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A relative filtration index and fibers of normal primes in extensions of finite type

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Abstract. For an extension $R \hookrightarrow A$ of commutative Noetherian rings the behavior of the associated morphism of topological spaces Spec $A \to$ Spec R is often measured by its behavior on each of its fibers. Specifically, one studies the 'splitting' (or 'branching') and the 'ramification' that occurs in each fiber. In the classical constructions of faithfully flat analytic extensions (e.g., completion or Henselization) of excellent local rings the splitting and ramification properties are fairly well understood; see EGA IV [6, 18.10], Nagata [13, Sect. 37] or Raynaud [15, Ch. IX]. The strongest results are usually achieved for fibers over a 'normal point' of Spec R, that is, over $\mathfrak{p} \in$ Spec R such that $R/\mathfrak{p}$ is a normal domain [e.g., the property of a normal prime $\mathfrak{p}$ in a local ring to be 'unibranched', i.e., the Henselization of $R/\mathfrak{p}$ is a (normal) domain].

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Introduction

Our attention in this article will be directed at module finite extensions of normal local domains. Here the ramification behavior for the most part has been sorted out (e.g., see Bourbaki [2, VI. 8] and Fossum [3, Ch. IV]). As a result, our central focus will concern the splitting behavior (or lack of it) of prime ideals. As perhaps suggested by the discussion in the previous paragraph, the best results will occur in the case of normal prime ideals in the base ring. To be specific, let $R \hookrightarrow A$ denote a module finite ring extension of excellent normal *local* domains for which the induced extension of fraction fields is separable. The question we address in Section 2 is: for $\mathfrak{p} \in$ Spec R under what circumstances can one expect a unique $P \in$ Spec A to contract to $\mathfrak{p}$? As we demonstrate in Corollary 2.2 an affirmative answer occurs surprisingly often for $\mathfrak{p}$ normal and depends, in part, on the requirement that $\mathfrak{p} \subseteq m_R^{\nu}$, for some positive integer ν which itself depends only on the extension $R \hookrightarrow A$. As a corollary (2.3) we extract for R regular a kind of weak 'Bertini principle' for prime ideals P of codimension one which contract

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to normal primes $fR = \mathfrak{p} \subseteq m_R^{\nu}$; in particular, $P = fA$ and so f represents an irreducible hypersurface in A.

In Section 1 we set the stage for (the theoretical) computation of the integer ν, described above, as a consequence of studying the behavior of 'norms' of elements in extensions of analytically normal rings for which a Galois group is present. The crucial technique rests on the interplay of the 'strong approximation property' of analytic local rings together with the Artin–Reese Lemma. The technique allows us to conclude that ideals in the local ring A which contract *deeply* into m_R, the maximal ideal of R, must be predictably *deep* inside the maximal ideal m_A of A. In fact, we actually construct in a theoretical sense a 'relative filtration index' which gives to some extent a measure of this phenomenon.

In Section 3 we consider a variant on Hironaka's Lemma which is basic to the theory of flat families parametrized by a nonsingular curve (see [7, p. 264]). After a slight adjustment in order to avoid one 'bad' case, we are able to show: if $\mathfrak{p}$ is a prime ideal of codimension one in the complete integral domain A with perfect residue field such that $A/\mathfrak{p}$ is normal, $\mathfrak{p} = tA + \mathfrak{p}^{(2)}$ and $\mathfrak{p}$ is locally principal in codimension $\leqslant 2$, then $\mathfrak{p} = tA$ and A is normal. In turn this provides us with a significant reduction in hypotheses for a result of Huneke [9, Prop. 2.3 (part (i))] concerning complete intersections. Huneke's result provides sufficient conditions on a prime $\mathfrak{p}$ of codimension one in order that the local normal domain A be a factorial complete intersection. The basic ingredients of our result (Theorem 3.7) require only that $A/\mathfrak{p}$ be a factorial complete intersection, $\mathfrak{p}$ be locally principal in codimension three and that $\mathfrak{p}/\mathfrak{p}^{(2)}$ be reflexive as an $A/\mathfrak{p}$-module (thus omitting the Gorenstein hypothesis on A and the reflexive hypothesis on each $\mathfrak{p}^n/\mathfrak{p}^{n+1}$).

0. Remarks on terminology and notation

Let $R \hookrightarrow A$ be a module finite extension of normal domains. When convenient we use the notation 'A/R' to signify the same setup. Usually our extensions will be *generically separable*, which means that the induced extension of fraction fields is separable. If $\mathfrak{p} \in \operatorname{Spec} R$, then the fiber over $\mathfrak{p}$ can be identified with Spec $(k(\mathfrak{p}) \otimes_R A)$, which in our case is finite. A prime $P \in \operatorname{Spec} A$ which contracts to $\mathfrak{p}$ (so P represents a point in the fiber over $\mathfrak{p}$) is said to *split* provided $|\operatorname{Spec}(k(\mathfrak{p}) \otimes_R A)| > 1$. The same prime P is said to be *unramified* over R provided

(i) $\mathfrak{p}A_P = PA_P$ and
(ii) $k^B(\mathfrak{p}) \hookrightarrow k^A(P)$ is a separable field extension.

In case the module finite extension A/R is generically Galois, that is, the induced extension of fraction fields $K_R \hookrightarrow K_A$ is not only separable but in fact a Galois extension, then more can be said concerning the splitting and ramification of P in Spec A. Let G denote the Galois group of K_A/K_R. Then G acts on A so that $A^G = R$. The *splitting group* H for P is defined as $H = \{\sigma \in G | \sigma(P) = P\}$

while the *splitting ring* $B = A^H$. Key properties associated with the extensions $R \hookrightarrow B \hookrightarrow A$ are

(i) $R/\mathfrak{p} \hookrightarrow B/P'$ is birational where $P' = P \cap B$.
(ii) P' is unramified over R.
(iii) P is the unique prime in Spec A which contracts to P'. (See [13, pp. 158, 159]).

We remark that only finitely many such splitting rings are available as intermediate normal domains since they are in one-to-one correspondence with the subgroups H of G. In addition, our assumption of generic separability for the extension A/R allows us to gain the advantages of a generically Galois extension by simply embedding A/R as an intermediate extension in a generically Galois extension S/R, where K_S/K_R represents the *Galois closure* of K_A/K_R. In case R is local and Henselian, then of course both A and S will be local also. With regard to obtaining extensions A/R within this context for which R is an analytic regular local ring, it suffices (when confronted with prime characteristic) that A be a local analytic k-algebra, where k is a perfect field (see Scheja–Storch [16] for details).

For other unexplained terminology and notation we refer the reader to Matsumura's book [12].

1. Strong approximation, Artin–Reese Lemma and the relative filtration index

Let (R, m) be a local ring and let R^N denote the free R-module of rank $N > 0$. In the tradition of [14, 17] the ring R is said to satisfy the *strong approximation property*, if for any positive integer N and polynomials $f_1, \dots, f_t$ in $R[X_1, \dots, X_N]$, there exists a function $s\colon \mathbb{N} \to \mathbb{N}$ (natural numbers), with $s(n) \geqslant n$ for each n, such that: for every $x = \langle x_1, \dots, x_N \rangle \in R^N$ satisfying $f_i(x_1, \dots, x_N) \equiv 0 \bmod m^{s(n)}$ for each i, there is $x' \in R^N$ satisfying $x' \equiv x \bmod m^n$ and $f_i(x'_1, \dots, x'_N) = 0$, for $i = 1, \dots, t$. The class of strong approximation rings is the same as the class of rings which satisfy Artin-approximation and includes the class of complete local rings as well as local analytic k-algebras where k is a perfect field (see [14, pp. 146, 148]).

Our use of strong approximation in ring extensions $R \hookrightarrow A$ will be to recover containment in ideal powers in A (e.g., $I \subseteq m_A^{\ell}$) from knowledge of 'similar' containments for contractions of these ideals in R (e.g., $I \cap R \subseteq m_R^{s(\ell)}$). Our first lemma is the key to our success in this endeavor. We remark that a module finite extension of a strong approximation ring is again a strong approximation ring (see [17, 4.2]).

LEMMA 1.1. *Let S/R be a generically Galois extension of local normal domains which are strong approximation rings. Let $f(X_1, \dots, X_d) = X_1 \cdots X_d \in$*

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$S[X_1,\dots,X_d]$, where $d=[S:R]$. Further, let $s:\mathbb{N}\to\mathbb{N}$ be the strong approximation function for f. For each $\ell\in\mathbb{N}$ there is a $\nu=s(\ell)\in\mathbb{N}$ such that, if $a\in S$ and if its norm satisfies

$$n(a)=\prod_{\sigma\in G}\sigma(a)\in m_R^\nu,$$

then $a\in m_S^\ell$.

Proof. Let $G=\{\sigma_1,\dots,\sigma_d\}$ denote the Galois group for S/R, i.e., $S^G=R$. We first note that $n(a)=f(\sigma_1(a),\dots,\sigma_d(a))$. Next we consider the possibility that $a\notin m_S^\ell$. Then each coordinate of the vector $v=\langle\sigma_1(a),\dots,\sigma_d(a)\rangle$ is not in m_S^ℓ. However, the coordinates of v represent a solution to $f(X)\equiv 0 \bmod m_S^\nu$, since of course $m_R^\nu\subseteq m_S^\nu$. By the strong approximation property there are $a_i\in S$ such that $a_i\equiv\sigma_i(a)\bmod m_S^\ell$ and such that $f(a_1,\dots,a_d)=a_1\dots a_d=0$ in S. However, each $a_i\neq 0$ since $a_i\equiv\sigma_i(a)\bmod m_S^\ell$ and since $\sigma_i(a)\notin m_S^\ell$ for each i. The fact that S is a domain coupled with the equation $f(a_1,\dots,a_d)=a_1\dots a_d=0$ yields a contradiction.

COROLLARY 1.2. (Notation as in 1.1): *Suppose that J is an ideal in S such that $J\cap R\subseteq m_R^\nu$. Then $J\subseteq m_S^\ell$.*

Proof. Let $a\in J$. Then $n(a)\in J\cap R\subseteq m_R^\nu$ which gives that $a\in m_S^\ell$.

Once again we consider a generically Galois extension S/R (see Sect. 0 for a definition) as above and consider one of the finitely many intermediate normal subrings B; so $B=S^H$ for some subgroup H of the Galois group G. Regarding the containment $B\subseteq S$ as a containment of R modules there is a positive integer $\kappa>0$ such that

$$(m_R^\ell S)\cap B=m_R^{\ell-\kappa}(m_R^\kappa SB)$$

for $\ell\geqslant\kappa$. The minimum such κ is referred to as being the Artin–Reese number for $B\subseteq S$ with respect to the ideal m_R. Actually for most rings in our context we could replace the R-ideal m_R by any R-ideal I and find a bound κ which works for all of the containments simultaneously, i.e., κ depends only on the R-module containment $B\subseteq S$. For details and more on the subject one should consult Huneke's article [10] on 'uniform bounds'.

THEOREM 1.3. *Let S/R be a generically Galois extension of strong approximation rings and let κ denote the maximum of the Artin–Reese numbers for the finitely many intermediate normal subrings. Given a natural number t choose ℓ such that $m_S^\ell\subseteq m_R^{\kappa+t}S$. If J is an ideal of S such that $J\cap R\subseteq m_R^\nu$, where $\nu=s(\ell)$, and if B is an intermediate normal subring, then for $\mathfrak{j}=J\cap B$ one has $\mathfrak{j}\subseteq m_R^tB$.*

Proof. From Corollary 1.2 we have that $J\subseteq m_S^\ell$. Therefore

$$\mathfrak{j}=J\cap B\subseteq m_S^\ell\cap B\subseteq m_R^{\kappa+t}S\cap B\quad\text{and}$$

$$(m_R^{\kappa+t}S)\cap B=m_R^t(m_R^\kappa SB)\subseteq m_R^tB.$$

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In future sections we will refer to the positive integer ℓ which occurs in the setup of Theorem 1.3 as the *relative filtration index of order* t and we denote it by $\ell_t = \ell_t(S/R)$. This index, which depends upon the extension $R \hookrightarrow S$, refers to the fact that an ideal J in S of which the contraction to R satisfies $J \cap R \subseteq m_R^{s(\ell)}$ must have the property that $J \cap B \subseteq m_R^t B$ for each of the finitely many intermediate normal subrings B (including S itself). Thus, this relative index to some extent measures the control one has over the 'order of vanishing' of the contracted ideal at m_R versus the order of vanishing that occurs at $m_R B$, for each intermediate B. Finally, one could pursue this same path with respect to other R-ideals but we have no need of this here.

2. How the relative filtration index affects the splitting of primes

Let $R \to S$ be a homomorphism of commutative Noetherian rings and let $P \in \operatorname{Spec} S$ have 'contraction' $\mathfrak{p} \in \operatorname{Spec} R$. Then the splitting of P (this would be called 'branching' of P with respect to the morphism $\operatorname{Spec} S \to \operatorname{Spec} R$ in the parlance of algebraic geometry) is measured by the 'size' of the fiber over $\mathfrak{p}$. Within our context this fiber is always finite. Our next result provides a criterion that insures this cardinality must be one, that is, insures that P is the unique prime in S which contracts to $\mathfrak{p}$.

THEOREM 2.1. *Let S/R be a generically Galois extension of excellent normal local domains. Then there is a positive integer ν such that, if $P \in \operatorname{Spec} S$ contracts to a normal prime $\mathfrak{p} = P \cap R$ and if $\mathfrak{p} \subseteq m_R^\nu$, then P is fixed by the Galois group of S/R and hence is the unique prime in Spec S which contracts to $\mathfrak{p}$.*

Proof. We first pass to the induced extension of completions $\hat{R} \hookrightarrow \hat{S}$ with respect to the maximal ideal topologies. The extension $\hat{S}/\hat{R}$ is also a generically Galois extension of normal local domains since the rings are excellent. Moreover the completion of the prime ideal $\mathfrak{p}$ in Spec R remains a prime ideal in $\hat{R}$ since $R/\mathfrak{p}$ is an excellent normal local domain. It suffices to show that there is a positive integer ν such that, if $Q \in \operatorname{Spec} \hat{S}$ has contraction $\hat{\mathfrak{p}} = Q \cap \hat{R}$ and if $\hat{\mathfrak{p}} \subseteq m_{\hat{R}}^\nu$, then Q is the unique prime ideal in Spec $\hat{S}$ which contracts to $\hat{\mathfrak{p}}$ (for then the contraction of Q to S will give $P = Q \cap S$ as the unique prime in Spec S with contraction $\mathfrak{p} = P \cap R$). Thus we may actually assume that $R \hookrightarrow S$ is an extension of complete local domains from the outset.

Under the assumption of completeness of S/R we have available the strong approximation property. Therefore, let $\ell = \ell_1(S/R)$, the filtration index of Theorem 1.3, and put $\nu = s(\ell)$ where $s\colon \mathbb{N} \to \mathbb{N}$ is the strong approximation function of Lemma 1.1. Next let B denote the intermediate normal subring of S/R which corresponds to the splitting group for the prime ideal P in Spec S which contracts to $\mathfrak{p}$. Let $\mathfrak{p}' = P \cap B$. We recall from our discussion in Section 0 that the induced extension $R/\mathfrak{p} \hookrightarrow B/\mathfrak{p}'$ is birational. However, the normality of $R/\mathfrak{p}$ gives that this induced ring homomorphism $R/\mathfrak{p} \hookrightarrow B/\mathfrak{p}'$ is in fact an isomorphism.

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This observation further yields that $B = R + \mathfrak{p}'$ (as R-modules). At this point we recall from our definition of the relative filtration index and Theorem 1.3 that $P \cap B = \mathfrak{p}' \subseteq m_R B$. This gives the containments $B = R + \mathfrak{p}' \subseteq R + m_R B \subseteq B$. It follows that $B = R + m_R B$ from which it follows that $B = R$ after an application of Nakayama's Lemma. However, the only reconciliation possible in a case where the splitting ring $B = R$ is that the splitting group H must be the entire Galois group, i.e., the prime P is fixed by the action of the Galois group of S/R.

COROLLARY 2.2. *Let A/R be a module finite extension of excellent local normal domains which is generically separable. Then there is a positive integer ν such that, if $P \in$ Spec A contracts to a normal prime $\mathfrak{p} = PR \subseteq m_R^\nu$, then P is the unique prime in Spec A which contracts to $\mathfrak{p}$.*

Proof. Since A/R is module finite and generically separable, the extension can be embedded (or enlarged) to a generically Galois extension S/R. Choose ν as in Theorem 2.1 for S/R. If $P \in$ Spec A has contraction $\mathfrak{p} = PR \subseteq m_R^\nu$, let $Q \in$ Spec S such that $Q \cap R = P$. From Theorem 2.1 the prime Q will be fixed by the Galois group of S/R; thus $P = Q \cap A$ is also unique.

In the following we give a typical application of Theorem 2.1 and Corollary 2.2. Let A be a normal local analytic algebra over a perfect field k or a complete local normal domain (with perfect residue field if A is of equal characteristic $p > 0$). Then A is a module finite extension of a regular local ring R such that A/R is generically separable (see discussion in Section 0 and specifically Scheja–Storch [16]). From Corollary 2.2 we obtain the existence of a positive integer ν so that, if $P \in$ Spec A contracts to a normal prime $\mathfrak{p} \subseteq m_R^\nu$, then P is the unique prime in Spec A which contracts to $\mathfrak{p}$ i.e., $\sqrt{\mathfrak{p}A} = P$. From this setup we deduce a rather weak theorem of 'Bertini type'. We remind the reader that the ramified primes in a module finite extension (in our context) of codimension one are finite in number.

THEOREM 2.3. *Let A be a local normal domain that is a module finite extension of an excellent regular local ring R and suppose that the extension is generically separable* (*see preceding discussion for existence of these*). *Then there is a positive integer ν such that, if $P \in$ Spec A is unramified of codimension one and if $P \cap R = fR$ is a normal prime in m_R^ν, then $P = fA$.*

Proof. From Corollary 2.2, it follows that P is the unique prime in Spec A which contracts to the principal prime $\mathfrak{p} = fR \in$ Spec R. Therefore $P = \sqrt{fA}$. However, f surely generates P locally in codimension one on Spec A. Since A is normal it follows that $P = fA$.

In order to illustrate part of the point of Theorem 2.3, let **C** denote the field of complex numbers and let $R = \mathbf{C}[[X_1, \ldots, X_n]]$ where $n \geqslant 3$. Suppose that A/R represents a module finite extension of R in which A is normal. Then, for some positive integer ν depending on A/R, the polynomials $\lambda_1 X_1^\ell + \lambda_2 X_2^\ell + \cdots + \lambda_n X_n^\ell$ in R must represent principal primes in A for $\ell \geqslant \nu$ and $\lambda_1 \lambda_2 \ldots \lambda_n \neq 0 (\lambda_i \in \mathbf{C})$.

We end this section by further analyzing the proof of Theorem 2.1 in the case of prime ideals of codimension one. To this end let P denote a prime ideal in S of

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codimension one. Keeping with the notation of Theorem 2.1, we assume that S/R is a generically Galois extension; in addition, we assume that R is *regular local* here (as in 2.3). Let B denote the splitting ring of P with respect to the Galois group for the extension S/R. Let $\mathfrak{p}' = B \cap P$ and $tR = R \cap P$ be the contractions of P to B and R, respectively. A crucial point in the proof of Theorem 2.1 is that $B = R + \mathfrak{p}'$ since the extension $R/tR \hookrightarrow B/\mathfrak{p}'$ is birational and thus equal in case R/tR is normal. Let $T = B/(R + \mathfrak{p}')$. Then $0 \to R/tR \to B/\mathfrak{p}' \to T \to 0$ is an exact sequence of R-modules and $ht_R(annT) \geqslant 2$. Moreover, because R/tR is Cohen–Macaulay one sees that $R/tR \hookrightarrow B/\mathfrak{p}'$ is an isomorphism if and only if this map is an isomorphism in codimension one (over R/tR) – or what amounts to the same thing – if and only if the R-module T has no associated primes of codimension two. Looking at this remark from the point of view of the R-short exact sequence $0 \to R+\mathfrak{p}' \to B \to T \to 0$ one further observes that the preceding statements are equivalent to the property that $R + \mathfrak{p}' \to B$ is an R-isomorphism in codimension two which is equivalent to the statement

$$R + \mathfrak{p}' \quad \text{is reflexive as an } R\text{-module.} \tag{$*$}$$

Statement $(*)$ has yet another interpretation. We consider the R-exact sequence

$$0 \to R \xrightarrow{i} R \oplus \mathfrak{p}' \xrightarrow{j} R + \mathfrak{p}' \to 0 \tag{$**$}$$

where $i(1) = (-t, t)$ and $j(r, \pi) = r + \pi$. Now $(*)$ holds if and only if the short exact sequence $(**)$ splits in codimension two. This of course has implications as to how the element t sits in $\mathfrak{p}'$ when the latter is viewed as an R-module. Noting that $R \oplus \mathfrak{p}'$ is a free R-module in codimension two, one observes that $(-t, t)$ will generate a free R-summand of $R \oplus \mathfrak{p}'$ in codimension two if and only if the order ideal of t in $\mathfrak{p}'$ (as an R-module) has grade at least three. This may be deduced from the splitting (in codimension $\leqslant 2$) of the bottom row in the commutative diagram

$$\begin{tikzcd} O \arrow[r] & R \arrow[r] \arrow[d, "t"'] & \mathfrak{p}' \arrow[d, "\cap" description] \arrow[dl] \\ & R \arrow[r, hook] & B \end{tikzcd} \qquad \text{where } 1 \mapsto t \in \mathfrak{p}'$$

showing that the element t belongs to the order ideal $0_{\mathfrak{p}'}(t)$ in codim $\leqslant 2$. Therefore, a prime ideal P of height two in Spec R will contain $0_{\mathfrak{p}'}(t)$ if and only if $P \supseteq (0_{\mathfrak{p}'}(t), t)$, that is, if and only if the order ideal of $(-t, t)$ in $R \oplus \mathfrak{p}'$ is contained in P. Thus we have found a third equivalent statement, namely,

$$\text{grade}_R 0_{\mathfrak{p}'}(t) \geqslant 3 \tag{$***$}$$

Returning to the original prime $P \in \operatorname{Spec} S$ one further observes that the condition

$$\operatorname{grade}_R 0_P(t) \geqslant 3 \qquad (* * **)$$

implies $(* * *)$ and hence also implies $(**)$ and $(*)$. In this setup as described above (and in [13, p. 159]) one has that $P = \sqrt{\mathfrak{p}' A}$; so if P is unramified over R, then one has $P = \mathfrak{p}' \otimes_B A$ *locally* in codimension two. It follows that $(* * *)$ and $(* * **)$ are equivalent in case P is unramified. We summarize these observations in the following statement.

THEOREM 2.4. *Let R be an excellent regular local ring and let A/R be a generically Galois extension in which A is local and normal. Let $P \in$ Spec A have codimension one and let $P \cap R = tR$. Then there is a positive integer ν such that, if P is unramified, grade$_R 0_P(t) \geqslant 3$ and if $t \in m_R^\nu$, then $P = tA$.*

The results which have just been presented exhibit a tendency for primes P which contract to normal primes in a generically Galois extension A/R of local normal domains to have ‘stability’ in the sense of being nonsplit. Of course this is not always the case. In particular, the situation may occur in which a prime $P \in$ Spec A contracts to a normal prime $\mathfrak{p} \in$ Spec R for which P is ‘maximally split’ (i.e., the splitting group for P consists of the identity). For example, if

$$A = \frac{k[[X, Y, Z, W]]}{(XY - ZW)} = k[[x, y, z, w]], \qquad R = k[[x, y, t]],$$

where char $k \neq 2$ and $t = 1/2(z - w)$, then A/R is a quadratic, generically Galois extension in which $P = (x, z) \in$ Spec A contracts to $\mathfrak{p} = xR$. The prime P is ‘maximally split’ with $Q = (x, w)$ being the other prime in the fiber over xR. Moreover, as the proof of Theorem 2.1 predicts, $A = R + P$ (as R-modules) and $k[[y, t]] \cong R/xR \cong A/P$ (see Example 3.1 for more on the ring A). The occurrence of a codimension one prime ideal P in a local normal domain A with A/P a complete intersection is the central topic in the forthcoming section.

3. A variant on Hironaka’s Lemma

In this section we consider the situation in which A is an excellent local domain and $\mathfrak{p} \in$ Spec A is a normal prime ideal with the property that $\mathfrak{p}/\mathfrak{p}^{(2)} \cong A/\mathfrak{p}$ as $A/\mathfrak{p}$-modules. This situation arises in its simplest form when $A/\mathfrak{p}$ is factorial $(= UFD)$ and $\mathfrak{p}/\mathfrak{p}^{(2)}$ is reflexive as an $A/\mathfrak{p}$-module. Also it occurs within the framework of Hironaka’s Lemma as we illustrate below.

Before getting into the essential details of this section, we make some observations concerning two classical examples. These observations will have a direct impact on our upcoming strategy for determining appropriate circumstances under which a prime ideal $\mathfrak{p}$ must be principal.

EXAMPLE 3.1. Let k denote a field which is not of characteristic 2.

(a) Let $A = k[[X, Y, X, W]]/(XY - ZW)$ and let lower case letters denote the cosets of X, Y, Z and W, respectively. The ideal $\mathfrak{p} = (x, z)$ is a prime ideal in A of codimension one such that $A/\mathfrak{p} \cong k[[Y, W]]$ is regular local. However, it is well known that $\mathfrak{p}^2 = \mathfrak{p}^{(2)} = (x^2, xz, z^2)$ (see [4, 8]) and that $\mathfrak{p}/\mathfrak{p}^{(2)}$ is a two-generated ideal in $A/\mathfrak{p}$; therefore $\mathfrak{p}/\mathfrak{p}^{(2)}$ is *not* reflexive.

Thus, in reference to our aforementioned strategy, we shall often impose the reflexive condition on $\mathfrak{p}/\mathfrak{p}^{(2)}$ (e.g., see 3.4 and 3.7).

(b) Let $A = k[[X, Y, Z]]/(Z^2 - XY)$ and let $\mathfrak{p} = (x, z)$. As in (a) the prime ideal $\mathfrak{p}$ has the property that $A/\mathfrak{p} = k[[Y]]$ is regular. Moreover, $\mathfrak{p}/\mathfrak{p}^{(2)}$ is principal and generated by the coset of Z. Thus A is a 2-dimensional normal local domain and $\mathfrak{p}$ is a nonprincipal prime ideal of codimension 1 such that $\mathfrak{p}/\mathfrak{p}^{(2)} \cong A/\mathfrak{p}$. The hypothesis in our main result of this section (see Theorem 3.4) avoids this situation.

LEMMA 3.2. *Let A be a local domain which is a strong approximation ring and suppose that A is a module finite and generically separable extension of a regular local ring R. Let $\mathfrak{p}$ be a prime ideal of A of codimension one. For any positive integer t we have the ideal containment $\mathfrak{p}^{(n)} \subseteq m_A^t\mathfrak{p}$ for $n \gg 0$.*

Proof. Let B denote the integral closure of A. Since A is a local strong approximation ring and since B is finite over A, then B is a local strong approximation ring as well. Let $q \in \operatorname{Spec} B$ be such that $\mathfrak{p} = A \cap q$. Then $\mathfrak{p}^{(n)} \subseteq q^{(n)}$ for each n. Of course the extension B/R may be embedded into a generically Galois (and module finite) extension S/R. Moreover, there is a codimension one prime ideal $\tilde{q} \in \operatorname{Spec} S$ such that $\tilde{q} \cap B = q$.

Now invoking Theorem 1.3 as well as the discussion which immediately follows that theorem, one sees that the relative filtration index, $\ell = \ell_t(S/R)$, of order t for S/R gives that $q^{(n)} \subseteq m_R^tB$ for $n \gg 0$, since $\tilde{q}^{(n)} \cap R = Rf^n$, where $\tilde{q} \cap R = Rf$, and since eventually $Rf^n \subseteq m_R^\nu$ where $\nu = s(\ell)$. Thus $\mathfrak{p}^{(n)} \subseteq m_R^tB \cap \mathfrak{p}$ for $n \gg 0$. By choosing t large enough to accommodate the Artin–Reese number κ for the R-module containment $\mathfrak{p} \subseteq B$ with respect to the m_R-adic filtration, one obtains the containments

$$\mathfrak{p}^{(n)} \subseteq m_R^tB \cap \mathfrak{p} \subseteq m_R^{t-\kappa}(m_R^\kappa B \cap \mathfrak{p})$$

that is, $\mathfrak{p}^{(n)} \subseteq m_R^{t-\kappa}\mathfrak{p}$, for $n \gg 0$. By choosing the initial t appropriately large as compared with the fixed integer κ we can assure that $t - \kappa$ can be made as large as any prescribed positive integer. Of course, if $\mathfrak{p}^{(n)} \subseteq m_R^{t-\kappa}\mathfrak{p}$, then $\mathfrak{p}^{(n)} \subseteq m_A^{t-\kappa}\mathfrak{p}$.

Remark 3.3. The hypotheses in a typical version of Hironaka's Lemma can be stated as follows: Let A be a local excellent domain and t be an element in A so that

(i) t is contained in a unique minimal prime $\mathfrak{p}$;
(ii) t generates $\mathfrak{p}$ locally at $A_\mathfrak{p}$;
(iii) $A/\mathfrak{p}$ is a normal domain,

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then $\mathfrak{p} = At$ and A is a normal domain. In our version we present here, we omit condition (i) entirely at the cost of strengthening condition (ii) to require that $\mathfrak{p}$ be principal locally in codimension two and that $\mathfrak{p}$ be generated by 't' modulo $\mathfrak{p}^{(2)}$. Moreover, if one is willing to grant the validity of Hironaka's Lemma for dimension $\leqslant 2$, then the two versions become equivalent.

THEOREM 3.4. *Let (A, m_A, k) be a complete local domain with perfect residue field k. Suppose that $\mathfrak{p}$ is a prime ideal in A of codimension 1 such that $A/\mathfrak{p}$ is a normal domain and such that $\mathfrak{p}/\mathfrak{p}^{(2)} \cong A/\mathfrak{p}$ as $A/\mathfrak{p}$-modules. If $\mathfrak{p}$ is locally principal in codimension 2, then $\mathfrak{p}$ is principal and A is normal.*

Proof. We may suppose that dim $A \geqslant 3$ and that A is a module finite and generically separable extension of a complete regular local ring R. Also note that $\mathfrak{p}/\mathfrak{p}^{(2)} \cong A/\mathfrak{p}$ gives that $\mathfrak{p} = tA + \mathfrak{p}^{(2)}$ and that t generates $\mathfrak{p}$ on the locus of primes Q such that $\mathfrak{p} \subseteq Q$ and $\mathfrak{p}_Q$ is a principal ideal.

Since $\mathfrak{p} = tA + \mathfrak{p}^{(2)}$ and since $A/\mathfrak{p}$ is normal, we obtain a short exact sequence

$$0 \to \mathfrak{p}/\mathfrak{p}^{(2)} \to A/\mathfrak{p}^{(2)} \to A/\mathfrak{p} \to 0.$$

It follows that $A/\mathfrak{p}^{(2)}$ satisfies the Serre condition S_2 (as a ring) since both $A/\mathfrak{p}$-modules on either end do (as $A/\mathfrak{p}$-modules). By means of induction we intend to argue that $\mathfrak{p} = tA + \mathfrak{p}^{(n)}$ and that $A/\mathfrak{p}^{(n)}$ is S_2 (as a module over itself) for $n > 0$. We consider the induction step $A/\mathfrak{p}^{(n+1)}$.

Viewing the short exact sequence

$$0 \to \frac{[(t) + \mathfrak{p}^{(n+1)}]}{\mathfrak{p}^{(n+1)}} \to \frac{A}{\mathfrak{p}^{(n+1)}} \to \frac{A}{(t, \mathfrak{p}^{(n+1)})} \to 0, \qquad (**)$$

we observe that the first term is isomorphic to $A/\mathfrak{p}^{(n)}$ since $\mathrm{Ass}(A/\mathfrak{p}^{(n)}) = \{\mathfrak{p}\}$ and since $\mathfrak{p}A_{\mathfrak{p}} = (t)$. Thus the first term is an S_2-module over $A/\mathfrak{p}^{(n+1)}$ by our induction hypothesis. Now consider $P \in \mathrm{Ass}(\mathfrak{p}/(t, \mathfrak{p}^{(n+1)}))$. Then $\mathfrak{p} \subseteq P$ and $htP \geqslant 3$ since $\mathfrak{p}$ is locally principal in codimension 2. Therefore localizing $(*)$ at P gives a 3 term sequence of depths (beginning from the left) $\geqslant 2, \geqslant 1, 0$, respectively. But this shows that the term $(\mathfrak{p}/(t, \mathfrak{p}^{(n+1)}))_P$ must be zero. It follows that the $\mathrm{Ass}(\mathfrak{p}/(t, \mathfrak{p}^{(n+1)})) = \emptyset$ and that $\mathfrak{p} = tA + \mathfrak{p}^{(n+1)}$. Moreover, we obtain a short exact sequence

$$0 \to \mathfrak{p}/\mathfrak{p}^{(n+1)} \to A/\mathfrak{p}^{(n+1)} \to A/\mathfrak{p} \to 0$$

in which the terms $\mathfrak{p}/\mathfrak{p}^{(n+1)} \cong A/\mathfrak{p}^{(n)}$ and $A/\mathfrak{p}$ satisfy the S_2-condition as $A/\mathfrak{p}^{(n+1)}$-modules. Hence so does $A/\mathfrak{p}^{(n+1)}$ satisfy the S_2-condition. This completes the required induction argument.

In order to complete our argument we simply appeal to Lemma 3.2 in regard to the module finite and generically separable extension A/R which was noted at the beginning of our proof. The result of Lemma 3.2 that $\mathfrak{p}^{(n)} \subseteq m_A\mathfrak{p}$, for $n \gg 0$, in conjunction with the fact that $\mathfrak{p} = tA + \mathfrak{p}^{(n)}$, for $n > 0$, yields that $\mathfrak{p} = At$ via Nakayama's Lemma. It follows that A is normal since A/tA is normal.

We note that one may delete the completeness portion of the hypothesis if one is willing to concede the normality of A.

COROLLARY 3.5. *Let A be an excellent local normal domain with perfect residue field and suppose that $\mathfrak{p}$ is a normal prime ideal of codimension* 1 *such that $\mathfrak{p}/\mathfrak{p}^{(2)} \cong A/\mathfrak{p}$. If $\mathfrak{p}$ is locally principal in codimension* 2 *then $\mathfrak{p}$ is principal.*

Proof. Since A is excellent and normal with perfect residue field, the completion $\hat{A}$ of A is a normal local domain which is a module finite and generically separable extension of a complete regular local ring R. We are now in a position to apply Theorem 3.4.

In our final remarks of the section we indicate how Theorem 3.4 and Corollary 3.5 provide certain criteria for a local normal domain to be a complete intersection. We start with the following result.

THEOREM 3.6. *Let A be an excellent local normal domain with perfect residue field. Suppose that $\mathfrak{p}$ is a normal prime of A such that $A/\mathfrak{p}$ is a complete intersection. If $\mathfrak{p}/\mathfrak{p}^{(2)}$ is reflexive as an $A/\mathfrak{p}$-module and if $\mathfrak{p}$ is locally principal in codimension* 4, *then $\mathfrak{p}$ is principal and A is a complete intersection.*

Proof. Our hypotheses on $\mathfrak{p}$ and $\mathfrak{p}/\mathfrak{p}^{(2)}$ guarantee that $\mathfrak{p}/\mathfrak{p}^{(2)}$ is locally principal on Spec $A/\mathfrak{p}$ in codimension $\leqslant 3$. By Grothendieck's Theorem [5] the module $\mathfrak{p}/\mathfrak{p}^{(2)}$ is principal. The result of Corollary 3.5 allows us to conclude that $\mathfrak{p}$ itself is principal.

Our final theorem on this matter is a generalization of a result of Huneke [9, Prop. 2.3]. To be specific, we are able to delete the 'Gorenstein' hypothesis on the ring in question. Further, we replace the reflexivity on the modules '$\mathfrak{p}^n/\mathfrak{p}^{n+1}$' by the weaker requirement that the single module $\mathfrak{p}/\mathfrak{p}^{(2)}$ be reflexive. The price for getting away with these milder conditions is that we ask the residue field to be perfect.

THEOREM 3.7. *Let A be an excellent normal local domain with perfect residue field and suppose that $\mathfrak{p}$ is a normal prime ideal of codimension* 1 *such that $\mathfrak{p}/\mathfrak{p}^{(2)}$ is reflexive as an $A/\mathfrak{p}$-module. If $A/\mathfrak{p}$ is a factorial complete intersection and if A is factorial in codimension* 3, *then A is a factorial complete intersection.*

Proof. From Corollary 3.5 we determine that $\mathfrak{p}$ is principal and from Lipman's result [11] we get that A is necessarily factorial.

NOTED ADDED IN PROOF

The material in Section 3 of our paper appears as an application of the main result (Th. 1.3) of Section 1. However, the crucial Lemma 3.2 is actually true in greater generality as a consequence of Chevally's eliminates the need for the assumption of 'generic separability' in the statement of Lemma 3.2. As a result of the 'Chevally version' of Lemma 3.2 one may omit the hypothesis of 'perfect residue field' in

3.4–3.7. The author is indebted to his colleague Sankar Dutta for this simplifying observation.

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