0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right]$, the radius is easily computed to be $\sqrt{7/8}$.

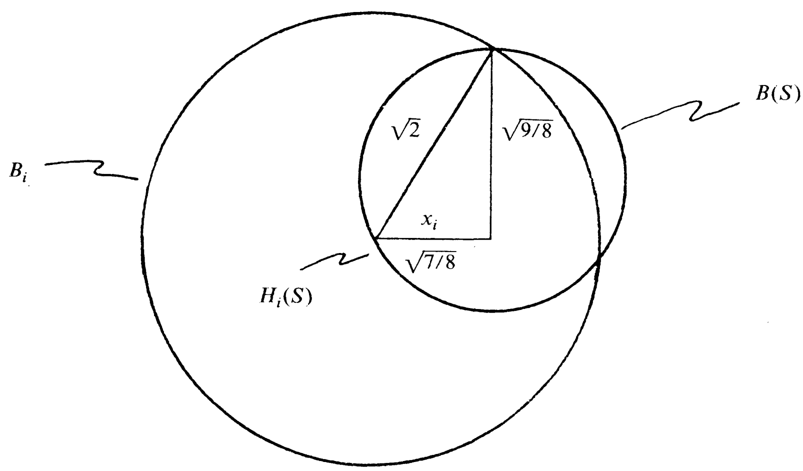

Figure 3. $H_i(S)$

Let $B(S)$ be the closed ball obtained by filling in the sphere circumscribing S. Let $H_i(S)$, $i = 1, 2, 3, 4$ be the closed half-ball $\{x \in B(S) \mid x \cdot x_i \geqq 0\}$, where x, x_i are referred to a coordinate system centered at barycenter $\left[0, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}\right]$, and x_i is the position vector of vertex i of the shared facet. Notice that the apex, $a = [0, 1, 1, 0]$ of S is contained in each of these half-balls. Since the minimal edge length is $\sqrt{2}$, x_i is the only lattice point interior to B_i, the closed ball of radius $\sqrt{2}$ centered at x_i. By figure 3 the only portion of $H_i(S)$ not interior to B_i is the apex a. Hence the only lattice points in

$$B(S) = \bigcup_{i=1}^{4} H_i(S)$$

are the vertices of S and S is an L-polytope.

Other than the facet shared with A, S has four which are 3-simplexes with a single long and five short edges; these are equivalent under the group of isometries of S.

3.3. Proposition. *Each facet of S having a single long edge is shared by an L-polytope which is an isometric copy of S.*

Proof. We need only consider the facet of S whose vertices have coordinates $[0, 1, 0, 0]$, $[0, 0, 1, 0]$, $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$, $[0, 1, 1, 0]$. The convex hull of this facet with $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$ is a simplex whose ten edge lengths are readily computed (see figure 4 where solid lines are long edges etc.).

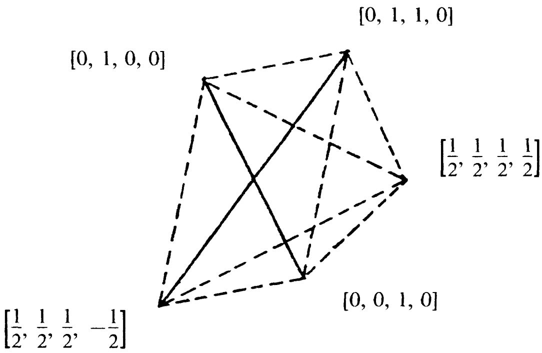

Figure 4. Copy of S

This simplex is clearly an isometric copy of S and an L-polytope.

3.4. Theorem. *In the red triangular lattice the star of L-polytopes at the origin consists of* 12 *isometric copies of A and* 90 *isometric copies of S.*

Proof. Translating the six vertices of A to the origin results in six copies of A. Inverting these through the origin leads to six additional since A is not invariant with respect to central inversion. Thus twelve distinct copies of A having two orientations have been accounted for.

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Lattice translations account for five copies of S at the origin. Since the nine 3-simplexes appearing as facets of A and the nine obtained from these by inversion are translationally inequivalent a factor of eighteen must be included. Thus there are ninety distinct isometric copies of S at the origin having eighteen different orientations.

Since it is impossible to construct additional isometric copies of the facets of A which are translationally inequivalent to the eighteen considered there are no further copies of A or S at the origin. That there are no other species of L-polytope follows from Proposition 3.3. Hence there are exactly 12 copies of A and 90 copies of S in the star of L-polytopes at the origin.

To analyze the central lattice we take the metrical form

$$\begin{aligned} f = \frac{1}{6}\{ &(x_1 + x_2)^2 + (x_1 + x_3)^2 + (x_1 + x_4)^2 + (x_2 + x_3)^2 \\ &+ (x_2 + x_4)^2 + (x_3 + x_4)^2 + (x_1 - x_2)^2 + (x_1 - x_3)^2 \\ &+ (x_1 - x_4)^2 + (x_2 - x_3)^2 + (x_2 - x_4)^2 + (x_3 - x_4)^2\} \\ = {} & x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{aligned}$$

which lies on the central axis of K. The elements of G are isometries of the central lattice. Since the metric matrix is the identity we no longer need distinguish between lattice vectors and coordinate vectors.

Starting with $\left[\frac{1}{2}, \frac{1}{2}, 0, 0\right]$ as center and successively adding and subtracting the four mutually orthogonal vectors $\left[\frac{1}{2}, \frac{1}{2}, 0, 0\right]$, $\left[\frac{1}{2}, -\frac{1}{2}, 0, 0\right]$, $\left[0, 0, \frac{1}{2}, \frac{1}{2}\right]$, $\left[0, 0, \frac{1}{2}, -\frac{1}{2}\right]$ we generate the vertices of a regular 4-dimensional *cross polytope* C:

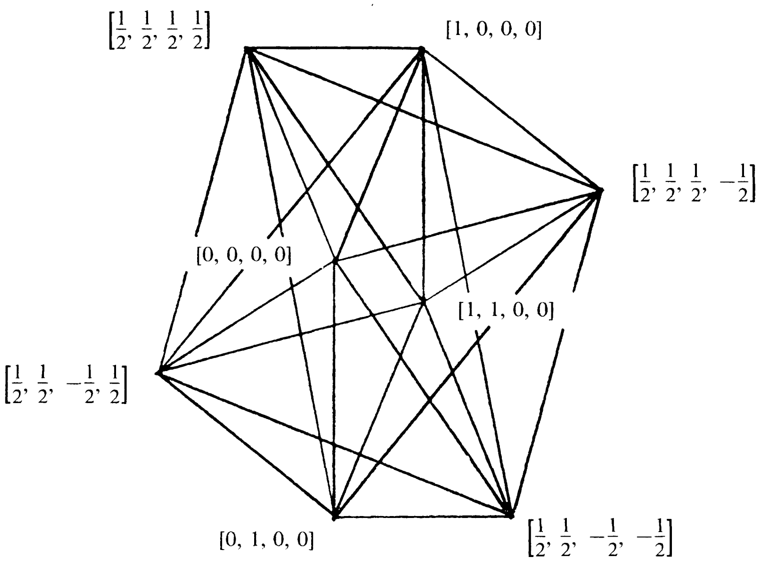

Figure 5. Cross Polytope C

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3.5. PROPOSITION. *The cross polytope C is an L-polytope in the central lattice.*

Proof. Since the eight vertices are at a distance $1/\sqrt{2}$ from the center of C, the coordinates of any additional lattice point lying in or on the circumscribed sphere must satisfy the inequality

$$\left(x_1 - \frac{1}{2}\right)^2 + \left(x_2 + \frac{1}{2}\right)^2 + x_3^2 + x_4^2 \leqq 1/2.$$

Since there are none such C is an L-polytope.

The axis of C attached to the origin coincides with the lattice vector $p = [1, 1, 0, 0]$. We will refer to this axis as the polar axis, the remaining three as equitorial. As can easily be verified, the G-orbit of p consists of the 24 lattice vectors having a pair of non-zero entries with values ± 1; these are the lattice vectors of length 2. Accordingly the action of G on C produces 24 distinct isometric copies of C at the origin.

The elements of the stability group of p, $G(p) \subset G$, fix the polar axis and permute the six equitorial vertices. The convex hull of these six forms a regular octahedron and the full group of isometries of C fixing the polar axis is the octohedral group. That $G(p)$ coincides with the full group can be deduced from its order which by Proposition 2.2 is $|G|/24 = 1152/24 = 48$ and is that of the octahedral group. As a consequence, the eight facets of C attached at the origin are $G(p)$-equivalent.

The element $s_{23} \in G$ maps C onto an isometric copy with polar axis $[1, 0, 1, 0]$. The facet of C with vertices $[0, 0, 0, 0]$, $[1, 0, 0, 0]$, $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$ is fixed by this action and thus shared with the copy. Since the eight facets of C attached to the origin are $G(p)$-equivalent, C is entirely surrounded by copies of itself in the star of L-polytopes at the origin. It follows that the star contains only the 24 copies of C enumerated above. Since C has eight vertices these fall into three groups of eight, elements being translationally equivalent if and only if they belong to the same group.

3.6. THEOREM. *In the central lattice the star of L-polytopes at the origin contains* 24 *L-polytopes, isometric copies of C. These belong to three equivalence classes of eight elements each under the group of lattice translations.*

The L-decomposition of $\mathbf{R}^4$ into non-overlapping copies of C is the regular honeycomb $\{3, 3, 4, 3\}$ described in [**1**, Section 7.8, 8.2]. By taking the convex hull of the centers of the 24 copies of C incident at the origin, a G-invariant polytope P is produced with the following interesting characterization: P is the collection of points of $\mathbf{R}^4$ which are at least as close to the origin as to any other lattice point. P is the regular polytope $\{3, 4, 3\}$ having twenty-four equivalent octahedral facets; the 24-cell. The collection of lattice translates of P forms a second regular honeycomb $\{3, 4, 3, 3\}$, the dual of $\{3, 3, 4, 3\}$.

4. The 4-dimensional *L*-polytopes. Here we complete our treatment of the 4-dimensional *L*-polytopes by enumerating those which are not type I. By Proposition 2.7 this involves examining a RT facet and portions of the interior of *K*, as well as the corresponding *L*-decompositions and *L*-polytopes. To determine whether a new species is found comparison must be made with the results of [**6**] where the type I 4-dimensional *L*-polytopes are pictured.

Since the RT facet and the central axis have already been examined (Theorems 3.4 and 3.6) we pursue our search by considering an arbitrary form $k \in \operatorname{int} K$ which lies off the central axis.

4.1. PROPOSITION. *Suppose that $k \in \operatorname{int} K$ does not lie on the central axis. The corresponding L-decomposition can be obtained (up to an affine transformation) from that of the central lattice by partitioning some or all of its cross polytopes, the vertices of the polytopes forming each partition being a subset of those of the original cross polytope. Hence this decomposition is a refinement of the L-decomposition of the central lattice.*

Proof. Note that the type II and III domains of *K* and all the faces of these domains which are interior to *K* contain points either on, or arbitrarily close to, the central axis. Hence we can assume that *k* lies close to the central axis. In fact it will be convenient to assume that *k* is close to $x_1^2 + x_2^2 + x_3^2 + x_4^2$, the metrical form of the central lattice.

Consider some cross polytope and its circumscribed empty sphere in the central lattice. In the perturbed lattice (corresponding to *k*) the vertices of this cross polytope are displaced and lie on one, two or even several distinct empty spheres nearly coincident with the original. If the perturbation is small enough lattice points not belonging to the original sphere will not lie on any disturbed copy. Hence *L*-polytopes in the disturbed lattice have vertices arising from subsets of those of cross polytopes in the central lattice.

Even though there may be some cross polytopes in the disturbed lattice the *L*-decomposition cannot consist entirely of cross polytopes. For in this case the *L*-decomposition would be affinely equivalent to that of the central lattice. Property (i) of Voronoi's partition of P^N (see Section 1) would imply that *k* lies on a one-dimensional face in this partition which is integrally equivalent to the central axis. But this is impossible since the central axis is the unique one-dimensional face in the partition of *K* which is interior to *K*. Hence the *L*-decomposition corresponding to *k* is as described in the proposition and the proof is complete.

By Proposition 4.1 our search for *L*-polytopes reduces to considering what effect a change in metric has on the individual cross polytopes in the central lattice. In fact, by symmetry (and by Theorem 3.6) it is sufficient to investigate the effect on only cross polytope *C*.

By perturbing the metrical form, angles between the four axes of C and their lengths can be altered. If angles alone are altered the vertices remain equidistant from the center and lie on a single circumscribed empty sphere. The resulting L-polytope is again a cross polytope but no longer regular.

If lengths of axes are altered, C fragments into smaller L-polytopes. Stretching a single axis of C results in vertices lying on a pair of distinct empty spheres. Corresponding L-polytopes are pyramids with a common regular octahedron as base. Polytope B of figure 6 results when the axis through [0, 0, 0, 0] and [1, 1, 0, 0] is stretched.

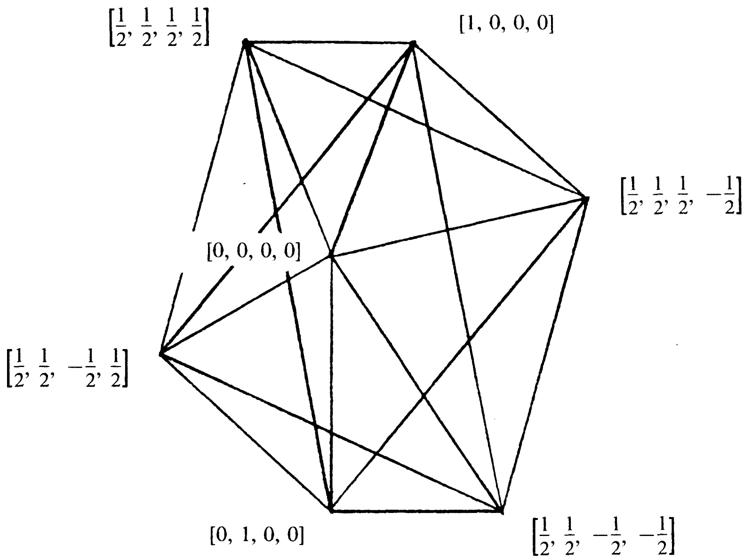

Figure 6. Pyramid B

Stretching a second axis produces similar results. The deformed octahedral base fragments into a pair of 3-pyramids with common square base, each circumscribed by a 3-sphere. The 4-dimensional cross polytope is partitioned into four copies of a pyramid with 3-pyramid as base. The convex hull of vertices [1, 0, 0, 0], [0, 1, 0, 0], $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right]$, $\left[\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right]$, [0, 0, 0, 0], $\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right]$, say D, would be such an L-polytope (see figure 6). Stretching a third axis produces again a doubling of L-polytopes, the result being a partition into eight simplexes.

A perturbation of the metric which both alters angles between axes and their lengths produces partitions which are affinely equivalent to one where lengths of axes alone are altered.

4.2. PROPOSITION. *In lattices corresponding to forms* $k \in \operatorname{int} K$, (*up to affine equivalence*) *only four types of L-polytopes occur*: B, C, D, *and simplexes.*

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In [6, Section 2] we pointed out that affine equivalence of L-polytopes is not sufficiently discriminating in our considerations. There we introduced the notion of z-equivalence. The classification of L-polytopes we seek is up to z-equivalence. By the definition of z-equivalence [6, Definition 2.2], two affinely equivalent L-polytopes are z-equivalent if, in both cases, the edge vectors generate the ambient lattice.

4.3. THEOREM. *Up to z-equivalence there are three 4-dimensional L-polytopes which are not type I. These are the cyclic polytope A of the red triangular lattice, the cross polytope C of the central lattice and pyramid B. Polytope A is type III alone whereas B, C belong to both types II and III.*

Proof. We will refer to Table V of [6] where the 16 4-dimensional type I L-polytopes are pictured. We remark that A, B, and C are pictured in Table VII of [6].

With our search of the relevant portions of K (Proposition 2.7) we have found (Theorem 3.4, Proposition 4.2) four non-simplicial L-polytopes; A, B, C, D. Since only D appears in Table V of [6] we conclude that A, B, C are not type I. It is easy to check that D is z-equivalent to No. 2 of this table, hence type I; in both cases the edge vectors include a basis for the ambient lattice.

It is clear that A is type III alone. The only lattice containing copies are those corresponding to forms on RT facets and these bound only type III domains. Since the central axis is contained in the boundary of every type II and III domain of K, C is both types II and III. A more detailed argument (which we omit) shows that the same holds for B.

All type II and type III simplicial L-polytopes are z-equivalent to No. 1 of Table V, hence type I. The edge vectors of all of these include a basis for the ambient lattice. We omit the details of the argument.

4.4. THEOREM. (Theorem 6.2 of [6]) *Up to z-equivalence there are 19 4-dimensional L-polytopes. Of these 16 are type I (Table V of [6]) and three are not; polytopes A, B, C.*

This concludes our treatment of the 4-dimensional L-polytopes.

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