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Homogeneous Einstein Manifolds with Vanishing S Curvature

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Abstract. Infinitely many new Einstein Finsler metrics are constructed on several homogeneous spaces. By imposing certain conditions on the homogeneous spaces, it is shown that the Ricci constant condition becomes an ordinary differential equation. The regular solutions of this equation lead to a two parameter family of Einstein Finsler metrics with vanishing S curvature.

1 Introduction

Einstein metrics play an important role in Riemannian geometry and general relativity. In some sense, they are the best metrics, because they are the critical points of the scalar curvature functional [4]. Akbar-Zadeh [1] generalizes this beautiful property to Finsler geometry. Since then, the construction of new Einstein manifolds has become a significant theme in Finsler geometry.

The first set of examples are those Finsler manifolds with constant flag curvature. A striking phenomenon is the abundance of such metrics. R. Bryant [6] constructs an n-parameter family of Finsler metrics on the n dimensional sphere with constant flag curvature +1. Bao, Robles, and Shen [3] show that there exist infinitely many non-isometric Randers metrics on spheres, whose flag curvatures are all +1. Using the same technique, one can construct many Einstein Randers metrics on spheres. Moreover, Wang, Huang, and Deng [11] observed that, among these metrics in odd dimensions there is a 1-parameter family admitting a transitive group of isometries. Hence, the number of non-homothetic homogeneous Einstein Finsler metrics on the sphere could be infinite. In contrast, it is conjectured by Böhm, Wang, and Ziller [5] that there are only finitely many homogeneous Einstein *Riemannian* metrics on compact coset space whose isotropy representation consists of pairwise inequivalent irreducible summands.

Up to now, the known examples of homogeneous Einstein Finsler metrics are rare, and most of them are of Randers type; see, for example, [11,12]. In this paper, we will construct a new class of Finsler metrics on some homogeneous spaces that depend on a one-variable function ϕ. This construction generalizes the Cheeger deformation in Riemannian geometry and seems suitable for further study.

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Now we briefly describe the construction as follows. Let $\pi : (M, g) \to (B, \breve{g})$ be a Riemannian submersion with totally geodesic fibers. Each tangent space $T_x M$ has a subspace $\ker \pi_{*x}$ that is tangent to the fiber; thus, every tangent vector y in $T_x M$ can be decomposed to $y_0 + y_1$, where $y_1 \in \ker \pi_{*x}$ and $y_0 \in (\ker \pi_{*x})^\perp$. Define a Finsler metric F on M by letting

$$F(y)^2 = g(y, y) \cdot \phi(s), \quad s = \frac{g(y_1, y_1)}{g(y, y)},$$

where the function $\phi : [0, 1] \to \mathbb{R}^+$ will satisfy some open conditions (see (2.4)). If $\phi(s) = 1 + \epsilon s$ for some constant $\epsilon > -1$, then the resulting metric F is just the Cheeger deformation of the original metric g.

An important class of Riemannian submersions is given by $\pi : G/H \to G/K$, where G is a Lie group and H, K are compact subgroups of G with $H \subset K$. One can consult [4, §9.79–§9.93] for concrete examples. The main result of this paper can be roughly stated as the following theorem.

Theorem *If the groups G, K, H satisfy conditions* (I) *and* (II) *in Section 2.1, then the above Finsler metric F on G/H is Einstein if and only if the function ϕ satisfies a second order* ODE.

The main technique used in this paper is the curvature formula introduced by the first author in [7]. On a homogeneous space G/H with Finsler metric F, this formula allows one to compute the curvatures at the Lie algebra level. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras of G and H, respectively. Let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ complementary to $\mathfrak{h}$. Then the metric F is simply a Minkowski norm on $\mathfrak{m}$. Let D be the trivial flat connection on $\mathfrak{m}$; then $D(\mathrm{d}(F^2/2))$, the Hessian of $F^2/2$, defines a Riemannian metric on $\mathfrak{m}\backslash\{0\}$. Traditionally, this metric is also defined by

$$g_y(u, v) = \frac{1}{2} \frac{\partial^2}{\partial s \partial t}(y + su + tv)\Big|_{s=t=0}, \quad u, v \in \mathfrak{m} \simeq T_y(\mathfrak{m}\backslash\{0\}), \ y \in \mathfrak{m}\backslash\{0\}.$$

For each y in $\mathfrak{m}\backslash\{0\}$, the spray vector η at y is uniquely determined by the relation

$$g_y(\eta, v) = g_y(y, [v, y]_\mathfrak{m}), \quad v \in \mathfrak{m}.$$

The connection operator N at y is defined by

$$Nv = \frac{1}{2} D_v \eta - \frac{1}{2}[y, v]_\mathfrak{m}, \quad v \in \mathfrak{m}.$$

By using the spray vector and the connection operator, one can compute the Ricci curvature via the following formula (*cf.* [7, Corollary 4.9]):

$$\mathrm{Ric}(y) = -\operatorname{tr}(\operatorname{ad}(y)\operatorname{ad}_\mathfrak{h}(y)) + D_\eta(\operatorname{tr} N) - \operatorname{tr}(N^2), \quad y \in \mathfrak{m}\backslash\{0\}.$$

The Finsler metric F is said to be *Ricci constant* (*Einstein* for short) if there is a constant number κ such that $\mathrm{Ric}(y) = \kappa F^2(y)$ holds for any y in $\mathfrak{m}\backslash\{0\}$.

It should be noted that there are several versions of Ricci curvature in Finsler geometry. For example, Li and Shen [10] introduced a version that shares many properties with its Riemannian counterpart. It is defined via Berwald connection as a contraction of the hh-curvature tensor

$$\mathrm{Ric}_{ij} = R_i{}^m{}_{mj}.$$

The Finsler metric F is said to be of *isotropic Ricci curvature* if $\mathrm{Ric}_{ij} = (n-1)\kappa(x)g_{ij}$ for some scalar function κ. In Riemannian geometry, Schur's lemma asserted that in dimensions greater than two, having isotropic Ricci curvature implies that it is actually constant. Up to now, it is not known whether Schur's lemma holds for any version of Ricci curvature tensor in Finsler geometry. However, it is proved in [10] that if the non-Riemannian quantity χ vanishes, the above-mentioned two versions of Ricci curvature coincide. We will show that our constructions naturally have vanishing S curvature, and thus $\chi = 0$. Consequently, these examples are Ricci constant for any version of Ricci curvature tensor.

The paper is arranged as follows. In Section 2, we introduce conditions (I) and (II) and give a detailed deduction of the Ricci curvature, which turns out to depend only on the function ϕ and its derivatives. In Section 3, we present several examples that satisfy conditions (I) and (II), thus leading to infinitely many new Einstein manifolds.

2 Computation of the Ricci Curvature

Let G/H be a compact homogeneous space, where G is a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$, and H is a closed connected subgroup with Lie algebra $\mathfrak{h}$. Let Q be a negative multiple of the Killing form of $\mathfrak{g}$; namely, the equality

$$Q(u,v) = -\frac{1}{c_2}\operatorname{tr}\big(\operatorname{ad}(u)\operatorname{ad}(v)\big), \quad u,v\in\mathfrak{g}$$

holds for some positive constant c_2. Then Q is an $Ad(G)$ invariant inner product on $\mathfrak{g}$. Consequently, we have

$$Q([u,v],w) + Q(v,[u,w]) = 0, \quad u,v,w\in\mathfrak{g}. \tag{2.1}$$

Let $\mathfrak{m}$ be the orthogonal complement of $\mathfrak{h}$ in $\mathfrak{g}$. The above equality implies that $[\mathfrak{h},\mathfrak{m}]$ is orthogonal to $\mathfrak{h}$, thus $[\mathfrak{h},\mathfrak{m}]\subset\mathfrak{m}$.

2.1 Conditions (I) and (II)

We say that the pair (G,H) satisfies condition (I) if

(1) $\mathfrak{m} = \mathfrak{m}_0 + \mathfrak{m}_1$ is an orthogonal decomposition with respect to Q, where the subspaces $\mathfrak{m}_0, \mathfrak{m}_1$ are $Ad(H)$ invariant, namely, $[\mathfrak{h},\mathfrak{m}_k]\subset\mathfrak{m}_k$, $k=1,2$;

(2) $[\mathfrak{m}_0,\mathfrak{m}_1]\subset\mathfrak{m}_0$, $[\mathfrak{m}_0,\mathfrak{m}_0]\subset\mathfrak{h}+\mathfrak{m}_1$, $[\mathfrak{m}_1,\mathfrak{m}_1]\subset\mathfrak{h}+\mathfrak{m}_1$.

Condition (I) implies that $\mathfrak{k} = \mathfrak{h}+\mathfrak{m}_1$ is a Lie subalgebra of $\mathfrak{g}$ and $(\mathfrak{g},\mathfrak{k})$ is a symmetric pair. Below we will always assume that (G,H) satisfies condition (I).

We say that the pair (G,H) satisfies condition (II) if the following hold:

(3) There is a constant c such that

$$Q\big([y_0,y_1],[y_0,y_1]\big) = c\cdot Q(y_0,y_0)Q(y_1,y_1),$$

for any y in $\mathfrak{m}$.

(4) There are constants c_0, c_1 such that

$$-\operatorname{tr}_{\mathfrak{h}}\left(\operatorname{ad}_{\mathfrak{h}}(y)\operatorname{ad}(y)\right) = c_0 Q(y_0, y_0) + c_1 Q(y_1, y_1),$$

for any y in $\mathfrak{m}$.

Here and after, we use y_0, y_1 to denote the $\mathfrak{m}_0$, $\mathfrak{m}_1$ component of $y \in \mathfrak{m}$, respectively. Let $\{e_i\}$ and $\{e_\alpha\}$ be orthonormal basis of $\mathfrak{m}_0$ and $\mathfrak{m}_1$, respectively. For each y in $\mathfrak{m}$, define

$$(2.2)\qquad \begin{aligned} A_0 &= \sum_\alpha Q\big([y_0, e_\alpha], [y_0, e_\alpha]\big), \\ A_1 &= \sum_i Q\big([y_1, e_i], [y_1, e_i]\big). \end{aligned}$$

It is easy to see that condition (3) implies that

$$A_0 = cn_1 Q(y_0, y_0), \quad A_1 = cn_0 Q(y_1, y_1),$$

where $n_1 = \dim(\mathfrak{m}_1)$ and $n_0 = \dim(\mathfrak{m}_0)$.

Remark Condition (4) is not restrictive. It will automatically hold if $\mathfrak{m}_0$, $\mathfrak{m}_1$ are $\operatorname{ad}(\mathfrak{h})$-irreducible. In that case c_0, c_1 are closely related to the Casimir constants. However, condition (3) is really restrictive.

2.2 The Finsler Metric and its Spray Vector

We will first consider homogeneous spaces satisfying condition (I). For such spaces, one can define a function F on $\mathfrak{m}\backslash\{0\}$ by letting

$$(2.3)\qquad F(y)^2 = Q(y, y)\cdot\phi(s), \quad s = Q(y_1, y_1)/Q(y, y), \quad y \in \mathfrak{m}\backslash\{0\},$$

where ϕ is a positively valued function defined on $[0, 1]$.

Lemma 2.1 *The above F is a Minkowski norm if and only if the function ϕ satisfies the following regularity conditions*

$$(2.4)\qquad \begin{aligned} &\phi + (1-s)\phi' > 0, && 2\phi + (1-2s)\phi' + 2s(1-s)\phi'' > 0, \\ &\phi - s\phi' > 0, && (\phi - s\phi')(\phi + (1-s)\phi') + 2s(1-s)\phi\phi'' > 0. \end{aligned}$$

Moreover, F is an Euclidean norm if and only if $\phi(s) = a + bs$, where the constants a and b satisfy $a > 0$ and $a + b > 0$.

Proof Let $\{e_i\}$ and $\{e_\alpha\}$ be Q-orthonormal basis of $\mathfrak{m}_0$ and $\mathfrak{m}_1$, respectively. Then we may write $y_0 = y^i e_i$ and $y_1 = y^\alpha e_\alpha$ for $y \in \mathfrak{m}$. The function $F(y)$ is now a positively valued function of $\{y^i, y^\alpha\}$ and it is homogeneous of degree 1. Direct computation shows that the Hessian matrix of $F^2/2$ is given by

$$\Theta = \begin{bmatrix} (\phi - s\phi')\delta_{ij} + 2s^2\phi'' l_i l_j & -2s(1-s)\phi'' l_i l_\beta \\ -2s(1-s)\phi'' l_\alpha l_j & (\phi + (1-s)\phi')\delta_{\alpha\beta} + 2(1-s)^2\phi'' l_\alpha l_\beta \end{bmatrix},$$

where $l_i = y^i Q(y, y)^{-1/2}$ and $l_\alpha = y^\alpha Q(y, y)^{-1/2}$.

We first consider the case $s \neq 0, 1$. In this case $y_0 \neq 0$ and $y_1 \neq 0$. It is easy to see that the orthogonal complement of y_0 in $\mathfrak{m}_0$ lies in the eigenspace of Θ with eigenvalue $(\phi - s\phi')$. Thus, the multiplicity of $(\phi - s\phi')$ is at least $n_0 - 1$. Similarly, $\phi + (1-s)\phi'$

is an eigenvalue with multiplicity at least $n_1 - 1$. Now let ζ, λ be a solution of the system

$$\begin{aligned}\lambda &= (\phi - s\phi') + 2s^2(1-s)\phi'' \cdot (1-\zeta)\\ &= \phi + (1-s)\phi' + 2s(1-s)^2\phi'' \cdot (1-1/\zeta).\end{aligned}$$

Then λ is a root of the following quadratic equation

$$\begin{aligned}\lambda^2 - \big(2\phi + (1-2s)\phi' + 2s(1-s)\phi''\big)\lambda \qquad\\ + (\phi - s\phi')\big(\phi + (1-s)\phi'\big) + 2s(1-s)\phi\phi'' = 0.\end{aligned} \tag{2.5}$$

It is straightforward to check that $y_0 + \zeta y_1$ is an eigenvector of Θ with eigenvalue λ. Since the discriminant of the above equation is

$$\big(2s(1-s)\phi'' + (1-2s)\phi'\big)^2 + 4s(1-s)(\phi')^2 \geq 0,$$

the two real solutions of (2.5) are the remaining eigenvalues of Θ. In this way we have found all the eigenvalues of Θ. By continuity, the eigenvalues have the same expressions when $s = 0$ or 1.

Recall that F is a Minkowski norm if and only if Θ is positive definite, if and only if all its eigenvalues are positive. Thus, the first assertion is proved.

In addition, F is Euclidean if and only if the Hessian matrix Θ is a constant matrix. Looking at the upper right corner shows that this can happen if and only if $\phi'' = 0$. So the second assertion is also proved. ∎

For each v in $\mathfrak{m}$, we have $D_v y = v$, $D_v y_1 = v_1$, thus

$$D_v\big(Q(y,y)\big) = 2Q(y,v), \quad D_v\big(Q(y_1,y_1)\big) = 2Q(y_1,v_1).$$

Direct differentiation yields

$$D_v s = \xi \cdot \big(Q(y_1,v_1) - sQ(y,v)\big), \tag{2.6}$$

where $\xi = 2/Q(y,y)$. By using (2.6), we have

$$g_y(y,v) = D_v(F^2/2) = (\phi - s\phi')Q(y,v) + \phi' \cdot Q(y_1,v_1). \tag{2.7}$$

Further differentiation yields

$$\begin{aligned}g_y(v,w) &= D_w D_v(F^2/2) = (\phi - s\phi')Q(v,w) + \phi' \cdot Q(v_1,w_1)\\ &\quad + \xi\phi'' \cdot \big(Q(y_1,v_1) - sQ(y,v)\big)\big(Q(y_1,w_1) - sQ(y,w)\big).\end{aligned} \tag{2.8}$$

Lemma 2.2 *If the homogeneous space G/H satisfies condition* (I) *and the Finsler metric is given by* (2.3)*, then the spray vector at y has the expression $\eta = \psi \cdot [y_0, y_1]$, where $\psi = \phi'/(\phi - s\phi')$.*

Proof We only need to check that the above η satisfies $g_y(v,\eta) = g_y(y,[v,y]_{\mathfrak{m}})$, for any v in $\mathfrak{m}$. Notice that since $[\mathfrak{m}_0, \mathfrak{m}_1] \subset \mathfrak{m}_0$, we have $\eta \in \mathfrak{m}_0$; so $\eta_1 = 0$. Consequently,

$$Q(y_1,\eta_1) - sQ(y,\eta) = -s\psi Q\big(y,[y_0,y_1]\big) = -s\psi Q\big(y_0,[y_0,y_1]\big) = 0,$$

where in the last equality we have used (2.1). By using (2.8) we have

$$g_y(v,\eta) = (\phi - s\phi')Q(v,\eta) = \phi' \cdot Q\big(v,[y_0,y_1]\big).$$

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By using (2.7), we have

$$g_y\big(y,[v,y]_{\mathfrak m}\big) = (\phi - s\phi')Q\big(y,[v,y]_{\mathfrak m}\big) + \phi'\cdot Q\big(y_1,[v,y]_{\mathfrak m_1}\big) = \phi'\cdot Q\big(y_1,[v,y]\big)$$
$$= \phi'\cdot Q\big(v,[y,y_1]\big) = \phi'\cdot Q\big(v,[y_0,y_1]\big).$$

It is clear that $g_y(y,[v,y]_{\mathfrak m}) = g_y(v,\eta)$. So the lemma is proved. ∎

2.3 The S Curvature

Recall that the connection operator N is defined by $N = \frac{1}{2}D\eta - \frac{1}{2}\operatorname{ad}_{\mathfrak m}(y)$ and is related to the S curvature via $S = \operatorname{tr}(N) + \operatorname{tr}\operatorname{ad}_{\mathfrak m}(y)$. Since G is compact, we have $\operatorname{tr}\operatorname{ad}_{\mathfrak m}(y) = 0$, so $S = \operatorname{tr}(N) = \frac{1}{2}\operatorname{tr}(D\eta)$.

For each v in $\mathfrak m$, we have

$$D_v\eta = D_v\psi\cdot[y_0,y_1] + \psi\cdot[v_0,y_1] + \psi\cdot[y_0,v_1]. \tag{2.9}$$

It is readily seen that $D_v\eta$ always belongs to $\mathfrak m_0$. When v belongs to $\mathfrak m_0$, we have

$$D_v\psi = \psi'\cdot D_v s = -s\xi\psi'\cdot Q(y,v) = -s\xi\psi'\cdot Q(y_0,v).$$

Hence, when v belongs to $\mathfrak m_0$, equation (2.9) can be written as

$$D_v\eta = -s\xi\psi' Q(y_0,v)\cdot[y_0,y_1] + \psi\cdot[v,y_1]. \tag{2.10}$$

Now, let $\{e_i\}$ be a basis of $\mathfrak m_0$ and let $\{e_\alpha\}$ be a basis of $\mathfrak m_1$, such that they consist an orthonormal basis of $\mathfrak m$ with respect to Q. Using (2.10), we have

$$\begin{aligned}\operatorname{tr}(D\eta) &= \sum_i Q(e_i, D_{e_i}\eta) + \sum_\alpha Q(e_\alpha, D_{e_\alpha}\eta) = \sum_i Q(e_i, D_{e_i}\eta)\\ &= \sum_i Q\big(e_i,\ -s\xi\psi' Q(y_0,e_i)\cdot[y_0,y_1] + \psi\cdot[e_i,y_1]\big)\\ &= \sum_i -s\xi\psi' Q(y_0,e_i)Q\big(e_i,[y_0,y_1]\big)\\ &= -s\xi\psi' Q\big(y_0,[y_0,y_1]\big) = 0.\end{aligned}$$

So we have proved the following proposition.

Proposition 2.3 *If the homogeneous space G/H satisfies condition* (I) *and the Finsler metric is given by* (2.3)*, then the S curvature vanishes identically, regardless of the function ϕ.*

2.4 The Ricci Curvature

Now we proceed to compute the Ricci curvature. Since S curvature vanishes, Ricci curvature has the simple expression

$$\begin{aligned}\operatorname{Ric} &= -\operatorname{tr}_{\mathfrak m}\big(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak h}(y)\big) - \operatorname{tr}(N^2)\\ &= -\operatorname{tr}_{\mathfrak m}\big(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak h}(y)\big) - \frac{1}{4}\operatorname{tr}\big(\operatorname{ad}_{\mathfrak m}(y)\circ\operatorname{ad}_{\mathfrak m}(y)\big)\\ &\quad - \frac{1}{4}\operatorname{tr}(D\eta\circ D\eta) + \frac{1}{2}\operatorname{tr}\big(D\eta\circ\operatorname{ad}_{\mathfrak m}(y)\big).\end{aligned}$$

On the right-hand side, the first two terms do not involve the metric F, but the last two terms do. We will treat the last two terms separately.

Lemma 2.4 *We have*

$$\mathrm{tr}(D\eta \circ D\eta) = 2s\xi\psi\psi' \cdot Q\big([y_0, y_1], [y_0, y_1]\big) - \psi^2 \cdot A_1,$$

where A_1 is the quantity defined in (2.2).

Proof For each v in $\mathfrak{m}$, the vector $D_v\eta$ belongs to $\mathfrak{m}_0$. It follows that $D\eta \circ D\eta$ maps $\mathfrak{m}$ into $\mathfrak{m}_0$. We only need to consider the trace of $D\eta \circ D\eta$ on the subspace $\mathfrak{m}_0$. For each v in $\mathfrak{m}_0$, we have from (2.10) that

$$\begin{aligned} Q(y, D_v\eta) &= Q(y_0, D_v\eta) = Q\big(y_0,\ -s\xi\psi' Q(y_0, v) \cdot [y_0, y_1] + \psi \cdot [v, y_1]\big) \\ &= \psi \cdot Q\big(y_0, [v, y_1]\big) = -\psi \cdot Q\big(v, [y_0, y_1]\big). \end{aligned}$$

It follows that

$$D_{D_v\eta}\psi = \psi' \cdot D_{D_v\eta}s = -s\xi\psi' Q(y, D_v\eta) = s\xi\psi\psi' Q(v, [y_0, y_1]).$$

By using this fact we obtain

$$\begin{aligned} D_{D_v\eta}\eta &= D_{D_v\eta}\psi \cdot [y_0, y_1] + \psi \cdot [D_v\eta, y_1] \\ &= s\xi\psi\psi' Q\big(v, [y_0, y_1]\big) \cdot [y_0, y_1] + \psi \cdot [D_v\eta, y_1]. \end{aligned}$$

Let $\{e_i\}$ be an orthonormal basis of $\mathfrak{m}_0$ with respect to Q. Utilizing the above equation yields

$$\begin{aligned} \mathrm{tr}(D\eta \circ D\eta) &= \sum_i Q(e_i, D_{D_{e_i}\eta}\eta) \\ &= \sum_i s\xi\psi\psi' Q\big(e_i, [y_0, y_1]\big) Q\big(e_i, [y_0, y_1]\big) + \psi \cdot Q\big(e_i, [D_{e_i}\eta, y_1]\big) \\ &= s\xi\psi\psi' Q\big([y_0, y_1], [y_0, y_1]\big) + \psi \cdot (X_1), \end{aligned}$$

where

$$\begin{aligned} (X_1) &= \sum_i Q\big(e_i, [D_{e_i}\eta, y_1]\big) = \sum_i Q\big(D_{e_i}\eta,\ [y_1, e_i]\big) \\ &= \sum_i Q\big(-s\xi\psi' Q(y_0, e_i) \cdot [y_0, y_1] + \psi \cdot [e_i, y_1],\ [y_1, e_i]\big) \\ &= -\psi A_1 + \sum_i -s\xi\psi' Q(y_0, e_i) \cdot Q\big([y_0, y_1], [y_1, e_i]\big) \\ &= -\psi A_1 - s\xi\psi' \sum_i Q(y_0, e_i) Q\big([[y_0, y_1], y_1], e_i\big) \\ &= -\psi A_1 - s\xi\psi' Q\big(y_0, [[y_0, y_1], y_1]\big) \\ &= -\psi A_1 + s\xi\psi' Q\big([y_0, y_1], [y_0, y_1]\big). \end{aligned}$$

Combining the above results completes the proof. ∎

Lemma 2.5 *We have*

$$\mathrm{tr}\left(D\eta \circ \mathrm{ad}_{\mathfrak{m}}(y)\right) = -\xi\psi' \cdot Q\big([y_0, y_1], [y_0, y_1]\big) + \psi \cdot A_1 - \psi \cdot A_0,$$

where A_0, A_1 are defined by (2.2).

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Proof For each v in $\mathfrak{m}$, we have

$$\begin{aligned} D_{[y,v]}s &= \xi \cdot \big(Q(y_1,[y,v]_1) - sQ(y,[y,v])\big) \\ &= \xi Q\big(y_1,[y,v]_1\big) = \xi Q\big(y_1,[y,v]\big) \\ &= -\xi Q\big(v,[y,y_1]\big) = -\xi Q\big(v,[y_0,y_1]\big). \end{aligned}$$

It follows that

$$\begin{aligned} D_{[y,v]}\eta &= \psi' D_{[y,v]}s \cdot [y_0,y_1] + \psi\big[[y,v]_0,y_1\big] + \psi\big[y_0,[y,v]_1\big] \\ &= -\xi\psi' Q\big(v,[y_0,y_1]\big)\cdot[y_0,y_1] + \psi\big[[y,v]_0,y_1\big] + \psi\big[y_0,[y,v]_1\big]. \end{aligned}$$

Using the above equation, we have

$$\operatorname{tr}\big(D\eta \circ \operatorname{ad}(y)\big) = -\xi\psi' \cdot (X_2) + \psi \cdot (X_3) + \psi \cdot (X_4),$$

where

$$\begin{aligned} (X_2) &= \sum_i Q\big(e_i,[y_0,y_1]\big) Q\big(e_i,[y_0,y_1]\big) = Q\big([y_0,y_1],[y_0,y_1]\big), \\ (X_3) &= \sum_i Q\big(e_i,[[y,e_i]_0,y_1]\big) = \sum_i Q\big([y_1,e_i],[y,e_i]_0\big) \\ &= \sum_i Q\big([y_1,e_i],[y,e_i]\big) \\ &= \sum_i Q\big([y_1,e_i],[y_1,e_i]\big) = A_1, \\ (X_4) &= \sum_i Q\big(e_i,[y_0,[y,e_i]_1]\big) = \sum_i Q\big(e_i,[y_0,[y_0,e_i]_1]\big) \\ &= \operatorname{tr}_{\mathfrak{m}_0}\big(\operatorname{ad}(y_0)\operatorname{ad}_{\mathfrak{m}_1}(y_0)\big) = \operatorname{tr}_{\mathfrak{g}}\big(\operatorname{ad}(y_0)\operatorname{ad}_{\mathfrak{m}_1}(y_0)\big) \\ &= \operatorname{tr}_{\mathfrak{g}}\big(\operatorname{ad}_{\mathfrak{m}_1}(y_0)\operatorname{ad}(y_0)\big) = \sum_\alpha Q\big(e_\alpha,[y_0,[y_0,e_\alpha]]\big) \\ &= \sum_\alpha Q\big([e_\alpha,y_0],[y_0,e_\alpha]\big) = -A_0. \end{aligned}$$

Notice that in the computation of (X_3), we have used the fact that

$$\sum_i Q\big([y_1,e_i],[y_0,e_i]\big) = 0.$$

One can prove this fact by expanding $Q(y_0,y_1) = 0$. Thus, the lemma is proved by plugging the above results. ■

Combining the above lemmas yields the following proposition.

Proposition 2.6 *If the homogeneous space G/H satisfies condtion* (I) *and the Finsler metric F is given by* (2.3)*, then the Ricci curvature satisfies*

$$\text{(2.11)} \quad \begin{aligned} \operatorname{Ric}(y) = &-\operatorname{tr}_{\mathfrak{m}}\big(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\big) - \frac{1}{4}\operatorname{tr}_{\mathfrak{m}}\big(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{m}}(y)\big) \\ &- \frac{(s\psi+1)\psi'}{Q(y,y)} Q\big([y_0,y_1],[y_0,y_1]\big) + \Big(\frac{1}{4}\psi^2 + \frac{1}{2}\psi\Big)\cdot A_1 - \frac{1}{2}\psi\cdot A_0, \end{aligned}$$

where $\psi = \phi'/(\phi - s\phi')$ and A_0, A_1 are defined by (2.2).

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2.5 The Main Theorem

Suppose further that the pair (G,H) satisfies condition (II); then we can immediately simplify the last three terms in the above curvature formula. Next we examine the first two trace terms.

Lemma 2.7 *If the pair* (G,H) *satisfies condition* (II)*, then we have*

$$-\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right)=c_0Q(y_0,y_0)+c_1Q(y_1,y_1).$$

Proof As an operator on $\mathfrak{g}$, the image of $\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)$ always lies in $\mathfrak{m}$. So we have

$$\begin{aligned}\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right)&=\operatorname{tr}_{\mathfrak{g}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right)\\&=\operatorname{tr}_{\mathfrak{g}}\left(\operatorname{ad}_{\mathfrak{h}}(y)\operatorname{ad}(y)\right)=\operatorname{tr}_{\mathfrak{h}}\left(\operatorname{ad}_{\mathfrak{h}}(y)\operatorname{ad}(y)\right).\end{aligned}$$

Using condition (II) proves the lemma. ■

Lemma 2.8 *If the pair* (G,H) *satisfies condition* (II)*, then we have*

$$-\operatorname{tr}_{\mathfrak{m}}(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{m}}(y))=c_2Q(y,y)-2c_0Q(y_0,y_0)-2c_1Q(y_1,y_1).$$

Proof The image of $\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}(y)$ lies in $\mathfrak{m}$, so we have on the one hand that

$$\begin{aligned}\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}(y)\right)&=\operatorname{tr}_{\mathfrak{g}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}(y)\right)=\operatorname{tr}_{\mathfrak{g}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{m}}(y)\right)\\&=\operatorname{tr}_{\mathfrak{g}}\left(\operatorname{ad}(y)\operatorname{ad}(y)\right)-\operatorname{tr}_{\mathfrak{g}}(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y))\\&=-c_2Q(y,y)-\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right).\end{aligned}$$

As operators on $\mathfrak{m}$, $\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)$ is the same as $\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{h}}(y)$, so we have on the other hand that

$$\begin{aligned}\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}(y)\right)&=\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{m}}(y)\right)+\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right)\\&=\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{m}}(y)\right)+\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right).\end{aligned}$$

Comparing the above two results yields

$$\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}_{\mathfrak{m}}(y)\operatorname{ad}_{\mathfrak{m}}(y)\right)=-c_2Q(y,y)-2\operatorname{tr}_{\mathfrak{m}}\left(\operatorname{ad}(y)\operatorname{ad}_{\mathfrak{h}}(y)\right).$$

Applying Lemma 2.7 completes the proof. ■

The above lemmas allow us to rewrite the Ricci curvature formula (2.11) as

$$\begin{aligned}\operatorname{Ric}(y)=\frac{1}{4}Q(y,y)\cdot\big\{&c_2+2c_0(1-s)+2c_1s-4c(s\psi+1)\psi's(1-s)\\&+(\psi^2+2\psi)cn_0s-2\psi cn_1(1-s)\big\}.\end{aligned}$$

Thus, we have the following theorem.

Theorem 2.9 *Let* G/H *be a homogeneous space that satisfies the conditions* (I) *and* (II)*. Define a homogeneous Finsler metric* F *on* G/H *by* (2.3)*. Then* F *is an Einstein*

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metric if and only if the function ϕ satisfies the following ODE*:*

$$\kappa\phi = c_2 + 2c_0(1-s) + 2c_1 s - 4c(s\psi+1)\psi' s(1-s) + (\psi^2+2\psi)cn_0 s - 2\psi c n_1(1-s), \tag{2.12}$$

where κ is a constant real number and $\psi = \phi'/(\phi - s\phi')$. In this case, $\mathrm{Ric} = \kappa F^2/4$.

At this point, we make some comments on the regularity of the solutions. Let

$$\Delta = \big(2c(n_0+n_1) - 2c_0 - c_2\big)^2 - 8c(2n_1+n_0)(c_1-c_0).$$

It is easy to show that if $\Delta \geq 0$, then equation (2.12) has solutions of the form $\phi(s) = 1 + ks$, where the coefficient k is given by

$$k = \frac{2c_0 + c_2 - 2c(n_0+n_1) \pm \sqrt{\Delta}}{2c(2n_1+n_0)}.$$

In this case, $\kappa = -2ckn_1 + c_2 + 2c_0$. Among the above two values of k, the bigger one is always > -1, so the corresponding function ϕ satisfies the regularity conditions (2.4). As a result, for each solution of the ODE (2.12), if it is sufficiently close to this special solution $1+ks$ (in L^2 sense), then it is also regular. One can consult [8] for another treatment on this issue. If $\Delta < 0$, then we are not able to obtain solutions of the specific form $1+ks$, but it is still possible to obtain other regular solutions.

3 Examples

In this section we will present some examples to which the above theorem could be applied. In the first two examples, $\mathfrak{m}_1$ is of dimension one, and the resulting Finsler metrics are of (α,β) type. To see this, we first note that the restriction of the quadratic form Q on $\mathfrak{m}_1$ is the square of a linear function β (which is dual to a unit vector in $\mathfrak{m}_1$); thus $Q(y_1,y_1) = \beta(y)^2$ and $Q(y,y) = \alpha(y)^2$, where α is the Euclidean norm associated with Q. It follows that the metrics (2.3) can be expressed by a Riemannian metric α and a one form β; thus, they are of (α,β) type. The other examples have $\mathfrak{m}_1$ of dimension greater than one.

3.1 The Stiefel Manifold $V_2(\mathbb{R}^{n+2}) = SO(n+2)/SO(n)$

Suppose $n \geq 2$. Let $G = SO(n+2)$, $H = SO(n)$, where H is embedded in G at the upper left corner. Let $Q(u,v) = -\operatorname{tr}(uv)$ for u, v in $\mathfrak{so}(n+2)$; then it is well known that $c_2 = n$. Define the subspaces $\mathfrak{m}_0$, $\mathfrak{m}_1$ as

$$\mathfrak{m}_0 = \left\{\begin{bmatrix} & A \\ -A^t & \end{bmatrix} \,\Big|\, A \in \mathbb{R}^{n\times 2}\right\} \quad\text{and}\quad \mathfrak{m}_1 = \left\{\begin{bmatrix} 0 & \\ & aJ \end{bmatrix} \,\Big|\, a \in \mathbb{R}\right\},$$

where J is a 2×2 anti-symmetric matrix whose right upper entry equals 1. One can show that condition (I) is satisfied by direct matrix computation.

Now we will prove that condition (II) is also satisfied. For y_0 in $\mathfrak{m}_0$ and y_1 in $\mathfrak{m}_1$, one can write

$$y_0 = \begin{bmatrix} & A \\ -A^t & \end{bmatrix} \quad\text{and}\quad y_1 = \begin{bmatrix} 0 & \\ & aJ \end{bmatrix}.$$

We have

$$Q(y_0,y_0) = 2\operatorname{tr}(AA^t), \quad Q(y_1,y_1) = 2a^2.$$

Moreover, since $[y_0, y_1] = \begin{bmatrix} & aAJ \\ aJA' & \end{bmatrix}$ and $J^2 = -I$, we find that

$$Q\big([y_0, y_1], [y_0, y_1]\big) = 2a^2 \operatorname{tr}(AA^t) = \frac{1}{2} Q(y_0, y_0) Q(y_1, y_1),$$

so condition (3) is satisfied with $c = 1/2$. Let $y = y_0 + y_1$ and $h = \operatorname{diag}(X, 0) \in \mathfrak{h}$, where $X \in \mathfrak{so}(n)$. Then we have

$$-\operatorname{ad}(y)(h) = \begin{bmatrix} & XA \\ A^t X & \end{bmatrix}, \quad -\operatorname{ad}_{\mathfrak{h}}(y) \operatorname{ad}(y)(h) = \begin{bmatrix} AA'X + XAA' & \\ & 0 \end{bmatrix}.$$

It is seen that the operator $P = -\operatorname{ad}_{\mathfrak{h}}(y) \operatorname{ad}(y)$ maps $X \in \mathfrak{so}(n)$ to $AA'X + XAA'$. Let $X_{ij} = E_{ij} - E_{ji}$, $1 \le i, j \le n$, where E_{ij} is the matrix whose (i, j) entry is 1 and all the other entries are 0's. Then $\{X_{ij} \mid 1 \le i < j \le n\}$ is a basis of $\mathfrak{so}(n)$. We have

$$PX_{ij} = \sum_{k=1}^{n} b_{jk} X_{ik} + b_{ki} X_{kj},$$

where b_{ij} is the (i, j) entry of the matrix AA'. It follows that

$$\operatorname{tr}(P) = \sum_{1 \le i < j \le n} (b_{jj} + b_{ii}) = (n-1) \sum_{i=1}^{n} b_{ii} = (n-1) \operatorname{tr}(AA').$$

So condition (4) is also satisfied with $c_0 = (n-1)/2$ and $c_1 = 0$.

Notice that $n_0 = \dim(\mathfrak{m}_0) = 2n$ and $n_1 = \dim(\mathfrak{m}_1) = 1$; it is straightforward to show that $\Delta = 4n^2$ and the ODE (2.12) has a special solution $\phi(s) = 1 + (n-1)s/(n+1)$. This solution corresponds to a Riemannian metric found by Arvanitoyeorgos [2]. The other linear solution $\phi(s) = 1 - s$ is not regular, according to Lemma 2.1. Since (2.12) is of second order, we conclude that the regular solutions of (2.12) depend on two parameters; namely, we obtain a two parameter family of regular (α, β) metrics on the Stiefel manifold $V_2(\mathbb{R}^{n+2})$, $n \ge 2$, with constant Ricci curvature and vanishing S curvature. Due to the nonlinearity of the ODE (2.12), we cannot find general solutions. However, we can find a one parameter family of special solutions:

$$\phi(s) = \left(\sqrt{1 + \left(\frac{n-1}{n+1} + \epsilon^2 \right) s} + \epsilon \sqrt{s} \right)^2.$$

The corresponding metrics are of Randers type, a rather special type of Finsler metrics.

3.2 The Sphere $S^{2n+1} = SU(n+1)/SU(n)$

Here the embedding of $SU(n)$ in $SU(n+1)$ is also at the upper left corner. Let $Q(u, v) = -\operatorname{tr}(uv)$ as above; then $c_2 = 2n + 2$. Define the subspaces $\mathfrak{m}_0$, $\mathfrak{m}_1$ as follows:

$$\mathfrak{m}_0 = \left\{ \begin{bmatrix} & \alpha \\ -\alpha^* & \end{bmatrix} \,\middle|\, \alpha \in \mathbb{C}^{n \times 1} \right\} \quad \text{and} \quad \mathfrak{m}_1 = \left\{ \begin{bmatrix} aI & \\ & -na \end{bmatrix} \,\middle|\, a \in \mathbb{C}, a + \overline{a} = 0 \right\}.$$

Thus, $n_0 = 2n$ and $n_1 = 1$. By direct matrix computation, one can show that conditions (I) and (II) are satisfied with

$$c = (n+1)/n, \quad c_0 = (n-1)/2, \quad c_1 = 0.$$

The ODE (2.12) has a special solution

$$\phi(s) = \frac{1}{2}\left(1 + \frac{n-1}{n+1} s\right)$$

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with $\kappa = 8n$, which corresponds to the canonical Riemannian metric on S^{2n+1}. Thus, we conclude that the ODE (2.12) has other regular solutions as well, which correspond to non-Riemannian Finsler metrics. Notice that the other linear solution $\phi(s) = 1 - s$ is not regular according to Lemma 2.1.

This family of examples is also discussed in [12] via a different method. Among the solutions, there is a one parameter family of Randers metrics with constant flag curvature; see [11].

3.3 The Sphere $S^{4n+3} = Sp(n+1)/Sp(n)$

Let $G = Sp(n+1)$ and $H = Sp(n)$, where the embedding of H in G is at the upper left corner. We can choose $Q(u,v) = -\operatorname{tr}(uv)$, so $c_2 = 4n+8$. The subspace $\mathfrak{m}_0$ consists matrices of the form $\left[\begin{smallmatrix} 0 & \xi \\ -\xi^* & 0 \end{smallmatrix}\right]$ and $\mathfrak{m}_1$ consists matrices of the form $\left[\begin{smallmatrix} a & \\ & 0 \end{smallmatrix}\right]$, where a is a pure imaginary quaternion and $\xi \in \mathbb{H}^n$. Thus $n_0 = 4n$ and $n_1 = 3$.

By direct matrix computation, one can show that conditions (I) and (II) are satisfied with $c = 1$, $c_0 = 2n+1$, and $c_1 = 0$. The ODE (2.12) has the following two special solutions:

$$\phi_1(s) = 1 + s \quad \text{and} \quad \phi_2(s) = 1 - \frac{(2n+1)s}{2n+3}.$$

The first solution is homothetic to the standard Riemannian metric on S^{4n+3}, and the second one corresponds to the metric found by G. Jensen [9]. Due to the difficulty in integrating the nonlinear ODE (2.12), we cannot find other explicit solutions that correspond to non-Riemannian metrics.

One can consult [8] for other descriptions of this family of examples.

3.4 The Wallach Space $F^6 = SU(3)/T^2$

Let $G = SU(3)$ and let H be a maximal torus in G. Set $Q(u,v) = -\operatorname{tr}(uv)$ so that $c_2 = 6$. We can assume that

$$\begin{aligned} \mathfrak{h} &= \left\{ \operatorname{diag}(a\mathrm{i}, b\mathrm{i}, -(a+b)\mathrm{i}) \;\middle|\; a, b \in \mathbb{R} \right\}, \\ \mathfrak{m}_0 &= \left\{ \left[\begin{smallmatrix} & \alpha \\ -\alpha^* & \end{smallmatrix}\right] \;\middle|\; \alpha \in \mathbb{C}^{2\times 1} \right\}, \\ \mathfrak{m}_1 &= \left\{ \left[\begin{smallmatrix} & c & \\ -\bar{c} & & \\ & & 0 \end{smallmatrix}\right] \;\middle|\; c \in \mathbb{C} \right\}. \end{aligned}$$

It is easy to check that conditions (I) and (II) are satisfied with

$$c = 1/2, \quad c_0 = 2, \quad c_1 = 2.$$

Together with $n_0 = 4$ and $n_1 = 2$, we find that $\Delta = 16$ and the ODE (2.12) has two special solutions: $\phi_1(s) = 1$ and $\phi_2(s) = 1+s$. The other solutions are non-Riemannian, and we cannot make them explicit.

This example can be generalized to $SU(n+2)/S(U(1)\times U(1)\times U(n))$. The determination of the constants c, c_0, and c_1 is direct via matrix computation, so we will omit the details.

3.5 The Wallach Space $F^{12} = Sp(3)/3Sp(1)$

In this case, we set $Q(u, v) = -\operatorname{tr}(uv)$ for u, v in $\mathfrak{sp}(3)$, so that $c_2 = 16$. We choose the subspaces $\mathfrak{h}$, $\mathfrak{m}_0$, $\mathfrak{m}_1$ as follows:

$$\mathfrak{h} = \left\{\operatorname{diag}(a_1, a_2, a_3) \mid a_i \in \mathbb{H}, a_i + \overline{a_i} = 0, i = 1, 2, 3\right\},$$
$$\mathfrak{m}_0 = \left\{\left[\begin{smallmatrix} & \alpha \\ -\alpha^* & \end{smallmatrix}\right] \mid \alpha \in \mathbb{H}^{2\times 1}\right\},$$
$$\mathfrak{m}_1 = \left\{\begin{bmatrix} & & \\ & & c \\ & -\bar{c} & 0 \end{bmatrix} \mid c \in \mathbb{H}\right\}.$$

By direct matrix computation, we find that conditions (I) and (II) are satisfied with

$$c = 1/2, \quad c_0 = 2, \quad c_1 = 2.$$

Together with $n_0 = 8$ and $n_1 = 2$, we find that $\Delta = 64$ and the ODE (2.12) has two special solutions: $\phi_1(s) = 1$ and $\phi_2(s) = 1+s$. The other solutions are non-Riemannian, and we cannot make them explicit.

This example can be generalized to $Sp(n+2)/(Sp(1) \times Sp(1) \times Sp(n))$. The determination of the constants c, c_0, and c_1 is direct via matrix computation, so we will omit the details.

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