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ISOMORPHISM PROBLEM FOR METACIRCULANT GRAPHS OF ORDER A PRODUCT OF DISTINCT PRIMES

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ABSTRACT. In this paper, we solve the isomorphism problem for metacirculant graphs of order pq that are not circulant. To solve this problem, we first extend Babai's characterization of the CI-property to non-Cayley vertex-transitive hypergraphs. Additionally, we find a simple characterization of metacirculant Cayley graphs of order pq, and exactly determine the full isomorphism classes of circulant graphs of order pq.

1. **Preliminaries.** Throughout the paper, p and q are distinct primes. For definitions and properties of permutation groups the reader is referred to [10], and for graph theoretic notation to [4]. Let $\mathbb{Z}_n$ be the ring of integers modulo n, and $\mathbb{Z}_n^*$ be the units of $\mathbb{Z}_n$. Let m, n be positive integers and set $\mu = \lfloor m/2 \rfloor$. Let $V = V(\Gamma) = \{v_j^i : i \in \mathbb{Z}_m, j \in \mathbb{Z}_n\}$, and $\alpha \in \mathbb{Z}_n^*$. Let $S_0, S_1, \ldots, S_\mu$ be subsets of $\mathbb{Z}_n$ satisfying the following conditions:

(1) $0 \notin S_0 = -S_0$,

(2) $\alpha^m S_r = S_r$ for $0 \leq r \leq \mu$,

(3) if m is even, then $\alpha^\mu S_\mu = -S_\mu$.

Let $E = \{(v_j^i, v_h^{i+r}) : 0 \leq r \leq \mu \text{ and } h - j \in \alpha^i S_r\}$. We define the *metacirculant graph* $\Gamma = \Gamma(m, n, \alpha, S_0, \ldots, S_\mu)$ to be the graph with vertex set V and edge set E. We will also refer to Γ as an (m, n)-*metacirculant*. Define two permutations ρ, τ on V by $\rho(v_j^i) = v_{j+1}^i$ and $\tau(v_j^i) = v_{\alpha j}^{i+1}$.

Metacirculant graphs were first introduced by Alspach and Parsons [2], where their elementary properties were discussed.

THEOREM 1 (ALPACH AND PARSONS, [2]). *The metacirculant* $\Gamma = \Gamma(m, n, \alpha, S_0, \ldots, S_\mu)$ *is vertex-transitive with* $\langle \rho, \tau \rangle \leq \mathrm{Aut}(\Gamma)$. *Conversely, any graph* Γ' *with vertex set* V *and* $\langle \rho, \tau \rangle \leq \mathrm{Aut}(\Gamma')$ *is an* (m, n)-*metacirculant.*

Although we are primarily interested in graphs, some of the results we will prove are true for more general objects than graphs. Let G be a finite set and $S \subseteq G \cup 2^G \cup 2^{2^G} \cup \cdots$, where 2^G is the collection of subsets of G, 2^{2^G} the collection of subsets of subsets, etc. The pair $X = (G, S)$ is called a *combinatorial object*. If $S \subseteq \mathbf{P}(G)$, the power set of G, then X will be called a *hypergraph*. We call G the *vertex set* of X and denote it by $V(X)$. Furthermore, S is called the *edge set* of X, and we denote it by $E(X)$. Let G be a set of cardinality n. Define the *trivial* combinatorial object E^n, to be the combinatorial object with $V(E^n) = G$ and $E(E^n) = \emptyset$. An *isomorphism* between two combinatorial objects (G, S) and (G', S') is a bijective function $f: G \longrightarrow G'$ such that

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$f(S) = S'$. Aut(X), the *automorphism group* of X, is the group of all isomorphisms from X to itself. We will say that a combinatorial object X is an (m, n, α)-*metacirculant object* ((m, n)-*metacirculant*) if $G = V$ and $\langle \rho, \tau \rangle \leq \mathrm{Aut}(X)$, where ρ, τ are defined as above. Let X be a vertex-transitive combinatorial object with $V(X) = G$, where G is some group. Let $G_L = \{g_L : G \longrightarrow G : g_L(x) = gx, g \in G\}$. We say X is a *Cayley object of* G (*Cayley object* if the group is unimportant) if and only if $G_L \leq \mathrm{Aut}(X)$. In [2], Alspach and Parsons gave sufficient conditions for an (m, n)-metacirculant graph to be a Cayley graph. Their proof is entirely group theoretic and hence also gives the following sufficient conditions for an (m, n)-metacirculant combinatorial object to be a Cayley object.

THEOREM 2 (ALSPACH AND PARSONS, [2]). *Let X be an (m, n, α)-metacirculant combinatorial object with $a = |\alpha|$, and $c = a/\gcd(a, m)$. If $\gcd(c, m) = 1$, then X is a Cayley object for the group $\langle \rho, \tau^c \rangle$. Furthermore, this group is abelian if $\gcd(a, m) = 1$ and it is cyclic if $\gcd(a, m) = 1 = \gcd(m, n)$.*

Marušič gave the full characterization of (q, p)-metacirculant graphs.

THEOREM 3 (MARUŠIČ, [8]). *A vertex-transitive graph Γ of order qp is metacirculant if and only if $\mathrm{Aut}(\Gamma)$ contains subgroups H and $K \neq 1$ such that H is transitive, $K \lhd H$ such that K is not transitive.*

This theorem can be generalized to the following result (see [5], p. 5).

THEOREM 4. *A vertex-transitive combinatorial object X of order qp is metacirculant if and only if $\mathrm{Aut}(X)$ contains subgroups H and $K \neq 1$ such that H is transitive, $K \lhd H$, K is not transitive and has orbits of size p, and the Sylow p-subgroups of K have order p.*

Let G be a transitive group of degree mk such that there exists a transitive subgroup $H < G$ such that H admits a complete block system $\mathcal{B}$ of m blocks each of size k. Enumerate the blocks $B_0, B_1, \ldots, B_{m-1}$. Define a map $\pi_1 : H \longrightarrow S^m$, the *symmetric group on m symbols*, by $\pi_1(\alpha) = \alpha/\mathcal{B}$ where $\alpha/\mathcal{B}(i) = j$ if and only if $\alpha(B_i) = B_j$. Clearly π_1 is a homomorphism. Let $H/\mathcal{B} = \mathrm{Im}(\pi_1)$. If $G = \mathrm{Aut}(\Gamma)$, for some vertex-transitive graph Γ, define a graph $\Gamma/\mathcal{B}$ with vertex set $V(\Gamma/\mathcal{B}) = \mathbb{Z}_m$ and edge set

$$E(\Gamma/\mathcal{B}) = \{(i, j) : \text{some vertex of } B_i \text{ is adjacent to some vertex of } B_j, i \neq j\}.$$

We observe that $H/\mathcal{B} \leq \mathrm{Aut}(\Gamma/\mathcal{B})$. In cases where no confusion will arise, we write α/k, H/k, *etc*., instead of $\alpha/\mathcal{B}$, $H/\mathcal{B}$, *etc*.

Let H be a transitive group on V and $K \lhd H$, $K \neq 1$ such that K is not transitive. Then H admits a complete block system $\mathcal{B}$ of m blocks each of size k, where the blocks are formed by the orbits of K. Let $B \in \mathcal{B}$. Define a map $\pi_2 : K|_B \longrightarrow S^k$ by $\pi_2(\alpha) = \alpha|_B$. Further, π_2 is also a homomorphism, and $K|_B \cong J \leq S^k$. If $\mathrm{Stab}_K(v_0^0) \neq 1$ and $\mathrm{Ker}(\pi_2) = 1$ we define an equivalence relation $\equiv$ on V by $v_j^i \equiv v_b^a$ if and only if $\mathrm{Stab}_K(v_j^i) = \mathrm{Stab}_K(v_b^a)$. We denote the equivalence classes of $\equiv$ by $E_0, E_1, \ldots, E_{m-1}$. One can easily show that each E_i is a block of H.

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Let X and Y be vertex-transitive hypergraphs. Let

$$A = \left\{ \left((x_1, y_1), (x_2, y_2), \ldots, (x_r, y_r) \right) : (x_1, \ldots, x_r) \in E(X), y_i \in V(Y) \right\},$$
$$B = \left\{ \left((x, y_1), (x, y_2), \ldots, (x, y_s) \right) : (y_1, y_2, \ldots, y_s) \in E(Y), x \in V(X) \right\}.$$

Define the *wreath (or lexicographic) product* of X and Y to be the hypergraph $X \wr Y$ such that $V(X \wr Y) = V(X) \times V(Y)$ and $E(X \wr Y) = A \cup B$. We observe that the wreath product of a circulant hypergraph of order m and a circulant hypergraph of order n is isomorphic to a circulant hypergraph.

2. **Characterization of metacirculant Cayley graphs.** Throughout this paper, we will make a clear distinction between "Cayley graph of G" and "isomorphic to a Cayley graph of G". As we will always be working with metacirculant graphs, a circulant graph with always be a graph Γ such that $\langle \rho, \tau \rangle \leq \mathrm{Aut}(\Gamma)$, where $\alpha = 1$. In some sense, this is a departure from normal practice, although given the content of this paper, necessary. This can also lead to some behavior which the reader may not have considered, as in many cases the distinction being made here is not necessary or even useful. For example, it is quite possible for the wreath product of two circulant graphs of order q and p to *not* be a circulant graph. For example, the metacirculant graph $\Gamma = \Gamma(3, 7, 2, \{1, 6\}, \mathbb{Z}_7)$ is the wreath product of a circulant graph of order 3 and a circulant graph of order 7, and is thus certainly isomorphic to a circulant graph. However, $\mathrm{Aut}(\Gamma)$ does not contain an appropriate τ for this graph to be a circulant graph (for any choice of τ, $\alpha = 2$ or 6).

We first give necessary and sufficient conditions for a (q, p)-metacirculant combinatorial object to be a Cayley object when $p^2 \nmid |\mathrm{Aut}(X)|$. Let $V_j = \{v_j^i : i \in \mathbb{Z}_q\}$ and $V^i = \{v_j^i : j \in \mathbb{Z}_p\}$.

THEOREM 5. *Let $X = X(q, p, \alpha)$ be a metacirculant combinatorial object, $p > q$, such that $p^2 \nmid |\mathrm{Aut}(X)|$. Then X is a Cayley object if and only if $X = X(q, p, \alpha')$ where $|\alpha'| = 1$ or $|\alpha'| = q$. Further, if $q^2 \mid |\alpha|$, then X is a Cayley object if and only if X is circulant.*

PROOF. By Theorem 1 it suffices to show necessity. Let $X = X(q, p, \alpha)$ satisfy the hypothesis and suppose that X is a Cayley object. As X is a Cayley object, $\mathrm{Aut}(X)$ contains the left translations of some group of order pq, and hence contains a regular subgroup, say G. As the Sylow p-subgroups of $\mathrm{Aut}(X)$ have order p, by conjugating G, if necessary, we may assume without loss of generality that $\langle \rho \rangle \leq G$. Further, $\langle \rho \rangle$ is also a Sylow p-subgroup of G and, as $|G| = pq$, $\langle \rho \rangle \lhd G$, and certainly $\langle \rho \rangle$ is not transitive. Hence G admits a complete block system of q blocks each of size p, where the blocks are formed by the orbits of $\langle \rho \rangle$. Note that we may assume that $G = \langle \rho, \tau_1 \rangle$ where $|\tau_1| = q$ as up to isomorphism there are exactly two groups of order pq, both of which can be generated in this fashion [6]. We conclude that $G = \langle \rho, \tau_1 \rangle$, where $\tau_1(v_j^i) = v_{\alpha_1 j + a_i}^{\sigma(i)}$, $\sigma \in S^q$, $\alpha_1 \in \mathbb{Z}_p^*$, and $a_i \in \mathbb{Z}_p$.

Let $N = N_{\mathrm{Aut}(X)}(\langle \rho \rangle)$ be the normalizer in $\mathrm{Aut}(X)$ of $\langle \rho \rangle$. Clearly N admits a complete block system B of q blocks of size p, where the blocks are orbits of $\langle \rho \rangle$. Then π_1 is

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defined. Let $K = \mathrm{Ker}(\pi_1)$. Note that $\tau, \tau_1 \in N$ where $\tau(v_j^i) = v_{\alpha j}^{i+1}$. If $q^2 \nmid |\alpha|$ we are done so assume that $q^2 \mid |\alpha|$. We must show that X is circulant. As $\tau^q(v_0^0) = v_0^0$ but $\tau^q \neq 1$, $\mathrm{Stab}_K(v_0^0) \neq 1$. As $p^2 \nmid |\mathrm{Aut}(X)|$, it is not difficult to show that (see [2]) $\mathrm{Ker}(\pi_2) = 1$, and hence the equivalence classes of $E_0, \ldots, E_{p-1}$ of $\equiv$ each have cardinality q, where each E_i contains exactly one element from each orbit of ρ. As τ^q fixes only $v_0^0, v_0^1, \ldots, v_0^{q-1}$ and $\rho \in K$, we may take $E_i = V_j$. As $\rho \in K$, we have that $a_i = a_j$ for every $i, j \in \mathbb{Z}_q$. Hence $\rho^{-a_0}\tau_1(v_j^i) = v_{\alpha_1 j}^{\sigma(i)}$. Let $\tau_2 = \rho^{-a_0}\tau_1$.

Note that $\langle\tau_2\rangle/p = \langle\sigma\rangle$ and $\langle\tau\rangle/p$ are Sylow q-subgroups of N/p. Thus there exists $\beta \in N$ such that $(\beta^{-1}/p)(\langle\tau_2\rangle/p)(\beta/p) = \langle\tau\rangle/p$. Then $\beta^{-1}\tau_2\beta \in N$ and $\beta^{-1}\tau_2\beta(V^i) = V^{i+w}$, for some $w \in \mathbb{Z}_q$. We conclude that $\beta^{-1}\tau_2\beta(v_j^i) = v_{\alpha_2 j + b_i}^{i+w}$, for some α_2 and $b_i \in \mathbb{Z}_p$. Further, by an argument analogous to an argument above, we have that $b_i = b_j$ for every $i, j \in \mathbb{Z}_q$. Hence we may assume $\beta^{-1}\tau_2\beta(v_j^i) = v_{\alpha_2 j}^{i+w}$. Let $t \in \mathbb{Z}_q$ such that $tw \equiv 1 \pmod q$. Then $\beta^{-1}\tau_2^t\beta(v_j^i) = v_{\alpha_2^t j}^{i+1}$. Let $\tau' = \beta^{-1}\tau_2^t\beta$ and $\alpha' = \alpha_2^t$. Then $\tau'(v_j^i) = v_{\alpha' j}^{i+1}$ and, as $|\tau_1| = q$ and $\gcd(t, q) = 1$, $|\tau'| = q$. Hence $|\alpha'| = 1$ or $|\alpha'| = q$. If $\alpha' = 1$, then X is circulant as required. If $|\alpha'| = q$, then, as $\mathbb{Z}_p^*$ is cyclic and $\tau \in \mathrm{Aut}(X)$, the function $\gamma : V \longrightarrow V$ by $\gamma(v_j^i) = v_{\alpha' j}^i$ is contained in $\mathrm{Aut}(X)$. Hence $\gamma^{-1}\tau' \in \mathrm{Aut}(X)$, and $\gamma^{-1}\tau'(v_j^i) = v_j^{i+1}$. We conclude that X is circulant. ∎

COROLLARY 6. *Let $\Gamma = \Gamma(q, p, \alpha, S_0, \ldots, S_\mu)$ be a Cayley graph, where p and q are primes satisfying $p > q$. Suppose that $q^2 \mid |\alpha|$ and for some r, $0 < r \leq \mu$, $0 < |S_r| < p$. Then Γ is a Cayley graph if and only if Γ is circulant.*

PROOF. If the Sylow p-subgroups of $\mathrm{Aut}(\Gamma)$ have order p, then the result follows from Theorem 5. If the Sylow p-subgroups of $\mathrm{Aut}(\Gamma)$ have order greater than p, then by [2], they have order p^q and $\Gamma \cong \Gamma_1 \wr \Gamma_2$, where Γ_1 is an order q-circulant and Γ_2 an order p-circulant. Then $|S_r| = 0$ or p for every $0 < r \leq \mu$ and the result follows. ∎

We illustrate this corollary with an example.

EXAMPLE 7. *The Petersen graph is not Cayley.*

PROOF. By [2], the Petersen graph is a $(2, 5, 2, \{1, 4\}, \{0\})$ metacirculant graph. Clearly, the Petersen graph satisfies the hypothesis of Corollary 6, and so is Cayley if and only if it is circulant. By inspection, the Petersen graph is not circulant with this labeling, and is thus not Cayley. ∎

3. **Isomorphisms of metacirulant Cayley graphs.** Let X be some Cayley object for some group G. We shall say that X is a *CI-object* if given any Cayley object Y of G such that X is isomorphic to Y, then X and Y are isomorphic by some $\alpha \in \mathrm{Aut}(G)$. Babai characterized this property in the following way:

LEMMA 8 (BABAI, [3]). *For a Cayley object X of G the following are equivalent.*

(i) X is a CI-object,

(ii) given a permutation $\phi \in S^G$ such that $\phi^{-1}G_L\phi \leq \mathrm{Aut}(X)$, G_L and $\phi^{-1}G_L\phi$ are conjugate in $\mathrm{Aut}(X)$.

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Let $p > q$, $q \mid p-1$, and $\alpha \in \mathbb{Z}_p^*$ such that $|\alpha| = q$. Then up to isomorphism, $\langle \rho, \tau \rangle$ is one of two groups of order pq, the other being $\mathbb{Z}_{pq}$.

THEOREM 9. *Let X be a Cayley object of G with $|G| = pq$, $p > q$ and q divides $p-1$, such that $p^2 \nmid |\mathrm{Aut}(X)|$. If $G = \mathbb{Z}_{pq}$ then X is a CI-object for G. If $G = \langle \rho, \tau \rangle$, for some α as above, then either X is a CI-object for G or X is also a Cayley object of $\mathbb{Z}_{pq}$.*

PROOF. If $G = \mathbb{Z}_{pq}$, then the result follows from the proof of Theorem 1, Case 1 [1].

If $G = \langle \rho, \tau \rangle$, for some α with $|\alpha| = q$, we will show that if $\phi \in S^{pq}$ such that $\phi^{-1}\langle \rho, \tau \rangle \phi \leq \mathrm{Aut}(X)$, then either $\phi^{-1}\langle \rho, \tau \rangle \phi$ and $\langle \rho, \tau \rangle$ are conjugate in $\mathrm{Aut}(X)$, or that X is also a Cayley object for $\mathbb{Z}_{pq}$. For brevity, let $\phi_1 = \phi^{-1}\tau\phi$. By hypothesis, $\langle \rho \rangle$ and $\langle \phi^{-1}\rho\phi \rangle$ are Sylow p-subgroups of $\mathrm{Aut}(X)$ and are thus conjugate. Let $\delta_1 \in \mathrm{Aut}(X)$ such that $\delta_1^{-1}\langle \phi^{-1}\rho\phi \rangle \delta_1 = \langle \rho \rangle$. Thus $\delta_1^{-1}\langle \phi^{-1}\rho\phi, \phi_1 \rangle \delta_1 = \langle \rho, \delta_1^{-1}\phi_1\delta_1 \rangle$. Let $\phi_2 = \delta_1^{-1}\phi_1\delta_1$. Clearly $\langle \rho \rangle \lhd \langle \rho, \phi_2 \rangle$ and so $\langle \rho, \tau, \phi_2 \rangle$ admits a complete block system of q blocks each of size p. Thus the map $\pi_1 : \langle \rho, \tau, \phi_2 \rangle \longrightarrow S^q$ is well defined. Hence $\langle \tau \rangle / p$ and $\langle \phi_2 \rangle / p$ are Sylow q-subgroups of $\langle \rho, \tau, \phi_2 \rangle / p$. Let $\delta_2 \in \langle \rho, \tau, \phi_2 \rangle$ such that $(\delta_2^{-1}/p)(\langle \phi_2 \rangle / p)(\delta_2 / p) = \langle \tau \rangle / p$. Let $\phi_3 = \delta_2^{-1}\phi_2\delta_2$. Let Q be a Sylow q-subgroup of $\langle \rho, \tau, \phi_3 \rangle$. If $|Q| = q$ then $\langle \tau \rangle$ and $\langle \phi_3 \rangle$ are Sylow q-subgroups of $\langle \rho, \tau, \phi_3 \rangle$ and so there exists $\delta_3 \in \langle \rho, \tau, \phi_3 \rangle$ such that $\delta_3^{-1}\langle \phi_3 \rangle \delta_3 = \langle \tau \rangle$. Let $\delta = \delta_3\delta_2\delta_1$. Then $(\delta\phi)^{-1}\langle \rho, \tau \rangle \delta\phi = \langle \rho, \tau \rangle$. Thus by Lemma 8, if $|Q| = q$, then X is a CI-object for $\langle \rho, \tau \rangle$.

If $|Q| = q^a$, $a > 1$, we will show that X is a Cayley object for $\mathbb{Z}_{pq}$. As $\langle \phi_3 \rangle / p = \langle \tau \rangle / p$, there exist $\beta \in \mathrm{Ker}(\pi_1)$ such that $|\beta| = q^b$, $b \geq 1$. Without loss of generality we may assume that $b = 1$. As $\langle \rho \rangle \lhd \langle \rho, \tau, \phi_2 \rangle$, $\beta(v_j^i) = v^i_{\alpha_i j + b_i}$ where $\alpha_i \in \mathbb{Z}_p^*$ and $b_i \in \mathbb{Z}_p$. As $p^2 \nmid |\mathrm{Aut}(X)|$, $\alpha_i = \alpha_0$ for all i, and as $|\beta| = q$, we must have $|\alpha_0| = q$. Now, $\mathbb{Z}_p^*$ is cyclic of order $p-1$ so there exists $r \in \mathbb{Z}_p^*$ such that $\alpha_0^r = \alpha^{-1}$. Hence $\beta^r\tau \in \langle \rho, \tau, \phi_2 \rangle$ and $\beta^r\tau(v_j^i) = v^{i+1}_{j+c_i}$, where each $c_i \in \mathbb{Z}_p$.

Let $K = \mathrm{Ker}(\pi_1)$. As the Sylow p-subgroups of $\mathrm{Aut}(X)$ have order p, by a previous argument $\mathrm{Ker}(\pi_2) = 1$. As $\beta \in K$, $\mathrm{Stab}_K(v_0^0) \neq 1$ and, again by a previous argument, we have that the equivalence classes $E_0, E_1, \ldots, E_{p-1}$ of $\equiv$ have cardinality q. We must then have that $\langle \rho, \tau, \phi_2 \rangle$ admits a complete block system of p blocks each of size q, where the blocks are the equivalence classes $E_0, E_1, \ldots, E_{p-1}$. As $\tau \in \langle \rho, \tau, \phi_2 \rangle$, $E_i = V^j$ for some j. We conclude that $c_i = c_j$ for all i, j. Hence X is a Cayley object for $\mathbb{Z}_{pq}$. ∎

COROLLARY 10. *Let $X = X(p, q, \alpha)$ and $X' = X'(p, q, \alpha')$ be metacirculant combinatorial objects such that X and X' are Cayley objects and $p^2 \nmid |\mathrm{Aut}(X)|$. Then X is isomorphic to X' if and only if*

(i) if X is circulant then there exists $\delta \in \mathrm{Aut}(\mathbb{Z}_{pq})$ such that $\delta(X) = X'$.

(ii) if X is not circulant then there exists $\delta \in \mathrm{Aut}(\langle \rho, \tau \rangle)$ and $\gamma : V \longrightarrow V$, where $\gamma(v_j^i) = v_j^{ri}$, $r \in \mathbb{Z}_q^*$ and $\gamma\delta(X) = X'$.

Further if X and X' are isomorphic, then X is circulant if and only if X' is circulant.

PROOF. (i) In view of Theorem 9, if $a = |\alpha|$ and $a' = |\alpha'|$, we may assume without loss of generality that $a = 1$ and $a' = 1$ or $a' = q$. If $a' = 1$, then X and X' are circulant and the result follows from Theorem 9. Hence we assume that $a' = q$, *i.e.*, that X is circulant

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and X' is not necessarily circulant. We will show that if X and X' are isomorphic, then X' is circulant implying (i) and by symmetry, that X is circulant if and only if X' is circulant.

Assume $X \cong X'$. Then $\mathrm{Aut}(X) \cong \mathrm{Aut}(X')$ and so $\mathrm{Aut}(X')$ contains a pq-cycle ω_0. As $p^2 \not| \, |\mathrm{Aut}(X)|$, $\langle \omega_0^q \rangle$ is a Sylow p-subgroup of $\mathrm{Aut}(X')$ and $N_{\mathrm{Aut}(X')}(\langle \omega_0^q \rangle)$ contains the pq cycle ω_0. Further $\langle \rho \rangle$ is also a Sylow p-subgroup of $\mathrm{Aut}(X')$ and so there exist $\beta_0 \in \mathrm{Aut}(X')$ such that $\beta_0^{-1}\langle \omega_0^q \rangle \beta_0 = \langle \rho \rangle$ and $\beta_0^{-1} N_{\mathrm{Aut}(X')}(\langle \omega_0^q \rangle)\beta_0^{-1} = N_{\mathrm{Aut}(X')}(\langle \rho \rangle)$. Let $\omega_1 = \beta_0^{-1}\omega_0\beta_0$ and $R = N_{\mathrm{Aut}(X')}(\langle \rho \rangle)$. Then $\langle \omega_1 \rangle \leq R$, $\langle \omega_1^q \rangle = \langle \rho \rangle$, and $\langle \omega_1 \rangle$ is cyclic of order pq. As $\langle \rho \rangle \lhd R$, R admits a complete block system of q blocks each of size p, where the blocks are formed by the orbits of $\langle \rho \rangle$. Hence the map π_1 is well defined, and $\langle \tau \rangle / p$ and $\langle \omega_1^p \rangle / p$ are Sylow q-subgroups of R/p. Thus there exist $\beta_1 \in R$ such that $(\beta_1^{-1}/p)(\langle \omega_1^p \rangle / p)(\beta_1/p) = \langle \tau \rangle / p$. Let $\omega = \beta_1^{-1}\omega_1\beta_1$. As $\langle \omega^p \rangle / p = \langle \tau \rangle / p$, $\omega(V_i) = V_{i+w}$ for some $w \in \mathbb{Z}_q$. As $\langle \rho \rangle \lhd \langle \omega \rangle$, $\omega(v_j^i) = v_{\phi j + b_i}^{i+w}$, where $\phi \in \mathbb{Z}_p^*$, and $b_i \in \mathbb{Z}_p$. Trivially, either the Sylow q-subgroups of R are either of order q or of order q^i, $i > 1$. In either case, the result follows with arguments analogous to those in Theorem 5.

(ii) If $p^2 \not| \, |\mathrm{Aut}(X)|$ and X is not circulant, then X is a Cayley graph for $\langle \rho, \tau \rangle$. Further, there exists $r \in \mathbb{Z}_q$ such that $\gamma^{-1}(X')$ is also a Cayley graph for $\langle \rho, \tau \rangle$. Hence by Theorem 9, X and $\gamma^{-1}(X')$ are isomorphic if and only if there exists $\delta \in \mathrm{Aut}(\langle \rho, \tau \rangle)$ such that $\delta(X) = \gamma^{-1}(X')$. Hence X and X' are isomorphic if and only if $\gamma\delta(X) = X'$. ∎

We now investigate the case where Γ can be written as the wreath product of an order q-circulant over an order p-circulant.

LEMMA 11. *Let $\Gamma = \Gamma(q, p, 1, S_0, \ldots, S_\mu)$ be a metacirculant graph with $\Gamma = \Gamma_1 \wr \Gamma_2$, where Γ_1 is an order q circulant graph and Γ_2 an order p circulant graph. Let $\Gamma' = \Gamma'(q, p, \alpha, S_0', \ldots, S_\mu')$ be a metacirculant graph such that $\Gamma \cong \Gamma'$ but Γ' is not circulant. Then $\Gamma = \Gamma(q, p, \alpha^q, S_0, \ldots, S_\mu)$.*

PROOF. Let $\delta \in S^V$ such that $\delta(\Gamma) = \Gamma'$. As $\Gamma = \Gamma_1 \wr \Gamma_2$, $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma')$ admit a complete block system B of q blocks of size p, formed by the orbits of $\langle \rho \rangle$. Also, as $\Gamma = \Gamma_1 \wr \Gamma_2$, $\langle \rho|_B : B \in B \rangle$ is a Sylow p-subgroup of $\mathrm{Aut}(\Gamma)$ and $\mathrm{Aut}(\Gamma')$. Hence $\delta^{-1}\langle \rho|_B : B \in B \rangle \delta$ is also a Sylow p-subgroup of $\mathrm{Aut}(\Gamma')$, so there exists $\omega \in \mathrm{Aut}(\Gamma')$ such that $\omega^{-1}\delta^{-1}\langle \rho|_B : B \in B \rangle \delta\omega = \langle \rho|_B : B \in B \rangle$. Replacing δ by $\delta\omega$, we assume that $\delta^{-1}\langle \rho|_B : B \in B \rangle \delta = \langle \rho|_B : B \in B \rangle$. Define $\pi_1 : \mathrm{Aut}(\Gamma') \longrightarrow S^q$ by $\pi_1(\gamma) = \gamma/B$. Then $\langle \pi_1(\tau) \rangle$ and $\langle \pi_1(\delta^{-1}\tau\delta) \rangle$ are Sylow q-subgroups of $\mathrm{Im}(\pi_1)$ so there exists $\omega \in \mathrm{Aut}(\Gamma')$ such that $\langle \pi_1(\omega^{-1}\delta^{-1}\tau\delta\omega) \rangle = \langle \pi_1(\tau) \rangle$. As before, by replacing $\delta\omega$ by δ, we assume that $\langle \pi_1(\delta^{-1}\tau\delta) \rangle = \langle \pi_1(\tau) \rangle$. Hence $\delta(v_j^i) = v_{\beta_i j + c_i}^{ri+b}$, $r \in \mathbb{Z}_q^*$, $b \in \mathbb{Z}_q$, $\beta_i \in \mathbb{Z}_p^*$, and $c_i \in \mathbb{Z}_p$. As $\rho \in \mathrm{Aut}(\Gamma) \cap \mathrm{Aut}(\Gamma')$, we assume without loss of generality that $c_0 = 0$. As Γ and Γ' are both metacirculant, we also assume that $b = 0$.

Now, $\tau^q \in \mathrm{Ker}(\pi_1)$ and $\delta\tau^q\delta^{-1} \in \mathrm{Aut}(\Gamma)$. Furthermore,

$$\delta\tau^q\delta^{-1}(v_j^0) = v_{\alpha^q j}^0.$$

As $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, the function $\iota : V \longrightarrow V$ by $\iota(v_j^i) = v_{\alpha^q j}^i$ is in $\mathrm{Aut}(\Gamma)$ so that $\Gamma = \Gamma(q, p, \alpha^q, S_0, \ldots, S_\mu)$ as required. ∎

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COROLLARY 12. *Let* $\Gamma = \Gamma(p, q, \alpha, S_0, \ldots, S_\mu)$ *and* $\Gamma' = \Gamma'(p, q, \alpha', S'_0, \ldots, S'_\mu)$ *be metacirculant graphs that are Cayley graphs. Then* Γ *is isomorphic to* Γ' *if and only if*

(i) if Γ *and* Γ' *are circulant then there exists* $\delta \in Aut(\mathbb{Z}_{pq})$ *such that* $\delta(\Gamma) = \Gamma'$,

(ii) if Γ *is not circulant and* $\Gamma \ncong \Gamma_1 \wr \Gamma_2$, Γ_1 *an order q-circulant and* Γ_2 *an order p-circulant, then there exists* $\delta \in \mathrm{Aut}(\langle \rho, \tau \rangle)$ *and* $\gamma: V \longrightarrow V$ *where* $\gamma(v_j^i) = v_j^{ri}$, $r \in \mathbb{Z}_q^*$, *and* $\gamma\delta(\Gamma) = \Gamma'$.

(iii) If Γ *is not circulant and* $\Gamma \cong \Gamma_1 \wr \Gamma_2$, *then define* $\gamma_1, \gamma_2: V \longrightarrow V$ *by* $\gamma_1(v_j^i) = v^i_{\alpha^i j}$ *and* $\Gamma_2(v_j^i) = v^i_{(\alpha^r)^i j}$. *Then* $\gamma_2\delta\gamma_1(\Gamma) = \Gamma'$, *for some* $\delta \in \mathrm{Aut}(\Gamma)$.

PROOF. (i) and (ii) follow from [2] and Corollary 12. (iii) If $\Gamma \cong \Gamma_1 \wr \Gamma_2$ and Γ is not circulant, then $\Gamma \cong \Gamma^*$, where Γ^* is a circulant graph. By Lemma 11, if $\iota: V \longrightarrow V$ by $\iota(v_j^i) = v^i_{\alpha^q j}$, then $\iota \in \mathrm{Aut}(\Gamma)$. As $\mathrm{Aut}(\Gamma^*) = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, the function $\tau^*: V \longrightarrow V$ by $\tau^*(v_j^i) = v^{i+1}_{\delta_i(j)}$, where $\delta_{q-1}(j) = \alpha^q j$ and $\delta_i(j) = j$, $0 \leq i \leq q-2$ is also contained in $\mathrm{Aut}(\Gamma^*)$. Then $\gamma_1\tau^*\gamma_1^{-1}(v_j^i) = v^{i+1}_{\alpha j}$ if $i \neq q-1$ and $\gamma_1\tau^*\gamma_1^{-1}(v_j^{q-1}) = v^0_{\alpha j}$. Clearly $\langle \rho \rangle \leq \gamma_1 \mathrm{Aut}(\Gamma^*)\gamma_1^{-1}$ so that $\gamma_1^{-1}(\Gamma^*)$ is a (q, p, α)-metacirculant graph. Thus $\gamma_1(\Gamma)$ is a circulant graph. Analogous arguments will show that $\gamma_2(\Gamma')$ is a circulant graph, in which case the result follows from [1]. ∎

4. **Isomorphisms of non-Cayley metacirculant graphs.** Let X and X' be (q, p)-metacirculant combinatorial objects that are not Cayley objects. Initially, determining necessary and sufficient conditions for X and X' to be isomorphic is hampered by no result corresponding to Babai's characterization of the CI-property for Cayley objects (Lemma 8). Sabidussi proved [9] that some 'multiple' $n\Gamma$ of Γ is a Cayley graph. We first generalize Sabidussi's result to vertex-transitive hypergraphs, and then use Babai's characterization of the CI-property for Cayley objects to characterize an analogous isomorphism result for non-Cayley hypergraphs.

LEMMA 13. *A combinatorial object X is isomorphic to a Cayley object of G if and only if* Aut(*X*) *contains a regular subgroup isomorphic to G.*

PROOF. If X is a Cayley object then G_L is a regular subgroup for some group G. If $\mathrm{Aut}(X)$ contains a regular subgroup S, then by Schur's method [10] we may relabel $V(X)$ with elements of S so that $S_L \leq \mathrm{Aut}(X)$. Hence X is a Cayley object. ∎

If there exists a vertex-transitive hypergraph Y and an integer $n > 1$ such that $X \cong Y \wr E^n$ then we say that X is *reducible*. Otherwise, X will be said to be *irreducible*.

LEMMA 14. *Let X be a reducible vertex-transitive hypergraph, Y an irreducible vertex-transitive hypergraph and n an integer such that* $X = Y \wr E^n$. *Then* $\mathrm{Aut}(X) = \mathrm{Aut}(Y) \wr S^n$, *and the orbits of* $1 \wr S^n$ *form a complete block system for* $\mathrm{Aut}(X)$.

PROOF. Clearly $\mathrm{Aut}(Y) \wr S^n \leq \mathrm{Aut}(X)$, and the orbits of $1 \wr S^n$ form a complete block system for $\mathrm{Aut}(Y) \wr S^n$. Denote the blocks of size n by $B_0, B_1, \ldots, B_k$. We note that it suffices to show that $B_0, \ldots, B_k$ are blocks of $\mathrm{Aut}(X)$. Assume not. Then there exists $\alpha \in \mathrm{Aut}(X)$ such that $\alpha(B_i) \cap B_i \neq \emptyset$ and $\alpha(B_i) \neq B_i$, for some $0 \leq i \leq k$. Then there exists $x \in B_i$ and $y \notin B_i$ such that $\alpha(x) \in B_i$ and $x' \in B_i$ such that $\alpha(x') = y$. If

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there exists an edge $(x_1, \ldots, x_r) \in E(X)$ such that $\alpha(x) = x_a$, $y = x_b$ for some a, b then $\alpha^{-1}(x_1, \ldots, x_r) \in E(X)$, contradicting the fact that $X = Y \wr E^n$.

Hence we assume that no such edge exists. Let $s = \max\{r : (y_1, y_2, \ldots, y_r) \in E(Y)\}$. Define $\tilde{Y}$ to be the hypergraph with vertex set $V(Y) = V(\tilde{Y})$ and

$$E(\tilde{Y}) = \big\{(y_1, \ldots, y_r) : 2 \leq r \leq s \text{ and } (y_1, \ldots, y_r) \notin E(Y)\big\}.$$

Let $\tilde{X} = \tilde{Y} \wr E^n$. We will show that $\mathrm{Aut}(X) = \mathrm{Aut}(\tilde{X})$. Note that this will imply the result as there exists $e = (x_1, x_2, \ldots, x_r) \in E(\tilde{X})$ such that $x_a = \alpha(x)$ and $x_b = y$.

Let $\beta \in \mathrm{Aut}(\tilde{X})$. If $e = (x_1, x_2, \ldots, x_r) \in E(X)$, then $\beta(e) \notin E(\tilde{X})$. Hence $\beta(e) \in E(X)$. Thus $\beta \in \mathrm{Aut}(X)$ and so $\mathrm{Aut}(\tilde{X}) \leq \mathrm{Aut}(X)$. Conversely, let $\beta \in \mathrm{Aut}(X)$ and e be an edge of $\tilde{X}$. Then $\beta(e) \notin E(X)$ so $\beta(e) \in E(\tilde{X})$. Thus $\mathrm{Aut}(X) = \mathrm{Aut}(\tilde{X})$. ∎

If X, Y, and n satisfy the hypothesis of Lemma 14, then Y will be denoted by X_* and $B_0, \ldots, B_k$ will be denoted by x_*, y_*, *etc*., where $x \in B_i$, $y \in B_j$, *etc*. Observe that Lemma 14 implies that if X and X' are isomorphic vertex-transitive hypergraphs and $\delta : X \longrightarrow X'$ is an isomorphism, then $\delta_* : X_* \longrightarrow X'_*$ is an isomorphism where $\delta_*(x_*) = y_*$ if and only if $\delta(x) \in y_*$. Further $n\delta : V(X \wr E^n) \longrightarrow V(X' \wr E^n)$ where $n\delta\big((x, a)\big) = \big(\delta(x), a\big)$ is also an isomorphism. Finally, if X and X' are irreducible, then $(n\delta)_* = \delta$.

THEOREM 15. *Let X be an irreducible vertex-transitive hypergraph, and $G \leq \mathrm{Aut}(X)$ be transitive. Let $n = |\mathrm{Stab}_G(x)|$, $x \in V(G)$. Then $X \wr E^n$ is isomorphic to a Cayley hypergraph of G.*

PROOF. By Lemma 13 it suffices to show that $\mathrm{Aut}(X \wr E^n)$ contains a regular subgroup isomorphic to G. Clearly $\mathrm{Aut}(X) \wr S^n \leq \mathrm{Aut}(X \wr E_n)$. We will show that $G \wr S^n$ contains a regular subgroup isomorphic to G. For the moment, assume that X is a graph. Then by Theorems 4 and 7 of [9], the result is true and $\mathrm{Aut}(X \wr E^n) = \mathrm{Aut}(X) \wr S^n$. Observe that $\mathrm{Aut}(X) \wr S^n$ admits a complete block system of $|V(X)|$ blocks of size n, where the blocks are formed by the orbits of $1 \wr S^n$, and hence the map $\pi_1 : \mathrm{Aut}(X) \wr S^n \longrightarrow \mathrm{Aut}(X)$ is surjective. By Theorem 2 of [9], we may label $V(X \wr E^n)$ with elements of G so that the blocks of size n are the left cosets in G of $\mathrm{Stab}_G(x)$, for fixed $x \in V(X)$, and the vertices of X may be labeled with left cosets in G of $\mathrm{Stab}_G(x)$. With this labeling, $\pi_1(g_L) = g$ for all $g \in G$. Hence $G_L \leq \pi_1^{-1}(G) = G \wr S^n$. Now, let X be an arbitrary irreducible vertex-transitive hypergraph and $n = |\mathrm{Stab}_G(x)|$, for some $x \in V(X)$. By Lemma 14, $\mathrm{Aut}(X \wr E^n) = \mathrm{Aut}(X) \wr S^n \geq G \wr S^n$. As $\mathrm{Aut}(K^r) = S^r$, the result follows. ∎

Let X be a vertex-transitive hypergraph, and G a transitive subgroup of $\mathrm{Aut}(X)$. Let $n = |\mathrm{Stab}_G(x)|$, $x \in V(X)$. Then $X \wr E^n$ is a Cayley hypergraph of G. We will refer to X as an n-Cayley hypergraph of G. Assume that X is irreducible, and that if X' is another n-Cayley hypergraph of G then X and X' are isomorphic by α_*, $\alpha \in \mathrm{Aut}(G)$. We then say that X is an (n, G)-*CI-hypergraph*. If whenever $G \leq \mathrm{Aut}(X)$ then X is an (n, G)-CI-hypergraph, we say that G is an (n, G)-*CI-group with respect to hypergraphs*. Of course, if $n = 1$, then X is isomorphic to a Cayley hypergraph and we revert to our earlier notation.

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THEOREM 16. *The following are equivalent:*

(i) X is an (n, G)-CI-hypergraph,

(ii) given a permutation $\phi \in S^{G_*}$, whenever $\phi^{-1}G\phi \leq \mathrm{Aut}(X)$, then $\phi^{-1}G\phi$ and G are conjugate in $\mathrm{Aut}(X)$.

PROOF. Let X and X' be irreducible vertex-transitive hypergraphs such that $X \cong X'$ and X and X' are n-Cayley hypergraphs of G. As X and X' are irreducible, by Lemma 14, if n is any integer then $\mathrm{Aut}(X \wr E^n) = \mathrm{Aut}(X) \wr S^n$. Let $n = |\mathrm{Stab}_G(x_0)|$, $x_0 \in V(X) = V(X')$. By Theorem 15, $X \wr E^n$ and $X' \wr E^n$ are both Cayley hypergraphs for G and so if $X \wr E^n$ and $X' \wr E^n$ are isomorphic, then by Lemma 8 they are isomorphic by $\alpha \in \mathrm{Aut}(G)$ if and only if whenever $\delta^{-1}G\delta \leq \mathrm{Aut}(X \wr E^n)$ then $\delta^{-1}G\delta$ and G are conjugate in $\mathrm{Aut}(X \wr E^n)$. Observe that as X and X' are irreducible, that if $(x, a) \in V(X \wr E^n) = V(X \wr E^n)$, then $(x, a)_* = \{x\} \times N$, where N is a set of cardinality n, and that by Lemma 14, these sets together form a complete block system of $\mathrm{Aut}(X \wr E^n)$ and $\mathrm{Aut}(X' \wr E^n)$.

Assume that whenever $\delta^{-1}G\delta \leq \mathrm{Aut}(X \wr E^n)$ then $\delta^{-1}G\delta$ and G are conjugate in $\mathrm{Aut}(X \wr E^n)$. Let $\delta' \in S^V$ such that $(\delta')^{-1}G\delta' \leq \mathrm{Aut}(X)$. Then $\left(n(\delta')^{-1}\right)G(n\delta') \leq \mathrm{Aut}(X \wr E^n)$ and so $\left(n(\delta')^{-1}\right)G(n\delta')$ and G are conjugate in $\mathrm{Aut}(X \wr E^n)$. Then $(n\delta')_* = \delta'$. We conclude that $(\delta')^{-1}G\delta'$ and G are conjugate in $\mathrm{Aut}(X)$.

Now assume that whenever $\delta^{-1}G\delta \leq \mathrm{Aut}(X)$ then $\delta^{-1}G\delta$ and G are conjugate in $\mathrm{Aut}(X)$. Let $\delta' \in S^G$ such that $(\delta')^{-1}G\delta' \leq \mathrm{Aut}(X \wr E^n)$. By reversing the argument above we conclude that $(\delta')^{-1}G\delta'$ is conjugate to $\beta^{-1}G\beta$ where $\beta_* = 1$. Hence $\beta^{-1} \in 1 \wr S^n$, $\beta \in \mathrm{Aut}(X \wr E^n)$ and so $\beta\beta^{-1}G\beta\beta^{-1} = G$. We conclude that $(\delta')^{-1}G\delta'$ is conjugate to G in $\mathrm{Aut}(X \wr E^n)$. Thus if $\phi \in S^{G_*}$ such that $\phi^{-1}G\phi \leq \mathrm{Aut}(X)$, then $\phi^{-1}G\phi$ and G are conjugate if and only if whenever $\phi' \in S^G$ and $(\phi')^{-1}G\phi' \leq \mathrm{Aut}(X \wr E^n)$, then $(\phi')^{-1}G\phi'$ and G are conjugate in $\mathrm{Aut}(X \wr E^n)$. Thus the result follows by Lemma 8. ∎

Define the *deviation*, $\mathrm{dev}(X)$, of a vertex-transitive hypergraph X, to be the smallest integer n such that $X \wr E^n$ is a Cayley hypergraph.

For the time being, we restrict our attention to (q, p)-metacirculant hypergraphs X such that $q < p$ and X is not Cayley.

LEMMA 17. *If $p^2 \nmid |\mathrm{Aut}(X)|$, then $\mathrm{dev}(X) = q^{k-1}$, where q^k is the smallest power of q that divides $|\alpha|$, for any choice of α such that $\tau \in \mathrm{Aut}(X)$.*

PROOF. It suffices to show that if $H \leq \mathrm{Aut}(\Gamma)$ and H is transitive, then there exists H' such that $|H'| \leq H$ and $\langle \rho, \tau \rangle \leq |H'|$ for some choice of α. This follows with a proof similar to that of Theorem 5. ∎

We remark that if Γ is a (q, p)-metacirculant graph that is not Cayley, then $p^2 \nmid |\mathrm{Aut}(\Gamma)|$.

THEOREM 18. *Let $X = X(q, p, \alpha)$ be an irreducible metacirculant hypergraph that is not a Cayley hypergraph such that $p^2 \nmid |\mathrm{Aut}(X)|$. Assume without loss of generality that $|\alpha| = q^k$. Then X is an $(q^{k-1}, \langle \rho, \tau \rangle)$-CI-hypergraph.*

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PROOF. We will show that whenever $\beta^{-1}\langle\rho,\tau\rangle\beta \leq \mathrm{Aut}(X)$ then $\beta^{-1}\langle\rho,\tau\rangle\beta$ and $\langle\rho,\tau\rangle$ are conjugate in $\mathrm{Aut}(X)$. Let $\beta \in S^V$ such that $\beta^{-1}\langle\rho,\tau\rangle\beta \leq \mathrm{Aut}(X)$. By arguments similar to those in Theorem 9, there exists $\sigma \in \mathrm{Aut}(X)$ such that $(\sigma\beta)^{-1}\langle\rho,\tau\rangle(\sigma\beta) = \langle\rho,\tau'\rangle$, where $\tau'(v_j^i) = v_{\alpha' j}^{i+c}$, $c \in \mathbb{Z}_q^*$, $\alpha' \in \mathbb{Z}_p^*$ and $|\alpha'| = |\alpha|$. Let $N = N_{\mathrm{Aut}(X)}(\langle\rho\rangle)$. Clearly $\tau,\tau' \in N$. If the Sylow q-subgroups of N have cardinality $q^{\mathrm{dev}(X)+1}$ then $\langle\tau\rangle$ and $\langle\tau'\rangle$ are Sylow q-subgroups of N and are thus conjugate in N. Hence $\beta^{-1}\langle\rho,\tau\rangle\beta$ and $\langle\rho,\tau\rangle$ are conjugate in $\mathrm{Aut}(X)$. Hence we assume that the Sylow q-subgroups of N have cardinality at least $q^{\mathrm{dev}(X)+2}$. Let $\pi_1 : N \longrightarrow S^q$. Then there exist $\psi \in \mathrm{Ker}(\pi_1)$ such that $|\psi| = q^{\mathrm{dev}(X)+1}$. Hence by an argument similar to an argument in Theorem 5, we conclude that X is circulant and so X is Cayley, a contradiction. ∎

COROLLARY 19. *Let $X = X(q,p,\alpha)$ and $X' = X'(q,p,\alpha')$ be irreducible metacirculant hypergraphs that are not Cayley hypergraphs such that p^2 does not divide $|\mathrm{Aut}(X)|$ or $|\mathrm{Aut}(X')|$. Then X and X' are isomorphic if and only if there exist $\delta \in \mathrm{Aut}(\langle\rho,\tau_1\rangle)$, where $\tau_1(v_j^i) = v_{\alpha_1 j}^{i+1}$, $|\alpha_1| = \mathrm{dev}(X) = \mathrm{dev}(X')$, and $\gamma : V \longrightarrow V$ by $\gamma(v_j^i) = v_j^{ri}$, $r \in \mathbb{Z}_q^*$, such that $\delta_*\gamma(X) = X'$.*

PROOF. It follows from arguments similar to arguments in Corollary 12 that there exists $\gamma \in S^V$, $\gamma(v_j^i) = v_j^{ri}$, $r \in \mathbb{Z}_q^*$, such that if $n = \mathrm{dev}(X)$ then $X \wr E^n$ and $X' \wr E^n$ are both Cayley hypergraphs for the group $\langle\rho,\tau_1\rangle$, where $\tau_1(v_j^i) = v_{\alpha_1 j}^{i+1}$. Hence by Theorem 18 $\gamma(X)$ and X' are isomorphic by δ_*, $\delta \in \mathrm{Aut}(\langle\rho,\tau_1\rangle)$. ∎

COROLLARY 20. *Let $\Gamma = \Gamma(q,p,\alpha,S_0,\ldots,S_\mu)$ and $\Gamma' = \Gamma'(q,p,\alpha',S_0',\ldots,S_\mu')$ be metacirculant graphs that are not Cayley graphs. Then Γ and Γ' are isomorphic if and only if there exists $\delta \in \mathrm{Aut}(\langle\rho,\tau_1\rangle)$, where $\tau_1(v_j^i) = v_{\alpha_1 j}^{i+1}$, $|\alpha_1| = \mathrm{dev}(\Gamma) = \mathrm{dev}(\Gamma')$ and $\gamma : V \longrightarrow V$ by $\gamma(v_j^i) = v_j^{ri}$, $r \in \mathbb{Z}_q^*$ such that $\delta_*\gamma(\Gamma) = \Gamma'$.*

PROOF. If $p^2 \mid |\mathrm{Aut}(\Gamma)|$, or $p^2 \mid |\mathrm{Aut}(\Gamma')|$, then Γ or Γ' is a Cayley graph. Hence the result follows from Corollary 19. ∎

Let G and G' be transitive permutation groups on Ω, $m = |\mathrm{Stab}_G(x)|$, $n = |\mathrm{Stab}_{G'}(x)|$, $x \in \Omega$. We say that G' is a *weak (n,G)-CI-group via G with respect to some class of hypergraphs* if and only if whenever X is an n-Cayley hypergraph of G' but X is not an (n,G')-CI-hypergraph, then X is isomorphic to an m-Cayley hypergraph of G and X is an (m,G)-CI-hypergraph. It follows by [1] that $\mathbb{Z}_{p^2}$ is a weak CI-group via $\mathbb{Z}_p^2$ with respect to graphs.

In [3], Babai proved that if $|G| = 2p$, then G is a CI-group with respect to graphs. It would seem natural to ask for what values of q dividing $p-1$, is $G = \langle\rho,\tau\rangle$ a (q^k,G)-CI-group, for $k \geq 0$?

THEOREM 21. *Let $q \mid p-1$, $\alpha \in \mathbb{Z}_p^*$ such that $|\alpha| = q^{k+1}$, and $\tau(v_j^i) = v_{\alpha j}^{i+1}$. Then $G = \langle\rho,\tau\rangle$ is a weak (q^k,G)-CI-group via $\mathbb{Z}_{pq}$ with respect to graphs, and is a (q^k,G)-CI-group with respect to graphs if and only if $q \leq 3$.*

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PROOF. Let Γ be a q^k-Cayley graph for G. By Theorems 9 and 18, G is a weak (q^k, G)-CI-group via $\mathbb{Z}_{pq}$ with respect to graphs. Hence we need only show that G is a (q^k, G)-CI-group with respect to graphs if and only if $q \leq 3$. As G is a weak (q^k, G)-CI-group via $\mathbb{Z}_{pq}$ with respect to graphs, we need only consider the case when Γ is also a Cayley graph for $\mathbb{Z}_{pq}$. Define $\tau_1 : V \longrightarrow V$ by $\tau_1(v_j^i) = v_j^{i+1}$. Hence $\tau_1 \in \mathrm{Aut}(\Gamma)$.

If $q > 3$, let $\alpha' \in \mathbb{Z}_p^*$ such that $|\alpha'| = q^{k+1}$. Define $\tau' : V \longrightarrow V$ by $\tau'(v_j^i) = v_{\alpha' j}^{i+1}$. Denote the orbits of $(\tau')^{-1}\tau_1$ of length q^{k+1} by $O_1, O_2, \ldots, O_s$. Let $1 \leq t \leq s$ be such that $O_t \subseteq V^1$. Let $T = O_t \cup \{v_{-j}^{-i} : v_j^i \in O_t\}$. Define a metacirculant graph Γ by $E(\Gamma) = \{v_b^a v_d^c : v_{c-d}^{a-c} \in T\}$. Then Γ is circulant, $\tau' \in \mathrm{Aut}(\Gamma)$, and, by Theorem 3.3 of [7] $\mathrm{Aut}(\Gamma) = \langle \tau_1, \rho, \tau'\tau, \iota \rangle$. Let $\tau_2 : V \longrightarrow V$ by $\tau_2(v_j^i) = v_{\alpha^2 j}^{i+1}$. Then $\tau_2 \in \mathrm{Aut}(\Gamma)$, $\langle \rho, \tau_2 \rangle \cong \langle \rho, \tau \rangle$, and it is not difficult to see that $\langle \rho, \tau_2 \rangle$ is not conjugate to $\langle \rho, \tau \rangle$ in $\mathrm{Aut}(\Gamma)$. Hence Γ is not a (q^k, G)-CI-graph and so G is not a (q^k, G)-CI-group with respect to graphs.

If $q \leq 3$, we first consider when $p^2 \not| \, |\mathrm{Aut}(\Gamma)|$. Let Γ' be a q^k-Cayley graph of $\langle \rho, \tau \rangle$ such that Γ' is isomorphic to Γ, and $\varphi : \Gamma \longrightarrow \Gamma'$ an isomorphism. By Corollary 10, Γ' is circulant and as $\mathbb{Z}_{pq}$ is a CI-group with respect to graphs, there exists $\delta \in \mathrm{Aut}(\Gamma)$ such that $\delta^{-1}\varphi^{-1}\langle \rho, \tau_1 \rangle \varphi\delta = \langle \rho, \tau_1 \rangle$, and $\varphi\delta(v_j^i) = v_{s^{-1}j}^{r^{-1}i}$ (as $\varphi\delta \in \mathrm{Aut}(\mathbb{Z}_{pq})$). Let $\varphi_1 = \varphi\delta$. Then $\varphi_1^{-1}\langle \rho \rangle \varphi_1 = \langle \rho \rangle$ and $\varphi_1^{-1}\tau\varphi_1(v_j^i) = v_{\alpha j}^{i+r}$. If $q = 2$, then $\varphi_1^{-1}\tau\varphi_1 = \tau$ so that $\varphi_1^{-1}\langle \rho, \tau \rangle \varphi_1 = \langle \rho, \tau \rangle$. If $q = 3$, then $r = 1$ or $r = 2$. If $r = 1$, then $\varphi_1^{-1}\langle \rho, \tau \rangle \varphi_1 = \langle \rho, \tau \rangle$. If $r = 2$, then $\iota^{-1}\varphi_1^{-1}\tau\varphi\iota = \tau$, so that $\iota^{-1}\varphi_1^{-1}\langle \rho, \tau_1 \rangle \varphi_1\iota = \langle \rho, \tau \rangle$.

If $p^2 \mid |\mathrm{Aut}(\Gamma)|$, then Γ is isomorphic to the wreath product of an order q-circulant over an order p-circulant, so it suffices to show that if $q = 2, 3$, then $\langle \rho, \tau \rangle$ is a CI-group with respect to graphs. First observe that if $\Gamma = \Gamma_1 \wr \Gamma_2$, then Γ_1 is complete or trivial. Let $\Gamma' \cong \Gamma$ such that Γ' is a (q^k, G)-Cayley graph.

If Γ is circulant, then Γ' is circulant. As $\mathbb{Z}_{pq}$ is a CI-group with respect to graphs, there exists $\delta \in \mathrm{Aut}(\mathbb{Z}_{pq})$ such that $\delta(\Gamma) = \Gamma'$. Furthermore, $\delta(v_j^i) = v_{\beta j}^{\kappa i}$. As Γ_1 is complete or trivial and $\Gamma = \Gamma_1 \wr \Gamma_2$, we may take $\kappa = 1$. It is then easy to verify that $\delta^{-1}\langle \rho, \tau \rangle \delta = \langle \rho, \tau \rangle$.

If Γ is not circulant, then Γ' is also not circulant. By Corollary 12, $\gamma_2\delta\gamma_1(\Gamma) = \Gamma'$, where γ_1, γ_2, and δ are as in Corollary 12. As above, we may assume that if $\delta(v_j^i) = v_{\beta j}^{\kappa i}$, then $\kappa = 1$. Then $\gamma_2\delta\gamma_1 = \delta$ and as above, $\delta^{-1}\langle \rho, \tau \rangle \delta = \langle \rho, \tau \rangle$. ∎

5. **Isomorphism classes of circulant graphs.** We first prove a lemma in more generality than is necessary for our purposes, characterizing the isomorphism class of a vertex-transitive combinatorial object in some circumstances.

Let X be a vertex-transitive combinatorial object of order m, G a transitive subgroup of $\mathrm{Aut}(X)$, $C_G = \{\phi^{-1}G\phi : \phi \in S^n\}$, and $X_G = \{\phi^{-1}G\phi : \phi^{-1}G\phi \leq \mathrm{Aut}(X)\}$. Let S^n act on C_G by conjugation and denote the permutation group induced by this action as Ω. Let $\alpha_0 = 1$, the identity permutation in S^n, and $\alpha_1, \ldots, \alpha_m \in S^n$ be such that $G \leq \mathrm{Aut}\big(\alpha_i(X)\big)$, $0 \leq i \leq m$, $\alpha_i(X) \neq \alpha_j(X)$ for any $i \neq j$, and if $\alpha \in S^n$ such that $G \leq \alpha(X)$, then $\alpha(X) = \alpha_i(X)$ for some $0 \leq i \leq m$.

Assume that X_G is a (possibly trivial) block of Ω. Let $X_G^0 = X_G$, and denote by $X_G^0, X_G^1, \ldots, X_G^r$ all blocks conjugate to X_G^0 in Ω. Let $\beta_i \in S^n$ be such that $\beta_i^{-1}X_G^0\beta_i = X_G^i$, $1 \leq i \leq r$ and $\beta_0 = 1$.

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LEMMA 22. *If X_G is a (possibly trivial) block of Ω, then the isomorphism class of X is $\cup_{i=0}^{r}\cup_{j=0}^{m}\beta_i\alpha_j(X)$, and if $a \neq b$ or $c \neq d$, then $\beta_a\alpha_c(X) \neq \beta_b\alpha_d(X)$.*

PROOF. Fix β_i, $0 \leq i \leq r$ and α_j, $0 \leq j \leq m$ as above. Clearly $\beta_i\alpha_j(X) \cong X$ for all $0 \leq i \leq r$, $0 \leq j \leq m$. Conversely, let Y be a combinatorial object isomorphic to X, with $\beta : X \longrightarrow Y$ an isomorphism. If $G \leq \mathrm{Aut}(Y)$, then $G \leq \mathrm{Aut}\big(\beta_0^{-1}(Y)\big)$ and there exists $0 \leq j \leq m$ such that $\alpha_j^{-1}\beta_0^{-1}(Y) = X$. Thus $Y = \beta_0\alpha_j(X)$. If $G \not\leq \mathrm{Aut}(Y)$, then $\beta^{-1}(X_G^0)\beta = X_G^i$ for some $1 \leq i \leq r$. Then $G \leq \beta_i^{-1}(Y)$ and so there exists $0 \leq j \leq m$ such that $\alpha_j^{-1}\beta_i^{-1}(Y) = X$. Thus $\beta_i\alpha_j(X) = Y$ as required. Finally, the last statement follows immediately from the definitions of β_i, α_j. ∎

THEOREM 23. *If X is an (n, G)-CI-object and $\mathrm{Aut}(G)_* \leq N_{S^V}\big(\mathrm{Aut}(X)\big)$, then X_G is a block of Ω.*

PROOF. Let $\delta \in S^V$ be such that there exists a transitive subgroup G' of $\mathrm{Aut}(X)$ isomorphic to G and $G' \leq \delta^{-1}\mathrm{Aut}(X)\delta$. We will show that $\delta \in N_{S^G}\big(\mathrm{Aut}(X)\big)$. As X is an (n, G)-CI-object, there exists $\phi \in \mathrm{Aut}(X)$ such that $\phi^{-1}G'\phi = G$. Now $G' \leq \mathrm{Aut}\big(\delta(X)\big)$, so $G \leq \mathrm{Aut}\big(\phi\delta(X)\big)$. Hence there exists $\alpha \in \mathrm{Aut}(G)_*$ such that $\alpha_*(X) = \phi\delta(X)$, and as $\mathrm{Aut}(G)_* \leq N_{S^{G_*}}\big(\mathrm{Aut}(X)\big)$, $\mathrm{Aut}\big(\alpha_*(X)\big) = \mathrm{Aut}\big(\phi\delta(X)\big)$. As $\phi \in \mathrm{Aut}(X)$, $\phi^{-1}\mathrm{Aut}(X)\phi = \mathrm{Aut}(X)$, and $\phi^{-1}\mathrm{Aut}(X)\phi = \mathrm{Aut}\big(\delta(X)\big)$. Hence $\mathrm{Aut}\big(\delta(X)\big) = \mathrm{Aut}(X)$ and so $\delta^{-1}\mathrm{Aut}(X)\delta = \mathrm{Aut}(X)$. Thus $\delta \in N_{S^{G_*}}\big(\mathrm{Aut}(X)\big)$. ∎

COROLLARY 24. *Let Γ be a circulant graph of order pq, and $G = \mathbb{Z}_{pq}$. Then Γ_G is a block of Ω.*

PROOF. By Theorem 2 of [1], Γ is a CI-graph of $\mathbb{Z}_{pq}$. Thus by Theorem 23 it suffices to show that $\mathrm{Aut}(\mathbb{Z}_{pq}) \leq N_{S^G}\big(\mathrm{Aut}(\Gamma)\big)$. By Theorem 3.3 of [7], $\mathrm{Aut}(\Gamma) = S^{pq}$, $S^p \times S^q$, $S^p \times A$, $S^q \times B$, $\mathrm{Aut}(\Gamma) \leq A \times B$, $A \leq \mathrm{AGL}(1, q)$, $B \leq \mathrm{AGL}(1, p)$, or $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, where Γ_1 and Γ_2 are circulant graphs of order p and q respectively. Note that $\mathrm{Aut}(G) \leq \mathrm{AGL}(1, q) \times \mathrm{AGL}(1, p)$, and that $N_{S^{pq}}(S^{pq}) = S^{pq}$, $S^p \times S^q \leq N_{S^{pq}}(S^p \times S^q)$, $S^p \times \mathrm{AGL}(1, q) \leq N_{S^{pq}}(S^p \times A)$, $S^q \times \mathrm{AGL}(1, p) \leq N_{S^{pq}}(S^q \times B)$, and $\mathrm{AGL}(1, q) \times \mathrm{AGL}(1, p) \leq N_{S^{pq}}(A \times B)$. Hence the result follows in the preceeding cases.

If Γ' is a circulant graph of prime order r, then $\mathrm{Aut}(\Gamma') = S^r$ or $\mathrm{Aut}(\Gamma') \leq N(r)$. Hence if $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\Gamma_1) \wr \mathrm{Aut}(\Gamma_2)$, then $\mathrm{Aut}(\Gamma) = S^p \wr S^q$, $S^p \wr A$, $B \wr S^q$, or $B \wr A$ where A and B are as above. In all of the these cases, we have that $N(p) \wr N(q) \leq N_{S^{pq}}\big(\mathrm{Aut}(\Gamma)\big)$. Then, as $N(p) \times N(q) \leq N(p) \wr N(q)$, $\mathrm{Aut}(G) \leq N_{S^{pq}}\big(\mathrm{Aut}(\Gamma)\big)$, and the corollary follows. ∎

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