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POSITIVE FORMS ON NUCLEAR $*$-ALGEBRAS AND THEIR INTEGRAL REPRESENTATIONS

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ABSTRACT. The main result of this paper establishes the existence and uniqueness of integral representations of KMS functionals on nuclear $*$-algebras. Our first result is about representations of $*$-algebras by means of operators having a common dense domain in a Hilbert space. We show, under certain regularity conditions, that (Powers) self-adjoint representations of a nuclear $*$-algebra, which admit a direct integral decomposition, disintegrate into representations which are almost all self-adjoint. We then define and study the class of self-derivative algebras. All algebras with an identity are in this class and many other examples are given. We show that if $\mathfrak{A}$ is a self-derivative algebra with an equicontinuous approximate identity, the cone of all positive forms on $\mathfrak{A}$ is isomorphic to the cone of all positive invariant kernels on $\mathfrak{A} \times \mathfrak{A}$. These in turn correspond bijectively to the invariant Hilbert subspaces of the dual space $\mathfrak{A}'$. This shows that if $\mathfrak{A}$ is a nuclear $\mathcal{LF}$-space, the positive cone of $\mathfrak{A}'$ has bounded order intervals, which implies that each positive form on $\mathfrak{A}$ has an integral representation in terms of the extreme generators of the cone. Given a continuous exponentially bounded one-parameter group of $*$-automorphisms of $\mathfrak{A}$, we can define the subcone of all invariant positive forms satisfying the KMS condition. Central functionals can be viewed as KMS functionals with respect to a trivial group action. Assuming that $\mathfrak{A}$ is a self-derivative algebra with an equicontinuous approximate identity, we show that the face generated by a self-adjoint KMS functional is a lattice. If $\mathfrak{A}$ is moreover a nuclear $\mathcal{LF}$ $*$-algebra the previous results together imply that each self-adjoint KMS functional has a unique integral representation by means of extreme KMS functionals almost all of which are self-adjoint.

1. Introduction. Positive forms on $*$-algebras and their integral representations by extreme generators are of considerable importance in Statistical Mechanics, Field Theory and Harmonic Analysis. For particular classes of functionals defined in terms of one-parameter groups of automorphisms one has obtained not only the existence but also the uniqueness of the representing measures. For instance, in Statistical Mechanics, the KMS states on appropriate C^*-algebras play an important role in the study of equilibrium as was shown by Haag, Hugenholtz and Winnink [**14**]. Their unique decomposition into extreme elements has been obtained by Ruelle [**24**]. For further references see [**7**: p.

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233–234]. Lanford and Ruelle [**17**] have obtained a similar result for asymptotically abelian functionals. On the other hand, the existence and uniqueness of Plancherel measures, equivalently the unique decomposition of L^2 spaces into irreducible components, is obviously of the very nature of Harmonic Analysis. The Bochner-Schwartz theorem and its various non-commutative analogues are examples of such results. In the case of Field Theory, theorems on the existence of integral representations have been obtained by Borchers and Yngvason [**4**], and Hegerfeld [**15**]. In this article we shall be interested in KMS functionals on a general class of topological $*$-algebras. KMS functionals on $*$-algebras of unbounded operators have been previously considered by Araki [**1**] and more recently by Fulling and Ruijsenaars [**11**].

The two examples of topological $*$-algebras which motivated us were the convolution algebra of Schwartz test functions on a Lie group and the Field algebra. They are both nuclear $\mathcal{LF}$-spaces but the convolution algebra has no unit while the Field Algebra does have one. We have not therefore systematically assumed the existence of a unit in the algebra. This lack however can be compensated by the presence of an approximate unit and the property of the algebra to be self-derivative (4.3).

In the case of topological $*$-algebras the GNS representation is in general realized by unbounded operators. Among the positive functionals on a $*$-algebra those for which the GNS representation is essentially self-adjoint in the sense of Powers [**22**] are particularly interesting. For convenience we have called such functionals self-adjoint.

To handle fields of unbounded operators we have found convenient to make a systematic use of Schwartz' theory of reproducing kernels [**25**]. We recall the relevant facts in 2.3.

Our main result is the theorem about the existence and uniqueness of integral representations of self-adjoint KMS functionals on self-derivative nuclear $\mathcal{LF}$ $*$-algebras having an equicontinuous approximate identity. As a particular case we obtain an integral representation for self-adjoint abelian (i.e. central) functionals on such $*$-algebras.

2. Preliminaries

2.1. Nuclear and $\mathcal{LF}$-spaces. We refer the reader to [**12**], [**13**] and [**21**] for the terminology and well-known properties of nuclear spaces and tensor products. Precise references will be given for deeper (or lesser known) results as they are used in the text. Let us warn the reader that our definition of an $\mathcal{LF}$-space is different from Grothendieck's definition. We say that a locally convex space E (always assumed Hausdorff), is an $\mathcal{LF}$-space if it is the *strict* inductive limit of a *sequence* $\{E_n\}$, of Fréchet spaces. We denote this by $E = \varinjlim_n E_n$. As noted in the introduction, the space of Schwartz test functions $\mathcal{D}(G)$, on a Lie group G, assumed countable at infinity, is an $\mathcal{LF}$-space.

There are three propositions which are part of the "folklore" that we would nonetheless like to state with a sketch of their proof because they are often used in the text.

PROPOSITION 2.1.1. *Let E, F and G be locally convex spaces such that E is a Fréchet space and F is metrizable. Then any separately continuous bilinear map $B : E \times F \longrightarrow G$ is continuous. In particular this is the case if both E and F are Féchet spaces.*

Proof. Let $(y_n)_{n\in\mathbf{N}}$ be a sequence converging to y in F. Then by the uniform boundedness principle $B(x, y_n)$ tends to $B(x, y)$ uniformly for x belonging to a compact subset of E. In particular, if $(x_n)_{n\in\mathbf{N}}$ converges to x in E, $B(x_k, y_n)$ converges to $B(x_k, y)$ uniformly with respect to k. This implies that $B(x_n, y_n)$ converges to $B(x, y)$. Thus, $E \times F$ being metrizable, B is continuous. □

PROPOSITION 2.1.2. *Let E or F be a nuclear space. Suppose also $E = \lim_{\overrightarrow{n}} E_n$ and $F = \lim_{\overrightarrow{n}} F_n$ with E_n, F_n Fréchet spaces. Then $E\bar{\otimes}F = \lim_{\overrightarrow{n}} E_n\hat{\otimes}F_n$. In particular if E and F are nuclear the space $E\bar{\otimes}F$ is nuclear.*

We will prove this proposition using the following well-known property of the ϵ-tensor product topology:

LEMMA 2.1.3. *Let E_1 be a linear topological subspace of the space F_1, equipped with the induced topology. Similarly $E_2 \subset F_2$. Let j and j' be the inclusion maps of E_1 in E_2 and of F_1 in F_2 respectively; they are injective linear topological homomorphisms. Then $j \otimes j'$: $E_1\hat{\otimes}_\epsilon F_1 \longrightarrow E_2\hat{\otimes}_\epsilon F_2$ is a one to one linear topological homomorphism, i.e. the first space may be regarded as a linear topological subspace of the second.*

Proof of 2.1.2. By [**13**: Produits tensoriels topologiques, Prop. 14, p. 76], $E\bar{\otimes}F = \lim_{\overrightarrow{n}} E_n\bar{\otimes}F_n$. By the lemma $E_{n_1}\hat{\otimes}_\epsilon F_{n_1}$ is a closed subspace of $E_{n_2}\hat{\otimes}_\epsilon F_{n_2}$ and hence it suffices to show that $E_n\bar{\otimes}F_n = E_n\hat{\otimes}F_n = E_n\hat{\otimes}_\epsilon F_n$. The first equality is a consequence of the fact that E_n and F_n are Fréchet spaces (cf. 2.1.1.). By assumption, E or F is nuclear. Let us assume it is E, then E_n being a (closed) subspace of E is also nuclear. Thus $E_n\hat{\otimes}F_n = E_n\hat{\otimes}_\epsilon F_n$ and the proposition is proved. □

PROPOSITION 2.1.4. *If E is a nuclear and $\mathcal{LF}$-space then E is separable.*

Proof. Let E be the strict inductive limit of a sequence E_n of Fréchet spaces. If E is nuclear then so are the subspaces E_n. If each E_n is separable so is E. Thus the proposition will follow if we prove that a nuclear and Fréchet space E is separable. Let $\{p_i\}_{n\in\mathbf{N}}$ be a fundamental family of seminorms on E. All the quotient maps π_i: $E \longrightarrow E_i := (E/p_i^{-1}(\{0\}))\hat{}$ are nuclear and therefore compact. Thus there exists a neighbourhood V in E such that $\overline{\pi_i(V)}$ is compact in E_i. Then $\pi_i(E) = \cup_{n\geq 0} n\overline{\pi_i(V)}$ is separable and so is its completion E_i. But E is isomorphic to a subspace of the countable product of the E_i which is also metrizable and separable. Therefore E itself is separable. □

2.2 $\mathcal{LF}$ $*$-algebras. Let $\mathfrak{A}$ be an algebra over **C**, equipped with an anti-linear involution $a \longmapsto a^*$ such that $(ab)^* = b^*a^*$. We assume that $\mathfrak{A}$ is a *topological $*$-algebra* i.e. an algebra equipped with a locally convex Hausdorff topology such that the product $(a, b) \longmapsto ab$ is separately continuous, and the involution continuous. $\mathfrak{A}$ will always be assumed to be barrelled. The cases of most interest to us will be when $\mathfrak{A}$ is in fact an $\mathcal{LF}$-space i.e. a strict inductive limit of a sequence of Fréchet spaces: $\mathfrak{A} = \lim\limits_{\vec{n}} \mathfrak{A}_n$. The defining spaces $\mathfrak{A}_n$ are then closed subspaces in $\mathfrak{A}$, not in general subalgebras. However;

PROPOSITION 2.2.1. *For every n and k in* **N**, *there exists $m \in$* **N** *such that* $\mathfrak{A}_n\mathfrak{A}_k \subset \mathfrak{A}_m$.

LEMMA 2.2.2. *Let F be a Fréchet space, and $u : F \longrightarrow E = \lim\limits_{\vec{n}} E_n$ a continuous linear map to any $\mathcal{LF}$-space. Then there exists an index n such that $u(F) \subset E_n$.*

Proof. The spaces E_n being closed in E, F is the countable union of closed subspaces $F = \cup_n u^{-1}(E_n)$. By Baire's theorem one of them has an interior point, hence equals the whole space F. □

Proof of 2.2.1. The product restricted to $\mathfrak{A}_n \times \mathfrak{A}_k$ is separately continuous, hence continuous ($\mathfrak{A}_n$ and $\mathfrak{A}_k$ being Fréchet spaces). Thus there exists a continuous linear map $u : \mathfrak{A}_n \hat{\otimes} \mathfrak{A}_k \longrightarrow \mathfrak{A}$ such that $u(x \otimes y) = x.y$. Applying the lemma to this map gives the result. □

Examples (1) $\mathfrak{A} = \mathcal{D}(G)$, G a Lie group: $\mathcal{D}_K(G)^*\mathcal{D}_H(G) \subset \mathcal{D}_{KH}(G)$.

(2) Let E be a locally convex space. Recall that the tensor algebra over E, is the locally convex direct sum:

$$T(E) = \mathbf{C} \oplus E \oplus (E\bar{\otimes}E) \oplus (E\bar{\otimes}E\bar{\otimes}E) \dots$$

The sum and the product by a complex scalar is defined componentwise. The product of two elements $x_1^k \otimes x_2^k \otimes \dots \otimes x_k^k \in E^{\otimes k}$ and $y_1^l \otimes y_2^l \otimes \dots \otimes y_l^l \in E^{\otimes l}$ is the element $x_1^k \otimes x_2^k \otimes \dots \otimes x_k^k \otimes y_1^l \otimes y_2^l \otimes \dots \otimes y_l^l$ of $E^{\otimes k} \times E^{\otimes l}$ obtained by concatenation. This product being bilinear and separately continuous extends to the completions and finally if $x = \{x_0, x_1, \dots, x_n, \dots\}$ and $y = \{y_0, y_1, \dots, y_n, \dots\} \in T(E)$ we define their product by $xy = \{x_0y_0, x_0y_1 + x_1y_0, \dots, \sum_{k+l=n} x_ky_l, \dots\}$. Clearly, if $\mathfrak{A}$ is the tensor algebra over a Fréchet space and $\mathfrak{A}_n$ denote the sum of the tensors of order $\leqq n$. Then $\mathfrak{A}_n\mathfrak{A}_k \subset \mathfrak{A}_{n+k}$. If E is Fréchet then $\mathfrak{A} = \lim\limits_{\vec{n}} \mathfrak{A}_n$ is therefore an $\mathcal{LF}$-algebra. When E has a continuous involution, denoted $*$ as usual, we define an involution on $E^{\otimes k}$ by

$$(x_1^k \otimes x_2^k \otimes \ldots \otimes x_k^k)^* = (x_k^k)^* \otimes (x_{k-1}^k)^* \otimes \ldots \otimes (x_1^k)^*$$

we extend to the completion by continuity and bilinearity, and we finally define it on $T(E)$ componentwise. $T(E)$ then becomes an $\mathcal{LF}$ $*$-algebra. For example, if $E = \mathcal{S}(\mathbf{R}^4)$, the space of rapidly decreasing functions, then the tensor algebra $\mathcal{B} = T(\mathcal{S}(\mathbf{R}^4))$ is the Field Algebra defined by Borchers in his algebraic formulation of Quantum Field Theory.

Approximate identities. We say that an algebra $\mathfrak{A}$ has an *approximate identity* if there exists a net $(e_\alpha)_{\alpha\in I}$ composed of hermitian elements such that $\lim_\alpha e_\alpha a = a$ for all $a \in \mathfrak{A}$ (and consequently $\lim_\alpha a e_\alpha = a\ \forall a$). If all the elements of the net $(e_\alpha)_{\alpha\in I}$ belong to a bounded set, we say that $\mathfrak{A}$ has a *bounded approximate identity*. We say that $\mathfrak{A}$ has an *equicontinuous approximate identity* if the family of operators $\{L_{e_\alpha}\}_{\alpha\in I}$ defined by left multiplication by e_α; $L_{e_\alpha}(a) = e_\alpha a$, is an equicontinuous family. If $\mathfrak{A}$ has an equicontinuous approximate identity and $\mathfrak{A}$ is separable then $\mathfrak{A}$ has a sequence which is an approximate identity (cf. 4.2.4). If $\mathfrak{A}$ is an $\mathcal{LF}$-space which has a sequential approximate identity then it is automatically an equicontinuous approximate identity by the principle of uniform boundedness. We will come back to these concepts in chapter 4.

2.3. Embedded Hilbert spaces. Let E be a quasi-complete locally convex Hausdorff space over $\mathbf{C}$. In this section we recall the main results from Schwartz's theory of Hilbert subspaces needed in the sequel [**25**]. Later we shall specialize to the case where E is the strong dual of the $*$-algebra $\mathfrak{A}$.

A Hilbert subspace $\mathcal{H}$ of E is a linear subspace, equipped with an inner product making it into a Hilbert space, and such that the inclusion map

(1) $$\mathcal{H} \underset{j}{\hookrightarrow} E$$

is continuous.

It will be convenient to introduce the space E^* conjugate to the dual E', i.e. a linear space over $\mathbf{C}$ together with an anti-linear bijection between E' and E^*. Thus all anti-linear maps on E' become linear on E^*. The elements of E^* will generally be denoted by greek letters (until we identify E^* with $\mathfrak{A}$). The canonical bilinear map on $E \times E'$ becomes a sesquilinear map on $E \times E^*$, which we denote as:

(2) $$\langle x, \xi \rangle$$

It is linear with respect to x, anti-linear with respect to ξ.

If E is a Hilbert space we shall usually identify E and E^* and replace the duality bracket (2) by the inner product.

Given a continuous linear map from the locally convex space E to a locally convex space $F, u : E \longrightarrow F$, we denote u^* the *adjoint* map, defined by the equation

$$(3) \qquad \langle ux, \eta \rangle = \langle x, u^* \eta \rangle$$

This is a linear map $u^* : F^* \longrightarrow E^*$ continuous with respect to the weak* and strong dual topologies.

If E and F are Hilbert spaces, identified with their conjugate dual spaces, the adjoint of a map $u : E \longrightarrow F$ is simply the usual Hilbert space adjoint.

If $u : E \longrightarrow F$ is antilinear the adjoint of u is defined similarly by

$$(3') \qquad \langle ux, \eta \rangle = \overline{\langle x, u^* \eta \rangle}$$

u^* is then a weak* and strongly continuous antilinear map from F^* to E^*.

Given the Hilbert subspace $\mathcal{H} \hookrightarrow E$ there exists by the Riesz representation theorem a unique element $j^*\xi \in \mathcal{H}$ such that

$$(4) \qquad (x | j^* \xi) = \langle jx, \xi \rangle$$

We denote $H\xi = jj^*\xi$ the same element regarded as element in E. The linear operator $H : E^* \longrightarrow E$, will be called the *reproducing operator* of the space $\mathcal{H}$. If E is a Hilbert space, identified with E^*, and $\mathcal{H}$ is a closed subspace of $E, j^*\xi$ is obviously the orthogonal projection of ξ on $\mathcal{H}$. Consequently in that case H is the orthogonal projector whose image is $\mathcal{H}$.

Replacing x by $j^*\eta$ in $\mathcal{H}$ we obtain:

$$(5) \qquad \langle H\eta, \xi \rangle = (j^*\eta | j^*\xi) \quad \forall \eta, \xi \in E^*$$

In particular, for $\xi = \eta$ one has

$$(6) \qquad \langle H\eta, \eta \rangle = \| j^*\eta \|^2 \quad \forall \eta \in E^*$$

The equations (5) and (6) show that H is hermitian, i.e.:

$$(7) \qquad \langle H\xi, \eta \rangle = \overline{\langle H\eta, \xi \rangle} \quad \forall \eta, \xi \in E^*$$

and that H is positive:

$$(8) \qquad \langle H\eta, \eta \rangle \geqq 0 \quad \forall \eta \in E^*$$

As a consequence one has the triangle inequality:

$$(9) \qquad |\langle H\xi, \eta \rangle| \leqq \langle H\xi, \xi \rangle^{1/2} \langle H\eta, \eta \rangle^{1/2}$$

Sometimes it is convenient to consider, instead of the reproducing operator H, the sesquilinear form $(\xi, \eta) \longmapsto \langle H\xi, \eta\rangle$. This we shall generally denote by the same symbol:

$$(10) \qquad H(\xi, \eta) = \langle H\xi, \eta\rangle$$

It is a non-negative (hermitian) sesquilinear form, which is seprately continuous with respect to the weak* topology. Conversely, any non-negative sesquilinear form on $E^* \times E^*$ which is separately weak* continuous, gives rise to a positive operator $H : E^* \longrightarrow E$, via formula (10). If H is the reproducing operator of $\mathcal{H}$, the sesquilinear form H is called the *reproducing kernel* of $\mathcal{H}$.

Let Hilb(E) be the set of Hilbert subspaces of E, and let $\mathcal{L}^+(E^*, E)$ be the set of linear operators $H : E^* \longrightarrow E$ satisfying (7) and (8), briefly: positive operators (recall that (7) is a consequence of (8)).

PROPOSITION 2.3.1 *The map $\mathcal{H} \longrightarrow H$, which associates with $\mathcal{H}$ its reproducing operator, is a bijection from* Hilb(E) *onto* $\mathcal{L}^+(E^*, E)$.

Let us sketch the proof: *Uniqueness*: i.e. $\mathcal{H}$ is determined by H. By (4) the subspace $j^*(E^*)$ is dense in $\mathcal{H}$, no element $\neq 0$ being orthogonal to it. The unit ball B of $\mathcal{H}$ which is weakly compact, is weakly closed in E. Being convex, B is closed. Thus B is the closure in E of the set $\{H\xi : \langle H\xi, \xi\rangle^{1/2} \leqq 1\}$. This proves that $\mathcal{H}$ is determined by H. Moreover it can be shown that one has, if $x \in \mathcal{H}$:

$$(11) \qquad \|x\| = \sup_{\xi} \frac{|\langle x, \xi\rangle|}{\langle H\xi, \xi\rangle^{1/2}}$$

(the supremum being taken over all $\xi \in E^*$ with $\langle H\xi, \xi\rangle > 0$) conversely if this expression is finite x belongs to $\mathcal{H}$ (cf. [**25**: p. 146]).

Existence. Let $N = \{\eta \in E^* : \langle H\eta, \eta\rangle = 0\}$. Equip E^*/N with the inner product structure derived from the sesquilinear form H. It is a consequence of the triangle inequality (9) that $N = \{\eta : H\eta = 0\}$ and there exists a continuous linear map $\dot{H} : E^*/N \longrightarrow E$ characterized by the relation $\dot{H}\dot{\xi} = H\xi$. This map has a one-to-one continuous extension $\hat{H} : (E^*/N)\hat{}\longrightarrow E$ to the completion. The image $\mathcal{H} = \operatorname{Im} \hat{H}$ with the Hilbert structure making $\hat{H}$ an isometry is a Hilbert subspace with reproducing operator H. (cf. [**25**: p. 154]).

Remark. One often limits this construction to the abstract space $(E^*/N)\hat{}$. Particularly in connection with fields of unbounded oprators it will be very useful to embed the space $\mathcal{H}$ in E. In the case, considered below, where H is a kernel defined on a $*$-algebra $\mathfrak{A}$ by means of a positive functional: $H(a, b) = \omega(b^*a)$, the Hilbert space $\mathcal{H}$ is the space Hilbert space occuring in the classical GNS construction. It will be naturally embedded in the dual of $\mathfrak{A}$.

Remark. We retain from the proof the following fact which will be repeatedly used in the sequel: Every Hilbert subspace $\mathcal{H} \hookrightarrow E$ has a privileged dense subspace $D_{\mathcal{H}} = j^*(E^*)$.

Note that the map $j^* : E^* \longrightarrow \mathcal{H}$ is one to one, i.e. $(E^*, \mathcal{H}, E)$ is a Gelfand triplet, if and only if $\mathcal{H}$ is dense in E.

Let E_b^* be the space E^* endowed with the topology of uniform convergence on bounded subsets of E. For future reference we note the following corollary.

Corollary 2.3.2. *Any positive kernel* $H : E^* \times E^* \longrightarrow \mathbf{C}$ *is continuous on* $E_b^* \times E_b^*$.

Proof. Since $j^* : E^* \longrightarrow \mathcal{H}$ is continuous with respect to the strong dual topology on E^* and the norm topology on $\mathcal{H}$, identified with the strong dual topology of $\mathcal{H}^*$, it is a consequence of (5) that the sesquilinear form H is continuous on $E_b^* \times E_b^*$ [**25**: p. 157]. □

Image spaces. Let $u : E \longrightarrow F$ be a continuous linear map to a second locally convex space. Let $\mathcal{H} \hookrightarrow E$ be a Hilbert subspace with reproducing operator H. Let N be the kernel of the restriction of u to $\mathcal{H}$. This is a closed linear subspace of $\mathcal{H}$. The image $u(\mathcal{H})$ is always equipped with the Hilbert structure making the restriction of u to the orthogonal complement $N^\perp$ of N in $\mathcal{H}$ an isometry. It follows that $u(\mathcal{H})$ is a Hilbert subspace of F.

Proposition 2.3.3. *The inner product in* $u(\mathcal{H})$ *of elements* ux_1 *and* ux_2, *with* $x_1, x_2 \in \mathcal{H}$, *is*

$$(ux_1|ux_2)_{u(\mathcal{H})} = (x_1|x_2)_{\mathcal{H}} \tag{12}$$

provided x_1 *or* x_2 *belongs to* $N^\perp$.

This is clear if both belong to $N^\perp$. If $x_1 \in N^\perp$ one can replace x_2 by its orthogonal projection on $N^\perp$ without changing either the inner product on the right or the image ux_2.

If the restriction of u to $\mathcal{H}$ is one-to-one $u : \mathcal{H} \longrightarrow u(\mathcal{H})$ is an isometric isomorphism.

The preceding proposition has an obvious analogue for antilinear maps u, the right hand side of twelve being replaced by its conjugate.

Proposition 2.3.4. *Let* $u : E \longrightarrow F$ *be a continuous linear or antilinear map. Then the reproducing operator of* $u(\mathcal{H})$ *is* uHu^*.

Proof. (For u linear). Let $y = ux$, with $x \in N^\perp$. Then

$$(y|uHu^*\xi)_{\mathcal{K}} = (x|Hu^*\xi) = \langle x, u^*\xi \rangle = \langle y, \xi \rangle.$$

Corollary 2.3.5. *Let* $\mathcal{H}$ *be a Hilbert space, identified with* $\mathcal{H}^*$, *and let* $\mathcal{K} \hookrightarrow \mathcal{H}$ *be a Hilbert subspace having the reproducing operator* $T : \mathcal{H} \longrightarrow \mathcal{H}$. *Then* $\mathcal{K} = T^{1/2}(\mathcal{H})$.

Proof. The reproducing operator of $T^{1/2}(\mathcal{H})$ is equal to $T^{1/2}(T^{1/2*}) = T^{1/2}T^{1/2} = T$. □

COROLLARY 2.3.6. *Under the previous assumption, let $j : \mathcal{H} \hookrightarrow E$ be a Hilbert subspace of E. The reproducing operator of $\mathcal{K}$ as subspace of E is jTj^*.*

Proof. $\mathcal{K}$ "as Hilbert subspace of E" is precisely $j(\mathcal{K})$. □

COROLLARY 2.3.7. *If $\mathcal{H}$ is an abstract Hilbert space identified to $\mathcal{H}^*$, and $u : \mathcal{H} \longrightarrow E$ is a continuous linear map, the image space $u(\mathcal{H})$ has the reproducing operator uu^*.*

The cone structure of Hilb(E). The *sum* of two positive operators from E^* to E is again a positive operator, as is the product of a positive operator by a non negative number. Following Schwartz let us give a direct description of the corresponding Hilbert spaces:

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert subspaces of E. Let $\mathcal{H}_1 \times \mathcal{H}_2$ as usual be equipped with the norm defined by $\|(x_1, x_2)\|^2 = \|x_1\|^2 + \|x_2\|^2$. Let $\Phi : \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow E$ be the map defined by $\Phi(x_1, x_2) = x_1 + x_2$. Then $\mathcal{H}_1 + \mathcal{H}_2$ is by definition the image Hilbert space $\Phi(\mathcal{H}_1 \times H_2)$. If $\mathrm{Ker}(\Phi) = (0)$ i.e. if $\mathcal{H}_1 \cap \mathcal{H}_2 = (0)$, the map $\Phi : \mathcal{H}_1 \times H_2 \longrightarrow \mathcal{H}$ is an isometric isomorphism. In that case the sum $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ is said to be direct and one writes $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$.

Let $\lambda \geqq 0$. Then if $\mathcal{H} \hookrightarrow E$ is a Hilbert subspace the space $\lambda\mathcal{H}$ is defined as follows: if $\lambda = 0$, $\lambda\mathcal{H} = (0)$. If $\lambda > 0$ the underlying linear space for $\lambda\mathcal{H}$ equals $\mathcal{H}$, the inner product on $\lambda\mathcal{H}$ is $(1/\lambda)$ times the inner product of $\mathcal{H}$.

Finally it is useful to introduce an *order relation* in Hilb(E) as follows: $\mathcal{H}_1 \leqq \mathcal{H}_2$ if $B_1 \subset B_2$ i.e. the unit ball of $\mathcal{H}_1$ is contained in the unit ball of $\mathcal{H}_2$. Equivalently $\mathcal{H}_1 \subset \mathcal{H}_2$, the inclusion being an operator of norm at most 1.

One then has:

PROPOSITION 2.3.8. ([**25** : §6]) *Let $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert subspaces of E with reproducing operators respectively equal to H, H_1 and H_2.*

1. *The reproducing operator of $\mathcal{H}_1 + \mathcal{H}_2$ is $H_1 + H_2$.*
2. *The reproducing operator of $\lambda\mathcal{H}$ is λH.*
3. *$\mathcal{H}_1 \leqq \mathcal{H}_2$ is equivalent to $H_1 \leqq H_2$ i.e. $H_2 - H_1 \in \mathcal{L}^+(E^*, E)$.*

Proof.
1. $\Phi(x_1, x_2) = j_1x_1 + j_2x_2, \Phi^*(\xi) = (j_1^*\xi, j_2^*\xi)$. Thus $\Phi\Phi^*\xi = H_1\xi + H_2\xi$.
2. $(f|\lambda H\xi)_{\lambda\mathcal{H}} = (f|H\xi)_{\mathcal{H}} = \langle f, \xi\rangle$.
3. This is a consequence of (11) (cf. [**25**: p. 160]). □

Example. Let $\mathfrak{A}$ be a nuclear $\mathcal{LF}$ space equipped with an anti-automorphism $x \longmapsto x^*$. Then if $E = \mathfrak{A}'_b$ we can identify E^*_b with $\mathfrak{A}$, the anti-duality being defined by:

$$\langle \ell, a\rangle = \ell(a^*).$$

The reproducing kernel H of a Hilbert subspace $\mathcal{H} \hookrightarrow \mathfrak{A}'$ corresponds to a continuous bilinear form B on $\mathfrak{A} \times \mathfrak{A}$ by 2.3.2 or to a continuous linear form L on $\mathfrak{A}\hat{\otimes}\mathfrak{A}$ by the formulas

$$H(a,b) = B(a,b^*) = L(a \otimes b^*)$$

Thus the cone $\mathcal{L}^+(E^*,E)$ can be identified to a closed convex subcone Γ of the conuclear space $(\mathfrak{A}\hat{\otimes}\mathfrak{A})' : \Gamma = \{L : L(a\otimes a^*) \geqq 0 \ \forall a \in \mathfrak{A}\}$. We shall later consider the case where $\mathfrak{A}$ is a topological $*$-algebra, i.e. an algebra with a separately continuous product and a continuous involution. Then, if $\omega \in \mathfrak{A}'$ is a positive form the sesquilinear form H_ω defined by $H_\omega(a,b) = \omega(b^*a)$ is positive and continuous!

2.4. Integrals of Hilbert subspaces. In this section we summarize the main properties of the integrals of Hilbert subspaces following [**29**].

We equip the space $\mathrm{Herm}(E^*,E)$ of all hermitian operators, i.e. operators satisfying condition (7), with the topology of pointwise convergence on E^* in E. It is a quasi-complete locally convex space over $\mathbf{R}$, subspace of E^{E^*}. The corresponding weak topology is the topology of pointwise convergence in E equipped with the weak topology. The dual of $\mathrm{Herm}(E^*,E)$ is generated by the linear forms $H \longmapsto \langle H\eta, \xi\rangle$ or even, by polarisation, the linear forms $H \longmapsto \langle H\xi,\xi\rangle$. The set $\mathcal{L}^+(E^*,E)$ is a closed convex cone in $\mathrm{Herm}(E^*,E)$.

Let Λ be a topological Hausdorff space, equipped with a Radon measure m (i.e. an inner regular locally finite measure on the Borel sets). Let F be a topological space (in the sequel E of $\mathrm{Herm}(E^*,E)$). A map $f : \Lambda \longrightarrow F$ is said to be *m-measurable* if for every compact set K and $\epsilon > 0$ there exists a compact set $K' \subset K$ such that $m(K\backslash K') \leqq \epsilon$, and such that the restriction of f to K' is continuous. If F has a countable base of open sets every Borel function is m-measurable (Lusin's theorem). In particular every lower semicontinuous function $f : \Lambda \longrightarrow [0,+\infty]$ is m-measurable. If F is a topological vector space it is clear that the sum of m-measurable functions is m-measurable. Also the product of a vector valued and a scalar m-measurable function is m-measurable.

Definition 2.4.1. A family $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ of Hilbert subspaces of E is continuous (resp. m-measurable) if the map $\lambda \longmapsto H_\lambda \in \mathrm{Herm}(E^*,E)$ is continuous (resp. m-measurable).

If $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ be an m-measurable family of Hilbert subspaces of E, a *measurable field* is an m-measurable map $\lambda \longmapsto f(\lambda) \in E$ such that $f(\lambda) \in \mathcal{H}_\lambda$ for all λ.

Example. $\lambda \longmapsto H_\lambda\xi$ is an m-measurable field for all $\xi \in E^*$.

LEMMA 2.4.2. *If $(\mathcal{H}_\lambda)_{\lambda\in K}$ and $f(.)_{|K}$ are continuous the map $\lambda \longmapsto \|f(\lambda)\|_\lambda$ (norm in $\mathcal{H}_\lambda$) is lower semicontinuous.*

Proof. Let

$$F_\xi(\lambda) = \frac{|\langle f(\lambda), \xi\rangle|}{\langle H_\lambda \xi, \xi\rangle^{1/2}}$$

if $\langle H_\lambda \xi, \xi\rangle > 0$, $F_\xi(\lambda) = 0$ if $\langle H_\lambda \xi, \xi\rangle = 0$. Then F_ξ is lower semicontinuous on K and by (10) $\|f(\lambda)\|_\lambda = \sup_\xi F_\xi(\lambda)$.

Consequently if $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ and $f(.)$ are measurable, the norm $\|f(\lambda)\|_\lambda$ is a measurable function of λ. We denote $L^2(m, \{\mathcal{H}_\lambda\})$ the space of equivalence classes, modulo fields almost everywhere 0, of square integrable fields:

$$(1) \qquad \int \|f(\lambda)\|_\lambda^2 dm(\lambda) < +\infty$$

with the norm defined as the square root of this expression. This is a Hilbert space (cf. [**29**]). □

PROPOSITION 2.4.3. *Let $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ be m-measurable and such that*

$$(2) \qquad \int \langle H_\lambda \xi, \xi\rangle dm(\lambda) < +\infty \quad \forall \xi \in E^*$$

Then there exists a Hilbert subspace $\mathcal{H} \hookrightarrow E$ whose reproducing kernel is defined by

$$(3) \qquad \langle H\eta, \xi\rangle = \int \langle H_\lambda \eta, \xi\rangle dm(\lambda)$$

Every square integrable field $f(.)$ is m-summable with values in E. We pose

$$(4) \qquad \Phi(f(.)) = \int f(\lambda) dm(\lambda)$$

The map $\Phi : L^2(m, \{\mathcal{H}_\lambda\}) \longrightarrow E$ is continuous and the space $\mathcal{H}$ is the image of $L^2(m, \{\mathcal{H}_\lambda\})$ under Φ i.e. $\mathcal{H} = \Phi(L^2(m, \{\mathcal{H}_\lambda\}))$ and the restriction of Φ to the orthogonal complement of its kernel is an isometric isomorphism.

Remark 2.4.4. If E is the strong dual of a nuclear $\mathcal{LF}$-space, which is necessarily separable, it can be shown that $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ is m-measurable if $H : \lambda \longmapsto H_\lambda$ is scalarly m-measurable, i.e. $\lambda \longmapsto \langle H_\lambda \eta, \xi\rangle$ or even $\lambda \longmapsto \langle H_\lambda \xi, \xi\rangle$ is m-measurable. The above proposition can then be deduced from Schwartz's results, particularly proposition 20. The general case is treated in [**29**].

Under the assumption of proposition 2.4.3 we shall say that the family $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ is *m-summable*. The space $\mathcal{H}$ is denoted $\mathcal{H} = \int H_\lambda dm(\lambda)$. If the kernel of Φ

is (0), in which case Φ is an isometric isomorphism of $L^2(m, \{\mathcal{H}_\lambda\})$ onto $\mathcal{H}$ the integral is said to be *direct* and one writes:

$$\mathcal{H} = \int^{\oplus} \mathcal{H}_\lambda dm(\lambda).$$

In this case it follows that every $f \in \mathcal{H}$ has an essentially unique decomposition (vector integral in E): $f = \int f(\lambda)dm(\lambda)$ with $f(.)$ square m-measurable. Moreover one then has:

$$\|f\|^2 = \int \|f(\lambda)\|_\lambda^2 dm(\lambda)$$
$$(f|g) = \int (f(\lambda)|g(\lambda))_\lambda dm(\lambda)$$

A particular example of such a decomposition is: $H\xi = \int H_\lambda \xi dm(\lambda)$. Let $A \subset \Lambda$ be a measurable subset. Then if $(\mathcal{H}_\lambda)$ is m-measurable one can analogously define $\mathcal{H}_A = \int_A \mathcal{H}_\lambda dm(\lambda)$ the space with kernel $H_A(\xi, \xi) = \int_A H_\lambda(\xi, \xi)dm(\lambda)$. Since $H_A \leqq H$, one has $\mathcal{H}_A \hookrightarrow \mathcal{H}$, the inclusion being an operator of norm $\leqq 1$. It is clear that if the integral is direct then for any measurable set $A \subset \Lambda$, $\mathcal{H}_A$ is a closed linear subspace of $\mathcal{H}$, $\mathcal{H}_A$ and $\mathcal{H}_B$ being orthogonal subspaces if A and B are disjoint. Conversely for the integral to be direct it suffices to have $\mathcal{H}_A \cap \mathcal{H}_B = (0)$ whenever A and B are disjoint (cf. [**29**]).

Let $\mathcal{H} = \int \mathcal{H}_\lambda dm(\lambda)$ be a not necessarily direct integral. Then, if R_A is the reproducing operator of $\mathcal{H}_A$ in $\mathcal{H}$, the map $A \longmapsto R_A$ is a semi-spectral measure, i.e. it takes value in the set of positive operators in $\mathcal{H}$, is countably additive with respect to the strong operator topology, and is such that $R_\Lambda = I$, the identity in $\mathcal{H}$. If the integral is direct R is a spectral measure.

Now consider the converse question. Let $j : \mathcal{H} \hookrightarrow E$ be a Hilbert subspace and let $A \longmapsto R_A$ be a semi-spectral measure in $\mathcal{H}$, and let $H_A = jR_Aj^*$ be the reproducing operator of the subspace $\mathcal{H}_A$ corresponding to R_A. Then we have:

PROPOSITION 2.4.5. *If E is a conuclear space (e.g. the strong dual of a barrelled nuclear space): Then under the previous assumptions one has:*

(1) *There exists a Radon measure m such that $m(A) = 0 \Rightarrow R_A = 0$, i.e. R is absolutely continuous with respect to m.*

(2) *If R is absolutely continuous with respect to m, there exists an m-summable family $(H_\lambda)_{\lambda \in \Lambda}$ of positive kernels such that $H_A = \int_A H_\lambda dm(\lambda)$ for all Borel sets $A \subset \Lambda$.*

In particular, if R is a spectral measure one has: $\mathcal{H} = \int^{\oplus} \mathcal{H}_\lambda dm(\lambda)$.

This is a generalization appropriate in the present context, of Maurin's nuclear spectral theorem [**18**]. For a proof we refer the reader to [**18**] or [**29**].

3. Representations of $*$-algebras.

3.1 Self-adjoint representations. Let $\mathfrak{A}$ be a topological $*$-algebra. We now assume that there is given a strongly continuous representation of $\mathfrak{A}$ on E, i.e. a separately continuous bilinear map $(a, x) \longmapsto ax$ from $\mathfrak{A} \times E$ to E satisfying:

(1) $\quad (ab)x = a(bx) \quad \forall a, b \in \mathfrak{A}, \forall x \in E$

We then define an action of $\mathfrak{A}$ on E^* as follows:

(2) $\quad \langle ax, \xi\rangle = \langle x, a^*\xi\rangle$

this is again a representation i.e.:

(3) $\quad (ab)\xi = a(b\xi) \quad \forall a, b \in \mathfrak{A}, \forall \xi \in E^*$

and the bilinear map $(a, \xi) \longmapsto a\xi$ is separately continuous if E^* is equipped with the weak* topology, or, $\mathfrak{A}$ being barelled, if E^* is equipped with the strong dual topology.

Note that if E^* is equipped with the weak* topology E can be viewed as $(E^*)^*$, the sesquilinear form being the complex conjugate of $\langle x, \xi\rangle$. The action of $\mathfrak{A}$ on $(E^*)^*$ then coincides with the original action of $\mathfrak{A}$ on E.

Definition 3.1.1. A Hilbert subspace $\mathcal{H} \hookrightarrow E$ will be said to be *invariant* if:

(4) $\quad aH\xi = Ha\xi \quad \forall a \in \mathfrak{A}, \forall \xi \in E^*$

Equivalently:

(4′) $\quad H(a\eta, \xi) = H(\eta, a^*\xi) \quad \forall a \in \mathfrak{A}, \forall \eta, \xi \in E^*$

Let $D_{\mathcal{H}} = H(E^*)$ be the corresponding dense subspace of $\mathcal{H}$. Then $D_{\mathcal{H}} \subset E$ is invariant under the operators a, and we define a representation π of $\mathfrak{A}$ on $D_{\mathcal{H}}$ by restriction. Thus

(5) $\quad \pi(a)\varphi = a\varphi \quad \varphi \in D_{\mathcal{H}}, a \in \mathfrak{A}$

Since $D_{\mathcal{H}}$ is a subspace of $\mathcal{H}$, we shall write, in view of (4):

(6) $\quad \pi(a)j^*(\xi) = j^*(a\xi)$

thus π is regarded as a representation of $\mathfrak{A}$ by, in general unbounded, operators in $\mathcal{H}$, with common invariant domain $D_{\mathcal{H}}$.

In this connection let us recall the notion of self-adjoint representation due to Powers [**22**] and [**23**].

Definition 3.1.2. A representation π of a $*$-algebra $\mathfrak{A}$ on a Hilbert space $\mathcal{H}$

is a mapping of $\mathfrak{A}$ into linear operators defined on a common dense domain $D_\pi \subset \mathcal{H}$, invariant under $\pi(a)$ for all $a \in \mathfrak{A}$, and such that π is linear and multiplicative on D_π. The representation is hermitian if $\pi(a^*) \subset \pi(a)^*$ for all $a \in \mathfrak{A}$ i.e.

$$(7) \qquad (\pi(a)\varphi|\psi) = (\varphi|\pi(a^*)\psi) \quad \forall \varphi, \psi \in D_\pi$$

A hermitian representation is also called a $*$-representation.

One defines the closure of a $*$-representation π as the representation $\bar{\pi}$ defined on $D_{\bar{\pi}} = \cap_a D_{\overline{\pi(a)}}$ by putting $\bar{\pi}(a) = \overline{\pi(a)}$, and a representation π^* on the domain $D_{\pi^*} = \cap_a D_{\pi(a)^*}$ by putting $\pi^*(a) = \pi(a^*)^*$.

One equips D_{π^*} with the "graph topology" i.e. the topology defined by the seminorms $\|f\|$ and $p_a(f) = \|\pi^*(a)f\|$. Then D_{π^*} is a complete topological vector space and $D_{\bar{\pi}}$ is the closure of D_π in D_{π^*}. Note that contrary to the assumption in [**22**] we have not stipulated the existence of a unit in $\mathfrak{A}$. But if one replaced $\mathfrak{A}$ by the algebra in $\mathcal{L}(D_\pi)$ by the algebra generated by the operators $\pi(a)$ and the identity one obtains precisely the situation considered by Powers.

Definition 3.1.3. A $*$-representation π is said to be closed if $\pi = \bar{\pi}$, self-adjoint if $\pi = \pi^*$, and essentially self-adjoint if $\bar{\pi} = \pi^*$.

Definition 3.1.4. The weak commutant $\{\pi(\mathfrak{A})\}'$ of the $*$-representation π is the set of bounded operators S in $\mathcal{H}$ such that

$$(8) \qquad (S\pi(a)\varphi|\psi) = (S\varphi|\pi(a^*)\psi) \quad \forall \varphi, \psi \in D_\pi, \forall a \in \mathfrak{A}$$

It is a weakly closed, symmetric, complex linear manifold in $\mathcal{L}(\mathcal{H})$.

The next proposition is proved in [**22**: Lemma 4.6, p. 96]. Its proof does not require the existence of a unit. The result is crucial for the work below.

PROPOSITION 3.1.5. *Let π be a self-adjoint representation with domain D_π. Then a bounded operator S belongs to the weak commutant $\{\pi(\mathfrak{A})\}'$ if and only if $SD_\pi \subset D_\pi$ and $S\pi(a)f = \pi(a)Sf$ for all $a \in \mathfrak{A}$ and $f \in D_\pi$. In particular $\{\pi(\mathfrak{A})\}'$ is an algebra, hence a von Neumann algebra.*

Proof. Let $\varphi, \psi \in D_\pi = \cap D_{\pi(a)}$. If $S \in \{\pi(\mathfrak{A})\}'$ using (8) we get that $\varphi \longmapsto (\pi(a)\varphi|S^*\psi) = (S\varphi|\pi(a^*)\psi)$ is continuous. Thus $S^*\psi \in D_{\pi(a)^*}$ and $(\varphi|\pi(a)^*S^*\psi) = (\varphi|S^*\pi(a^*)\psi)$ for all φ. Thus $\pi(a)^*S^*\psi = S^*\pi(a^*)\psi$ for all ψ i.e. $S^*\psi \in D_\pi = D_{\pi^*}$. Moreover $\pi(a)^*(S)^* = S^*\pi(a^*)$ for all $a \in \mathfrak{A}$. Replacing S by S^* we obtain the result.

COROLLARY 3.1.6. *The weak commutant $\{\pi(\mathfrak{A})\}'$ of an essentially self-adjoint representation is a von Neumann algebra.*

In fact $\{\pi(\mathfrak{A})\}' = \{\bar{\pi}(\mathfrak{A})\}'$.

Note that there exist examples of weak commutants which are not algebras (cf. [**22**: p. 92]).

COROLLARY 3.1.7. *$S : D_\pi \longrightarrow D_\pi$ is continuous with respect to the graph topology.*

In fact: $p_a(Sf) = \|\pi(a)Sf\| = \|S\pi(a)f\| \leqq \|S\|p_a(f)$.

Let us return to the specific situation at hand where $\mathcal{H} \hookrightarrow E$ is a Hilbert subspace of E, and π is defined by restricting the representation of $\mathfrak{A}$ in E to $D_{\mathcal{H}} = j^*(E^*)$.

For $a \in \mathfrak{A}$ we pose:

$$\mathcal{M}_a = \{f \in \mathcal{H} : af \in \mathcal{H}\} \tag{9}$$

$$\mathcal{M} = \bigcap_{a\in\mathfrak{A}} \mathcal{M}_a \tag{10}$$

This is a linear subspace of $\mathcal{H}$ containing $D_{\mathcal{H}}$.

PROPOSITION 3.1.8. *One has $D_{\pi(a)^*} = \mathcal{M}_{a^*}$, the maximal domain for a^*, and*

$$\pi(a)^*f = a^*f \quad f \in \mathcal{M}_{a^*} \tag{11}$$

Proof. Let $\varphi = j^*(\xi)$ belong to $D_{\mathcal{H}}$, and $f \in \mathcal{M}_{a^*}$. Then one has

$$(f|\pi(a)\varphi) = (f|j^*(a\xi)) = \langle f, a\xi\rangle = \langle a^*f, \xi\rangle = (a^*f|\varphi)$$

Thus f belongs to the domain of $\pi(a)^*$ and $\pi(a)^*f = a^*f$. Conversely, let f belong to the domain of $\pi(a)^*$ and let $g = \pi(a)^*f$. Then

$$\langle g, \xi\rangle = (g|j^*(\xi)) = (f|{}^*j(a\xi)) = \langle f, a\xi\rangle = \langle a^*f, \xi\rangle$$

Thus $a^*f = g$ belongs to $\mathcal{H}$, i.e. $f \in \mathcal{M}_{a^*}$. □

We get as an immediate consequence of the above:

PROPOSITION 3.1.9. *π is a $*$-representation of $\mathfrak{A}$: $\pi(a^*) = \pi(a)^*|_{D_{\mathcal{H}}}$.*

A second consequence is that the domain of the representation π^* is the space $\mathcal{M}$ and

$$\pi^*(a)f = af \quad f \in \mathcal{M} \tag{12}$$

Thus $\mathcal{M}$ equipped with its graph topology defined by the seminorms: $\|f\|$, $p_a(f) = \|af\|, a \in \mathfrak{A}$, is a complete topological vector space. Recall that the domain of $\bar{\pi}$ is the closure of $D_{\mathcal{H}}$ in the space $\mathcal{M}$.

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PROPOSITION 3.1.10. *Let $\mathcal{H} \hookrightarrow E$ be an invariant Hilbert subspace. Let $\mathcal{K} \hookrightarrow \mathcal{H}$ be a Hilbert subspace of $\mathcal{H}$, and let $T \in L(\mathcal{H})$ be its reproducing operator. Then $\mathcal{K} \hookrightarrow E$ is an invariant Hilbert subspace if and only if the operator T belongs to the weak commutant of the representation π.*

Proof. Let $j : \mathcal{H} \hookrightarrow E$ be the injection. Then the reproducing operator of $\mathcal{K}$ in E is the operator $K = jTj^*$. Let T belong to the weak commutant $\{\pi(\mathfrak{A})\}'$ i.e. assume:

$$(13) \qquad (T\pi(a)\varphi|\psi) = (T\varphi|\pi(a^*)\psi) \quad \forall \varphi, \psi \in D_{\mathcal{H}}$$

Then one has

$$\begin{aligned} K(a\eta, \xi) &= (Tj^*a\eta, j^*\xi) = (T\pi(a)j^*\eta, j^*\xi) = (Tj^*\eta, \pi(a^*)j^*\xi) \\ &= K(\eta, a^*\xi) \end{aligned}$$

i.e. K is invariant. Conversely the invariance of K implies the equality of the third and fourth term, i.e. $T \in \{\pi(\mathfrak{A})\}'$. □

Under the hypotheses of proposition 3.1.10 there exists a representation of $\mathfrak{A}$ on the dense subspace $D_{\mathcal{K}}$ of $\mathcal{K}$ defined analogously. If necessary we distinguish these representations by denoting them $\pi_{\mathcal{H}}$ and $\pi_{\mathcal{K}}$ respectively.

The following theorem has been obtained by Powers [**22**: Theorem 4.7, p. 97] in the case where π is self-adjoint and the space $\mathcal{K}$ is a closed subspace of $\mathcal{H}$.

THEOREM 3.1.11. *Let $\mathcal{H}$ and $\mathcal{K}$ be invariant Hilbert subspaces of E with $\mathcal{K} \hookrightarrow \mathcal{H}$. If $\pi_{\mathcal{H}}$ is essentially self-adjoint, so is $\pi_{\mathcal{K}}$.*

LEMMA 3.1.12. *Let π be a self-adjoint representation of $\mathfrak{A}$ with domain D_π. Let S be a one-to-one self-adjoint bounded operator on $\mathcal{H}$ belonging to the weak commutant $\{\pi(\mathfrak{A})\}'$ of π. Then $S(D_\pi)$ is a dense subspace of D_π equipped with its graph topology.*

Proof. For notational simplicity we assume S to be positive as well (the only case used below). Let F be the resolution of the identity for S, i.e. $S = \int_0^{+\infty} \lambda F(d\lambda)$. Let $F_n = F[(1/n), n], B_n = \int_{1/n}^{n} (1/\lambda) F(d\lambda)$. Then F_n and B_n belong to the von Neumann algebra $\{\pi(\mathfrak{A})\}'$. Let $f \in D_\pi$, $f_n = F_n f$ and $g_n = B_n f_n$. Then f_n and g_n belong to D_π and $f_n = Sg_n$. Thus $f_n \in S(D_\pi)$. Also f_n tends to f in $\mathcal{H}$, and $\pi(a)f_n = F_n\pi(a)f$ tends to $\pi(a)f$ for all $a \in \mathfrak{A}$. Thus f_n converges to f in the topological vector space D_π. □

Proof of 3.1.11. Let $\pi = \pi_{\mathcal{H}}$. Then $D_{\mathcal{K}}$ as linear subspace of $\mathcal{H}$ equals $Tj^*(E^*)$ i.e. it equals $T(D_{\mathcal{H}})$. On the other hand we know (2.3.4) that $\mathcal{K} = T^{1/2}(\mathcal{H})$. Since $\{\bar{\pi}(\mathfrak{A})\}' = \{\pi(\mathfrak{A})\}'$ is a von Neumann algebra $T^{1/2}$ belongs to it. Thus the space $\mathcal{N} = T^{1/2}(\mathcal{M})$ is invariant under the action of $\mathfrak{A}$. Let ρ be the representation of $\mathfrak{A}$ in $\mathcal{N}$ defined by $\pi(a)f = af$ (i.e. the restriction of the closure of $\pi_{\mathcal{K}}$ to $\mathcal{N}$). □

LEMMA 3.1.13. *ρ is a self-adjoint $*$-representation.*

Proof. a. ρ is hermitian: let $g_i = T^{1/2}f_i, i = 1, 2$, with $f_i \in \mathcal{M} = D_{\bar{\pi}}$. Since the orthogonal projection on $\mathrm{Ker}(T^{1/2})$ belongs to $\{\pi(\mathfrak{A})\}'$ and so leaves $\mathcal{M}$ invariant (3. 1. 5), we may take f_i orthogonal to $\mathrm{Ker}(T^{1/2})$. Then we have

$$(\rho(a)g_1|g_2)_{\mathcal{K}} = (\bar{\pi}(a)f_1|f_2)_{\mathcal{H}} = (f_1|\bar{\pi}(a^*)f_2)_{\mathcal{H}} = (g_1|\rho(a^*)g_2)_{\mathcal{K}}.$$

b. Let $g_i = T^{1/2}f_i, i = 1, 2$, with $f_1 \in \mathcal{M}$ and $f_2 \in \mathcal{H}$ orthogonal to $\mathrm{Ker}(T^{1/2})$. Let g_2 belong to D_{ρ^*} i.e. assume that for every $a \in \mathfrak{A}$ there exists a constant c_a such that

$$|(\rho(a)g_1|g_2)_{\mathcal{K}}| \leqq c_a\|g_1\|_{\mathcal{K}} \quad \forall g_1 \in \mathcal{N}$$

Then we have

$$|(\bar{\pi}(a)f_1|f_2)_{\mathcal{H}}| \leqq c_a\|f_1\|_{\mathcal{H}} \quad \forall f_1 \in \mathcal{M}.$$

Thus f_2 belongs to $D_{\pi(a)^*}$ for all $a \in \mathfrak{A}$. Therefore $\bar{\pi}$ being self-adjoint, f_2 belongs to $\mathcal{M}$ and so g_2 belongs to $T^{1/2}(\mathcal{M}) = D_\rho$.

Thus we have $D_{\rho^*} = D_\rho$, which implies that ρ is self-adjoint. □

Now $D_{\mathcal{K}} - T(D_{\mathcal{H}}) = T^{1/2}T^{1/2}(D_{\mathcal{H}}) \subset T^{1/2}(\mathcal{M}) = \mathcal{N}$, since $T^{1/2}(D_{\mathcal{H}}) \subset T^{1/2}(\mathcal{M}) \subset \mathcal{M}$. To prove that the representation $\pi_{\mathcal{K}}$, i.e. ρ restricted to $D_{\mathcal{K}}$, is essentially self-adjoint it is sufficient to prove: $D_{\mathcal{K}}$ is dense in $\mathcal{N}$, equipped with its graph topology. We first prove:

LEMMA 3.1.14.
(1) *$T^{1/2}$ is a bounded operator in $\mathcal{K}$.*
(2) *$T^{1/2}$ is a positive operator in $\mathcal{K}$.*
(3) *$T^{1/2}$ is a one-to-one on $\mathcal{K}$.*
(4) *$T^{1/2}$ belongs to the weak commutant of ρ.*

Proof. Let $g = T^{1/2}f$, $f \in \mathcal{H}$ orthogonal to $\mathrm{Ker}(T^{1/2})$. (1) Then $g \in \mathcal{H}$ so $T^{1/2}g$ belongs to $\mathcal{K} = T^{1/2}(\mathcal{H})$. Also $\|T^{1/2}g\|_{\mathcal{K}} = \|Tf\|_{\mathcal{K}} \leqq \|Tf\|_{\mathcal{H}} \leqq \|T\|\ \|f\|_{\mathcal{H}} = \|T\|\ \|g\|_{\mathcal{K}}$. (2) $(T^{1/2}g|g)_{\mathcal{K}} = (g|f)_{\mathcal{H}} = (T^{1/2}f|f)_{\mathcal{H}} \geqq 0$. (3) If $T^{1/2}g = 0, Tf = 0$, so $f \in \mathrm{Ker}(T) = \mathrm{Ker}(T^{1/2})$; hence $g = 0$. (4) We have $T^{1/2}(\mathcal{N}) = T^{1/2}T^{1/2}(\mathcal{M}) \subset T^{1/2}(\mathcal{M}) = \mathcal{N}$ since $T^{1/2}(\mathcal{M}) \subset \mathcal{M}$. Also $T^{1/2}$ commutes with $\rho(a)$ because this is the restriction of $\bar{\pi}(a)$ to $\mathcal{N}$. Therefore $T^{1/2}$ belongs to the weak commutant by proposition 3.1.5. □

We can now terminate the proof of theorem 3.1.11: We know that $D_{\mathcal{H}}$ is dense in $\mathcal{M}$. Now $T^{1/2} : \mathcal{M} \longrightarrow \mathcal{N}$ is continuous with respect to the graph topologies because $T^{1/2}$ is continuous and intertwines $\bar{\pi}$ and ρ. Consequently $T^{1/2}(D_{\mathcal{H}})$ is dense in $T^{1/2}(\mathcal{M}) = \mathcal{N}$. Therefore $T^{1/2} : \mathcal{N} \longrightarrow \mathcal{N}$ being continuous by lemma 3.1.7, $D_{\mathcal{K}} = T^{1/2}T^{1/2}(D_{\mathcal{H}})$ is dense in $T^{1/2}(\mathcal{N})$ for the topology induced by $\mathcal{N}$. Finally $T^{1/2}(\mathcal{N})$ being dense in $\mathcal{N}$ by lemma 3.1.12, it follows that $D_{\mathcal{K}}$ is dense in $\mathcal{N}$. □

Consequence 3.1.15. $\mathcal{N} = T^{1/2}(\mathcal{M})$ is the maximal domain for $\pi_{\mathcal{K}}$ and $\bar{\pi}_{\mathcal{K}} = \rho$.

In fact, every self-adjoint representation is maximal [**22**: p. 95].

3.2. Desintegration of self-adjoint representations. Now we assume that $\mathcal{H}$ is the direct integral of invariant Hilbert subspaces:

$$(1) \qquad \mathcal{H} = \int_{\Lambda}^{\oplus} \mathcal{H}_{\lambda} dm(\lambda)$$

Let H and H_{λ} denote the corresponding reproducing operators, $D = H(E^*)$ and $D_{\lambda} = H_{\lambda}(E^*)$ the privileged dense subspaces, and π and π_{λ} the representations of $\mathfrak{A}$ obtained as before in D and D_{λ} respectively.

PROPOSITION 3.2.1. *Let π be essentially self-adjoint. The following conditions on $\mathcal{M}$ are equivalent:*

(1) *$\mathcal{M}$ equipped with its graph topology is metrizable.*

(2) *There exists a countable set $\Delta \subset \mathfrak{A}$ such that for all $a \in \mathfrak{A}$ there exist $a_1, a_2 \dots, a_n \in \Delta$ such that $\|af\| \leqq \|f\| + \sum_{i=1}^{n} \|a_i f\|$ for all $f \in \mathcal{M}$.*

(3) *There exists a countable set $\Delta \subset \mathfrak{A}$ such that for all $a \in \mathfrak{A}$ there exist $a_1, a_2 \dots, a_n \in \Delta$ such that $\|a\varphi\| \leqq \|\varphi\| + \sum_{i=1}^{n} \|a_i \varphi\|$ for all $\varphi \in D_{\mathcal{H}}$.*

(4) *There exists a countable subset S of $\mathfrak{A}$ such that for all $a \in \mathfrak{A}$ there exist $b \in S$ and a number $k \geqq 0$ such that $\|a\varphi\| \leqq k(\|\varphi\| + \|b\varphi\|)$ for all $\varphi \in D_{\mathcal{H}}$.*

Proof. 1. $\Leftrightarrow$ 2. is obvious. 2. $\Leftrightarrow$ 3. because D_H is dense in $\mathcal{M}$. 3. $\Rightarrow$ 4. with S the set of finite sums $b = \sum_{i=1}^{n} a_i^* a_i$, with $a_i \in \Delta$. 4. $\Rightarrow$ 1. obvious. □

Note that these conditions on $\mathcal{M}$ are satisfied if $\mathfrak{A}$ is algebraically generated by a countable subset.

More generally, assume $\mathfrak{A}$ contains subalgebras $\mathfrak{D}$ and $\mathfrak{M}$ such that $\mathfrak{D}$ is countably generated, and such that every element in $\mathfrak{A}$ has a decomposition as finite sum $a = \sum_i d_i b_i$, with $d_i \in \mathfrak{D}, b_i \in \mathfrak{M}$. If $\pi(d)$ is bounded for all $d \in \mathfrak{M}$, $\mathcal{M}$ is still metrizable. A situation in which this occurs, is the case where $\mathfrak{A}$ is the convolution algebra of distributions with compact support on a Lie group G, $\mathfrak{D}$ is the set of distributions with support in the neutral element, and $\mathfrak{M}$ is the set of measures with compact support. Then if π is the representation of $\mathfrak{A}$ associated to a unitary representation of G, π is not bounded, $\mathfrak{A}$ is not countably generated, but $\mathcal{M}$ is metrizable.

We do not at present know of any example where π is essentially self-adjoint and where $\mathcal{M}$ is not metrizable (cf. [**22**] and [**23**]).

THEOREM 3.2.2. *Assume $\mathfrak{A}$ is a nuclear $\mathcal{LF}$ $*$-algebra. If π is essentially self-adjoint, and the space $\mathcal{M}$ is metrizable, π_λ is essentially self-adjoint for m-almost all λ.*

Let us introduce some notations. Recall that $\mathcal{M}$ is the maximal domain for the representation π in $\mathcal{H}$. Thus $\mathcal{M}$ is the domain of the self-adjoint closure of π, i.e. D is dense in $\mathcal{M}$ for the graph topology. Let $\mathcal{M}_\lambda$ denote the maximal domains in $\mathcal{H}_\lambda$ respectively. We need to prove that $\mathcal{D}_\lambda$ is dense in $\mathcal{M}_\lambda$ for almost all λ.

The self-adjoint representation $\bar{\pi}$ is the restriction to $\mathcal{M}$ of the given representation in E.

If $B \subset \Lambda$ is a measurable subset, let

$$\mathcal{H}_B = \int_B^{\oplus} \mathcal{H}_\lambda dm(\lambda)$$

be the subspace with reproducing kernel

$$H_B(\eta, \xi) = \int_B H_\lambda(\eta, \xi) dm(\lambda).$$

Then $\mathcal{H}_B$ is a closed subspace of $\mathcal{H}$. Let $P_B : \mathcal{H} \longrightarrow \mathcal{H}_B$ be the orthogonal projection. The invariance $H_\lambda(a\eta, \xi) = H_\lambda(\eta, a^*\xi)$ implies the invariance of the kernels H_B. Thus by proposition 3.1.10 P_B belongs to the weak commutant $\{\pi(\mathfrak{A})\}'$. In particular the maximal space $\mathcal{M}$ is invariant under P_B.

Let $f \in \mathcal{M}$ and let $f = \int_\Lambda f(\lambda) dm(\lambda)$ be its decomposition with $f(.)$ square summable. Then $P_B f = \int_B f(\lambda) dm(\lambda)$.

LEMMA 3.2.3. *The map $\lambda \longmapsto af_\lambda$ is a square summable field for all $a \in \mathfrak{A}$ and*

$$(1) \qquad aP_B f = \int_B af_\lambda dm(\lambda)$$

Proof. Since the map $x \longmapsto ax$ is continuous in E, we have (1) as a vector integral in E. Let $g = \overline{\pi(a)}f = af$ then $g = \int g(\lambda) dm(\lambda)$ with $g(.)$ square integrable, and

$$(2) \qquad P_B g = \int_B g(\lambda) dm(\lambda).$$

Since $aP_B f = \bar{\pi}(a) P_B f = P_B g$ the expressions (1) and (2) are equal for any measurable set B. Therefore $af(\lambda) = g(\lambda)$ for $a.a.\lambda$ and we get the desired conclusion.

We now want to prove that $f \in \mathcal{M}$ implies that $f_\lambda \in \mathcal{M}_\lambda$ for almost all λ. So let $f \in \mathcal{M}$. For B measurable in Λ, let H_B^f be the kernel defined on $\mathfrak{A} \times \mathfrak{A}$ by:

$$H_B^f(a,b) = (P_B af | bf)_{\mathcal{H}}.$$

The map $a \longmapsto af \in \mathcal{H}$ is continuous by the closed graph theorem. Thus $\mathcal{H}_B^f$ is a continuous and positive kernel on $\mathfrak{A} \times \mathfrak{A}$. This defines a measure $B \longmapsto H_B^f \in (\mathfrak{A} \hat{\otimes} \mathfrak{A})'$. By proposition 2.4.5 we know that $\mathfrak{A}$ being a nuclear $\mathcal{LF}$ space this measure has a Radon-Nikodym derivative $\lambda \longmapsto \mathcal{H}_\lambda^f$ with values in the continuous and positive kernels:

$$H_B^f = \int_B H_\lambda^f dm(\lambda).$$

Now by the previous lemma we have:

$$\int_B \langle H_\lambda^f a, a \rangle dm(\lambda) = \langle H_B^f a, a \rangle = (P_B af | af) = \int_B (af_\lambda | af_\lambda) dm(\lambda).$$

Hence

(3) $$\langle H_\lambda^f a, a \rangle = (af_\lambda | af_\lambda)$$

for all λ not belonging to a null set N_a depending on a. □

LEMMA 3.2.4. *$f \in \mathcal{M} \Rightarrow f \in \mathcal{M}_\lambda$ for a.a. λ.*

Proof. Let $\mathfrak{A}_0 = \{a_n\}$ be a countable dense set of elements in $\mathfrak{A}$. Applying the preceding reasoning for each a_n we get $\langle H_\lambda^f a_n, a_n \rangle = (a_n f_\lambda | a_n f_\lambda)$ for all λ not belonging to the null set $N = \cup_n N_{a_n}$. Let $a \in \mathfrak{A}$ and (a_ℓ) be a sequence in $\mathfrak{A}_0$ such that $a = \lim_\ell a_\ell$. Then $af_\lambda = \lim_l a_l f_\lambda$ for all λ. The map $x \longmapsto \|x\|_\lambda$ being lower semicontinuous on E we have for each $\lambda \in \Lambda$:

$$\|af_\lambda\|_\lambda^2 \leqq \underline{\lim}_{\ell} \|a_\ell f_\lambda\|_\lambda^2 = \underline{\lim}_{\ell} \langle H_\lambda^f a_\ell, a\ell \rangle = \langle H_\lambda^f a, a \rangle < +\infty$$

for all $\lambda \in \Lambda \setminus N$. In particular $f_\lambda \in \mathcal{M}_\lambda$ for all $\lambda \in \Lambda \setminus N$.

We now prove $\mathcal{M}_\lambda = \mathcal{N}\lambda$ for almost every $\lambda \in \Lambda$. Let $\{p_i\}_{i \in I}$ be a fundamental system of seminorms on $\mathfrak{A}$. Let $\mathfrak{A}_i = (\mathfrak{A}/p_i^{-1}(0))\hat{}$ be the Banach space associated with p_i. Then by the nuclearity of $\mathfrak{A}$ the canonical map $a \longmapsto \dot{a}$ from $\mathfrak{A}$ to $\mathfrak{A}_i$ is nuclear. Thus

(4) $$\dot{a} = \sum_n \alpha_n L_n(a) \dot{a}_n$$

where $(\alpha_n) \in \ell_1$, $\{L_n\}$ is an equicontinuous sequence in $\mathfrak{A}'$ and $\|\dot{a}_n\| = p_i(a_n) \leqq 1$. For each $i \in I$, let $q_i(f) = \sup\{\|af\|_{\mathcal{H}} : p_i(a) \leqq 1\}$. Let $\mathcal{M}_i = \{f \in \mathcal{M} :$

$q_i(f) < +\infty\}$. Then $\mathcal{M} = \cup_{i\in I}\mathcal{M}_i$. In fact, if $f \in \mathcal{M}$ the map $a \longmapsto \|af\|_{\mathcal{H}}$ is a lower semicontinuous seminorm on $\mathfrak{A}$. The space $\mathfrak{A}$ being barrelled, it is continuous. Thus there exists $i \in I$ and $c \geqq 0$ such that $\|af\|_{\mathcal{H}} \leqq cp_i(a)$ for all $a \in \mathfrak{A}$ and so $q_i(f) \leqq c$.

If $f \in M_i$ we have $\|af\|_{\mathcal{H}} \leqq q_i(f)p_i(a)$. Therefore the map $a \longmapsto af$ factors via a continuous linear map $\dot{a} \longmapsto \dot{a}f = af$ defined on $\mathfrak{A}_i$. Applying this to the expansion (4) we get

$$af = \sum_n \alpha_n L_n(a) a_n f. \tag{5}$$

By the metrizability of $\mathcal{M}$, there exists a sequence $\{\varphi_k\} \subset D_{\mathcal{H}}$ such that $a\varphi_k \longrightarrow af$ in $\mathcal{H}$ for all $a \in \mathfrak{A}$. Then the maps $a \longmapsto a\varphi_k$ from $\mathfrak{A}$ to $\mathcal{H}$ are equicontinuous by the uniform boundedness principle. This means that there exists a continuous seminorm p_i on $\mathfrak{A}$ and a constant M such that $\|a\varphi_k\| \leqq Mp_i(a)$ $\forall a \in \mathfrak{A}$ i.e. $q_i(\varphi_k) \leqq M_i$ $\forall k$. Hence

$$\|a_n(f - \varphi_k)\| \leqq M + q_i(f) \quad \forall n \text{ and } \forall k.$$

Define the function $F_k : \Lambda \longrightarrow [0, +\infty]$ as follows:

$$F_k(\lambda) = \sum_n |\alpha_n|\ \|a_n f_\lambda - a_n \varphi_{k,\lambda}\|_\lambda.$$

Then

$$\left(\int F_k(\lambda)^2 dm(\lambda)\right)^{1/2} \leqq \sum_n |\alpha_n| \left(\int \|a_n f_\lambda - a_n\varphi_{k,\lambda}\|_\lambda^2 dm(\lambda)\right)^{1/2}$$
$$\leqq \sum_n |\alpha_n|\ \|a_n(f - \varphi_k)\| < +\infty$$

This first of all implies that $F_k(\lambda) < +\infty$ for all $k \in \mathbf{N}$ except possibly for λ belonging to a null set N. Secondly, since the right hand side goes to zero as k goes to ∞, there exists a subsequence (still denoted F_k) going to 0 for all λ outside a null set $N' \supset N$. This implies that

$$\|af_\lambda - a\varphi_{k,\lambda}\|_\lambda \leqq \sum_n |\alpha_n|\ |L_n(a)|\ \|a_n f_\lambda - a_n\varphi_{k,\lambda}\|_\lambda \leqq \sup_{n\in\mathbf{N}} |L_n(a)| F_k(\lambda)$$

which goes to 0 for all $a \in \mathfrak{A}$ and all $\lambda \in \Lambda \setminus N'$. Hence, $f_\lambda \in \mathcal{N}_\lambda$ for all $\lambda \in \Lambda \setminus N'$ i.e. almost all λ. Thus π_λ is essentially self-adjoint for almost all λ, which was to be proved. □

Remark. We have used the metrizability of $\mathcal{M}$ only to obtain an approximation of f in $\mathcal{M}$ by a subset of $D_{\mathcal{H}}$ which is bounded in $\mathcal{M}$. One would like to

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do without the metrizability assumption but it is an unsolved question whether in general, π being essentially self-adjoint, every $f \in \mathcal{M}$ belongs to the closure of a bounded subset of $D_{\mathcal{H}}$.

The analogous result for bounded representations is easier to prove. In this case, we do not need the nuclearity assumption on $\mathfrak{A}$ and $\mathcal{M} = \mathcal{H}$ is automatically metrizable.

PROPOSITION 3.2.5. *Let $\mathfrak{A}$ be an $\mathcal{LF}$ $*$-algebra and let π, π_λ and m be defined as above. If π is bounded then π_λ is bounded for m-almost all λ.*

Proof. Since π is bounded, there exists for each $a \in \mathfrak{A}$, a constant $M_a = \|\pi(a)\|$ such that

$$\|\pi(a)\varphi\|_{\mathcal{H}} \leqq M_a \|\varphi\|_{\mathcal{H}} \quad \forall \varphi \in D_{\mathcal{H}}.$$

Then for each measurable set A, we have

$$\|aP_A\varphi\|_{\mathcal{H}} \leqq M_a \|P_A\varphi\|_{\mathcal{H}} \quad \forall \varphi \in D_{\mathcal{H}}. \tag{6}$$

Assume first that $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$ is a continuous family, Λ being equal to the support of m. If $\varphi = j^*\xi$ for $\xi \in E^*$ then it follows from (6) that,

$$H_A(a\xi, a\xi) = (P_A j^* a\xi | j^* a\xi) \leqq M_a^2 H_A(\xi, \xi).$$

Thus, for each $a \in \mathfrak{A}$, we have

$$H_\lambda(a\xi, a\xi) \leqq M_a^2 H_\lambda(\xi, \xi) \tag{7}$$

for almost all $\lambda \in \Lambda$, hence for all $\lambda \in \Lambda$, and for all $\xi \in E^*$. Rewriting (7) as follows

$$\|aj_\lambda^*\xi\|_{\mathcal{H}_\lambda} \leqq M_a \|j_\lambda^*\xi\|_{\mathcal{H}_\lambda}$$

it is seen that for all $\lambda \in \Lambda$, the representation π_λ is bounded, and $\|\pi_\lambda(a)\| \leqq \|\pi(a)\|$ for all $\lambda \in \Lambda$. In the general case the conclusion follows from the Lusin measurability of the family $(\mathcal{H}_\lambda)_{\lambda\in\Lambda}$. □

4. Invariant kernels and positive forms.

4.1. Kernels associated to positive forms. The GNS construction. Let $\mathfrak{A}$ be a topological $*$-algebra (cf. 2.2) and let $\omega \in \mathfrak{A}'$ be a *positive form*, i.e. a continuous linear functional satisfying the condition:

$$\omega(a^*a) \geqq 0 \quad \forall a \in \mathfrak{A}. \tag{1}$$

Then one defines a sesquilinear form H_ω on $\mathfrak{A} \times \mathfrak{A}$ by:

(2) $$H_\omega(a, b) = \omega(b^* a).$$

We shall identify $\mathfrak{A}$ with its conjugate space by means of the involution $x \longmapsto x^*$. If $E = \mathfrak{A}'$ equipped with the weak dual topology, $\mathfrak{A}$ may be identified with E^*, the anti-duality between E and $\mathfrak{A}$ being defined by:

(3) $$\langle f, a \rangle = f(a^*).$$

With this notation (2) becomes:

(4) $$H_\omega(a, b) = \langle \omega, a^* b \rangle.$$

The algebra $\mathfrak{A}$ naturally acts on its dual $\mathfrak{A}'$ as follows:

(5) $$\langle af, x \rangle = \langle f, a^* x \rangle \quad a, x \in \mathfrak{A}.$$

This defines a representation of $\mathfrak{A}$ on $\mathfrak{A}'$:

(6) $$a(bf) = (ab)f$$

which is separately continuous when $\mathfrak{A}'$ is equipped with the weak (or, $\mathfrak{A}$ being barrelled, the strong) dual topology.

The associated representation on $\mathfrak{A} = E^*$ (cf. 3.1) is the natural action of $\mathfrak{A}$ on $\mathfrak{A}$ by multiplication from the left (left regular representation).

Before proceeding with the invariance properties of kernels let us point out that in this case there is also a natural action of $\mathfrak{A}$ on $\mathfrak{A}'$ by multiplication from the right:

(6) $$\langle fb, x \rangle = \langle f, xb^* \rangle \quad x, b \in \mathfrak{A}.$$

One then has the following associativity rule:

(7) $$(af)b = a(fb) \quad \forall a, b \in \mathfrak{A}, \forall f \in \mathfrak{A}'.$$

The map $(f, b) \longmapsto fb$ is bilinear and separately continuous, but it is not a representation.

Let $\overline{\mathfrak{A}}$ be the space conjugate to $\mathfrak{A}$, equipped with the $*$-algebra structure which makes the conjugation $x \longmapsto \bar{x}$ a $*$-homomorphism:

(8) $$\bar{a}\bar{x} = \overline{ax}; \quad \overline{a^*} = \bar{a}^*.$$

Then the map $(f, \bar{b}) \longmapsto fb^*$ is representation of the algebra $\overline{\mathfrak{A}}$.

Example. Let G be a unimodular Lie group equipped with a Haar measure dg. Let $\mathcal{C}_c^\infty(G)$ be the space of test functions equipped with the convolution product and the involution $\varphi^*(g) = \overline{\varphi(g^{-1})}$. Then $\mathfrak{A}' = \mathcal{D}'(G)$ is the space of distributions on G and the left and right actions of $\mathfrak{A}$ and $\mathfrak{A}'$ are the usual operators of left and right regularization of a distribution by test functions.

If ω is a positive functional on $\mathfrak{A}$, the kernel $H = H_\omega$ defined by (2) or (4) is associated with the reproducing operator $H : \mathfrak{A} \longrightarrow \mathfrak{A}'$ defined by:

(9) $\quad Ha = a\omega$

Clearly the operator and kernel H are invariant:

(10) $\quad aHx = Hax \quad \forall a, x \in \mathfrak{A},$

(11) $\quad H(ac, b) = H(c, a^*b) \quad \forall a, b, c \in \mathfrak{A}.$

Let $\mathcal{H}_\omega \hookrightarrow \mathfrak{A}'$ be the Hilbert subspace associated with the reproducing operator $\mathcal{H}_\omega : a \longmapsto a\omega$. The subspace $D_{\mathcal{H}_\omega}$ (cf. 2.3) will now be denoted D_ω. Thus the space

(12) $\quad D_\omega = \{a\omega : a \in \mathfrak{A}\}$

is dense in $\mathcal{H}_\omega$. The inner product of $\mathcal{H}_\omega$ on D_ω has the expression:

(13) $\quad (a\omega|b\omega) = \langle a\omega, b\rangle;$

alternatively:

(13′) $\quad (a\omega|b\omega) = \langle \omega, a^*b\rangle = \omega(b^*a).$

The space D_ω is obviously invariant under left multiplication by elements of $\mathfrak{A}$. We denote π_ω the representation of $\mathfrak{A}$ obtained by restricting to D_ω the representation in $\mathfrak{A}'$ i.e.:

(14) $\quad \pi_\omega(a)\varphi = a\varphi, \quad \varphi \in D_\omega.$

Then π_ω is a $*$-representation of $\mathfrak{A}$ by, in general unbounded, operators in $\mathcal{H}_\omega$ with common domain D_ω (3.1.9). The representation $(\mathcal{H}_\omega, D_\omega, \pi_\omega)$ is called the GNS representation associated to the positive functional ω.

In the case of the example $\mathfrak{A} = \mathcal{C}_c^\infty(G) = \mathcal{D}(G)$, the positive forms on $\mathfrak{A}$ are just the positive definite distributions, denoted $\mathcal{P}(G)$. If ω equals δ, the Dirac delta, H_ω is the space $L^2(G; dg)$ which, as usual, is considered as a Hilbert subspace of $\mathcal{D}'(G)$.

A natural question at this point is whether every invariant operator $H : \mathfrak{A} \longrightarrow \mathfrak{A}'$, i.e. satisfying (10), is associated with some positive functional ω on $\mathfrak{A}$, and if so whether this ω is uniquely defined by the operator or kernel H. More precisely we shall require to know when the correspondence $\omega \longmapsto H_\omega$ is homeomorphism with respect to appropriate topologies.

Let us merely note here that this has an obvious answer if $\mathfrak{A}$ has a unit 1. Since $1 = 1^*$, ω is determined by H:

$$(15) \qquad \omega(a) = H(a, 1).$$

Conversely if H is invariant and ω is determined by (15), we have $H(a, b) = H(b^*a, 1) = \omega(b^*a)$.

In the case $\mathfrak{A} = \mathcal{D}(G)$, $\mathfrak{A}$ has no unit and it is in general impossible to extend ω as positive functional to the algebra obtained by adjoining a unit to $\mathfrak{A}$. This will be shown in the next section. The other sections of this chapter will be devoted to the correspondence $\omega \longmapsto H_\omega$ when $\mathfrak{A}$ does not necessarily have a unit.

Let us end this introductory section by applying section 3.1 to the GNS representation.

Definitions 4. 1. 1. 1. *A positive functional ω on $\mathfrak{A}$ will be called self-adjoint if its associated $*$-representation π_ω on $\mathcal{H}_\omega$ is essentially self-adjoint (cf.* 3. 1).

2. *ω will be called bounded if $\pi_\omega(a)$ is bounded for all $a \in \mathfrak{A}$*

Obviously, any bounded positive functional is a fortiori self-adjoint. Recall the following result [**2**: §37]:

PROPOSITION 4.1.2. *If ω is a positive functional on a Banach $*$-algebra, then ω is bounded.*

Proof. Let ω be a state on a Banach $*$-algebra and let $(D_\omega, \pi_\omega, \mathcal{H}_\omega)$ denote, as usual, the GNS construction associated to ω. For $a \in \mathfrak{A}$, then $\pi_\omega(a)$ is bounded if $\exists M_a > 0$ such that

$$(16) \qquad \|\pi_\omega(a)b\omega\|_{\mathcal{H}_\omega} \leqq M_a\|b\omega\|_{\mathcal{H}_\omega} \quad \forall b \in \mathfrak{A}$$

i.e.

$$\omega(b^*a^*ab) = M_a^2\omega(b^*b) \quad \forall b \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a Banach $*$-algebra with a unit, 1, we know that ω is bounded and $\|\omega\| = \omega(1)$. If $\mathfrak{A}$ has no unit, we define for $b \in \mathfrak{A}$, $\omega_b(a) := \omega(b^*ab)$. As before, let $\mathcal{B} = \mathfrak{A} \oplus \mathbf{C}\,1$ be the algebra obtained from $\mathfrak{A}$ by adjunction of a unit. Because $\mathfrak{A}$ is an ideal in $\mathcal{B}$, we can define $\tilde{\omega}(c) = \omega(b^*cb)$ for all $c \in \mathcal{B}$ and $\tilde{\omega}$ is a positive functional on the algebra $\mathcal{B}$ which is equal to ω_b on $\mathfrak{A}$ (one could check easily that $\tilde{\omega}$ is the extension given by proposition 4.2.1 with $k = \omega(b^*b)$). Thus $\|\tilde{\omega}\| = \tilde{\omega}(1) = \omega(b^*b)$. Hence,

$$\|\pi_\omega(a)b\omega\| = \omega_b(a^*a)^{1/2} \leqq \|a^*a\|^{1/2}\|b\omega\|_{\mathcal{H}_\omega} \quad \forall a, b \in \mathfrak{A}. \qquad \square$$

Remark 4.1.3. In the case of $C_c^\infty(G)$, π_ω is bounded for each ω since it is the smeared version of the unitary representation of G by left translation in $\mathcal{H}_\omega$.

However, even though the tensor algebra $T(\mathcal{S}(\mathbf{R}^4))$ has a unit it is well-known that in that case not all representations of the form π_ω are given by bounded operators.

PROPOSITION 4.1.4. *Let ω and ω' be functionals on $\mathfrak{A}$ such that $0 \leqq \omega' \leqq \omega$. Then if ω is self-adjoint so is ω'.*

Proof. Let $\omega'' = \omega - \omega'$. Then $\mathcal{H}_\omega = \mathcal{H}_{\omega'} + \mathcal{H}_{\omega''}$. Hence $\mathcal{H}_{\omega'} \leqq \mathcal{H}_\omega$. This is therefore a direct consequence of theorem 3.1.11. □

Remark. It is natural to think that a similar statement is valid with the word "self-adjoint" replaced by "bounded" but we have not proved this.

4.2. Algebras with and without unit. Let $\mathfrak{A}$ be a $*$-algebra without unit, and let $\mathcal{B} = \mathfrak{A} + [1]$ be the algebra obtained by adjoining a unit. Recall that for each linear functional f, we define another one by: $f^*(a) := \overline{f(a^*)}$. If $f = f^*$, we say naturally that f is hermitian. The following proposition is well-known.

PROPOSITION 4.2.1. (cf. [**20**: p. 187]) *The necessary and sufficient conditions for a positive functional f on $\mathfrak{A}$ to be the restriction to $\mathfrak{A}$ of a positive linear functional on $\mathcal{B}$, are that*

(a) *there exists a constant $k \geqq 0$ such that:*

(1) $$|f(b)|^2 \leqq kf(b^*b) \quad \forall b \in \mathfrak{A}.$$

(b) *f is hermitian.* □

If $\mathfrak{A}$ has an approximate identity (e_α), then in particular the products of elements of $\mathfrak{A}$ form a total subset of $\mathfrak{A}$. This is equivalent to the fact that each positive functional on $\mathfrak{A}$ is hermitian.

PROPOSITION 4.2.2. *If $\mathfrak{A}$ is barrelled and has a bounded approximate identity i.e. $(e_\alpha)_{\alpha\in I}$ is a bounded subset of $\mathfrak{A}$, then every continuous positive functional on $\mathfrak{A}$ satisfies conditon (1).*

Proof. For each $y \in \mathfrak{A}$ and $\alpha \in I$: $|f(e_\alpha b)|^2 \leqq f(e_\alpha e_\alpha)f(b^*b)$. Now the form $(a, b) \longmapsto f(ab)$ being separately continuous, it is hypocontinuous, and so bounded on a product of two bounded sets. Thus $k = \sup_\alpha f(e_\alpha e_\alpha)$ is finite. Since $|f(e_\alpha b)|^2 \leqq kf(b^*b)$, we get (1) in the limit. (cf. [20: p. 188]) □

Example 1. Let $\mathfrak{A}$ be the convolution algebra $L^1(G)$ on a locally compact unimodular group. Then $\mathfrak{A}$ has a bounded approximate identity, and so every continuous positive form is hermitian and satisfies (1).

PROPOSITION 4.2.3. *Let ω be positive and hermitian on $\mathfrak{A}$. Let $\mathcal{H}_\omega$ be the left* *invariant Hilbert subspace of* $\mathfrak{A}'$ *whose reproducing operator is* $H : a \longmapsto a\omega$. *Then* $\omega \in \mathcal{H}_\omega$ *if and only if* ω *satisfies condition:*

$$(1) \qquad |\omega(a)|^2 \leqq k\omega(a^*a) \quad \forall a \in \mathfrak{A}.$$

Proof. We pose as usual $\langle T, a\rangle = T(a^*)$. Then (1) becomes, given the fact that a positive form is hermitian:

$$(2) \qquad |\langle \omega, a\rangle|^2 \leqq k\langle Ha, a\rangle$$

Equivalently

$$(3) \qquad |\langle \omega, a\rangle| \leqq M\|j^*a\| \quad (M = \sqrt{k}).$$

If this is satisfied there exists an element $h \in \mathcal{H}_\omega$ such that $\langle \omega, a\rangle = (h|j^*a)$, which equals $\langle jh, a\rangle$. Thus $\omega = h \in \mathcal{H}$. Conversely this implies (3) with $M = \|\omega\|$.

Example 2. Since $\mathcal{D}(G)$ has an approximate identity every element of $\mathcal{P}(G)$ is hermitian. In this particular case, (1) is equivalent to two further properties.

PROPOSITION 4.2.4. *Let* $\omega \in \mathcal{P}$. *Then the following are equivalent:*
(1) $\omega \in \mathcal{H}_\omega$.
(2) ω *is a continuous function (i.e.* $\omega = T_f$, *with* $f \in \mathcal{C}(G)$*).*
(3) ω *is the restriction to* $\mathcal{D}(G)$ *of a positive linear functional on* $L^1(G)$.

Proof. (2) $\Rightarrow$ 3. If $\omega = T_f$ is a positive functional it is well known that f is a continuous positive definite function. Thus ω extends to a positive functional on L^1.

(3) $\Rightarrow$ 2. Conversely it is well known that a positive functional on L^1 is associated to a positive definite continuous function.

(2) $\Rightarrow$ 1. Lt $K(t,s) = f(t^{-1}s)$. Then K is the reproducing kernel of a Hilbert subspace $\mathcal{H} \hookrightarrow \mathcal{C}(G)$, with $L_t f \in \mathcal{H}$. Now if H is the reproducing operator of $\mathcal{H}$ as a subspace of $\mathcal{D}'$, we have $H\varphi = \int \varphi(t)K_t dt = \varphi * f = \varphi * \omega = H_\omega \varphi$. Thus $\mathcal{H} = \mathcal{H}_\omega$, and so $\omega = T_f \in \mathcal{H}_\omega$.

(1) $\Rightarrow$ 2. We have for all $T \in \mathcal{H}_\omega : (T|\varphi * \omega) = \langle T, \varphi\rangle$. Then it is easily verified that T is the regular distribution associated with the function $t \longmapsto (T|\delta_t * \omega)$.

Example 3. Let L be any countable set. Let $\mathfrak{A}$ be the set of matrices $(a_{ij})_{i\in L, j\in L}$, vanishing except for a finite number of terms. Let 1 be the identity matrix, $(\delta_{ij})_{i\in L, j\in L}$ and let $\mathcal{B}$ be the algebra generated by $\mathfrak{A}$ and 1 in the algebra of all matrices whose rows and columns have finite length. Then $\mathfrak{A}$ has an approximate identity, namely the sequence (1_Λ) where $1_\Lambda = \delta_{ij}$ if $i, j \in \Lambda$, and 0 elsewhere, Λ being a finite subset of L. We equip $\mathfrak{A}$ with the inductive limit topology $\mathfrak{A} = \cup \mathfrak{A}_\Lambda, \Lambda$ finite $\mathfrak{A}_\Lambda = \mathbf{C}^{\Lambda \times \Lambda}$. Then it is a strict inductive limit. The approximate identity is therefore not bounded.

Let $\omega(A) = \text{trace}(A)$. Then ω is a positive form and $\omega(A^*A) = \|A\|_2^2$, the square of the Hilbert-Schmidt norm. Now if we had (5) i.e.: $|tr(A)| \leqq M\|A\|_2$ it would follow that every Hilbert-Schmidt matrix is of trace class. This is not the case. Consequently the trace does not extend to a positive form on $\mathcal{B}$.

Since $\mathfrak{A} = \mathbf{C}^{(L \times L)}$, the anti-dual of $\mathfrak{A}$ is the space of all matrices: $\mathbf{C}^{L \times L}$, the anti-duality is:

$$\langle A, B \rangle = tr(AB^*) = \sum_{ij} A_{ij} \bar{B}_{ij}.$$

The space $\mathcal{H}_\omega$ associated to the trace is clearly the space of all Hilbert-Schmidt matrices, i.e. $\ell^2(L \times L)$. In the anti-duality between $\mathfrak{A}$ and $\mathbf{C}^{L \times L}$ the form ω is identified with the identity matrix $I = (\delta_{ij}) : tr(A) = \langle A, I \rangle$. Thus the condition is not satisfied because the identity matrix is not Hilbert-Schmidt.

Conclusion. These examples show that conditon (1) is in general too restrictive. But the existence of an equicontinuous approximate identity is not unusual in an $\mathcal{LF}$-algebra.

4.3. The Derived Algebra. Let $\mathfrak{A}$ be a topological algebra, i.e. an algebra equipped with a locally convex topology for which the product is separately continuous. We assume $\mathfrak{A}$ to be complete, as is the case when $\mathfrak{A}$ is an $\mathcal{LF}$-space. Let $p : \mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$ be the map $(x, y) \longmapsto xy$. Then p induces a continuous linear map $P = P_\mathfrak{A}$:

(1) $\qquad P : \mathfrak{A} \bar{\otimes} \mathfrak{A} \longrightarrow \mathfrak{A}.$

(Recall that for all matters concerning topological tensor products we refer to [**13**: Produits tensoriels topologiques]). Let $\mathcal{D}\mathfrak{A} = P(\mathfrak{A} \bar{\otimes} \mathfrak{A})$ be the image, and $\mathcal{N} = \text{Ker}(P)$. Then we have a linear bijection:

(2) $\qquad \dot{P} : \mathfrak{A} \bar{\otimes} \mathfrak{A} / \mathcal{N} \longrightarrow \mathcal{D}\mathfrak{A}.$

We equip $\mathcal{D}\mathfrak{A}$ with the quotient topology, i.e. with the topology for which $\dot{P}$ becomes a homeomorphism.

PROPOSITION 4.3.1. (1) *$\mathcal{D}\mathfrak{A}$ is an ideal in $\mathfrak{A}$ and a topological algebra in its own right, with continuous inclusion:*

(3) $\qquad \mathcal{D}\mathfrak{A} \hookrightarrow \mathfrak{A}$

More precisely: if $a \in \mathfrak{A}$ the operators $L_a : x \longrightarrow ax$, and $R_a : x \longrightarrow xa$ leave $\mathcal{D}\mathfrak{A}$ invariant and are continuous in $\mathcal{D}\mathfrak{A}$.

(2) *If* $\mathfrak{A}$ *is a Banach algebra (resp. a Fréchet algebra, resp. a nuclear Fréchet algebra)* $\mathcal{D}\mathfrak{A}$ *has the same property.*

(3) *If* $\mathfrak{A}$ *is a topological* $*$*-algebra,* $\mathcal{D}\mathfrak{A}$ *is invariant under the involution and a topological* $*$*-algebra in its own right.*

(4) *If* $u : \mathfrak{A} \longrightarrow \mathcal{B}$ *is a continuous homomorphism there exists a unique continuous homomorphism* $\mathcal{D}u : \mathcal{D}\mathfrak{A} \longrightarrow \mathcal{D}\mathcal{B}$ *such that* $\mathcal{D}u \circ P_{\mathfrak{A}} = P_{\mathcal{B}} \circ u \otimes u$. *We have* $u(\mathcal{D}\mathfrak{A}) \subset \mathcal{D}\mathcal{B}$, *and* $\mathcal{D}u$ *is precisely the restriction-corestriction of* u *to* $\mathcal{D}\mathfrak{A}$ *and* $\mathcal{D}\mathcal{B}$.

(5) *If* $\mathfrak{A}$ *and* $\mathcal{B}$ *are* $*$*-algebras and* u *is a* $*$*-homomorphism,* $\mathcal{D}u$ *is a* $*$*-homomorphism.*

(6) *If* $U = \{u\}$ *is an equicontinuous set of homorphisms* $\mathfrak{A} \longrightarrow \mathcal{B}$ *the set* $\{\mathcal{D}u\}_{u\in U}$ *is equicontinuous.*

(7) *Let* $(e_\alpha)_{\alpha\in I}$ *be an equicontinuous approximate identity in the algebra* $\mathfrak{A}$. *Then the maps* L_{e_α} *(resp.* R_{e_α}*) are equicontinuous in* $\mathcal{D}\mathfrak{A}$, *and converge pointwise to the identity of* $\mathcal{D}\mathfrak{A}$. *The family* $(e_\alpha^2)_{\alpha\in I}$ *is an equicontinuous approximate identity in* $\mathcal{D}\mathfrak{A}$.

LEMMA 4.3.2. *Let* $w : \mathfrak{A} \longrightarrow \mathcal{B}$ *be a linear map with the property that there exists a continuous linear map* $v : \mathfrak{A}\bar{\otimes}\mathfrak{A} \longrightarrow \mathcal{B}\bar{\otimes}\mathcal{B}$ *such that* $w \circ P = P \circ v$. *Then* $w(\mathcal{D}\mathfrak{A}) \subset \mathcal{D}\mathcal{B}$ *and the restriction-corestriction of* w *to* $\mathcal{D}\mathfrak{A}$ *and* $\mathcal{D}\mathcal{B}$ *is continuous.*

$$
\begin{array}{ccc}
\mathfrak{A}\,\bar{\otimes}\,\mathfrak{A} & \xrightarrow{\;\;v\;\;} & \mathcal{B}\,\bar{\otimes}\,\mathcal{B} \\
{\scriptstyle P}\big\downarrow & & \big\downarrow{\scriptstyle P} \\
\mathcal{D}\mathfrak{A} & \xrightarrow{\;\;w\;\;} & \mathcal{D}\mathcal{B}
\end{array} \qquad (4)
$$

Moreover, if $(w_i)_{i\in I}$ *and* $(v_i)_{i\in I}$ *are sets of linear maps such that* $w_i \circ P = P \circ v_i$ *for all* $i \in I$, $(w_i)_{i\in I}$ *is equicontinuous in* $\mathcal{L}(\mathcal{D}\mathfrak{A}, \mathcal{D}\mathcal{B}))$ *if* $(v_i)_{i\in I}$ *is equicontinuous in* $\mathcal{L}(\mathfrak{A}\bar{\otimes}\mathfrak{A}, \mathcal{B}\bar{\otimes}\mathcal{B})$.

Proof. Clearly w maps $\mathrm{Im}(P)$ to $\mathrm{Im}(P)$. Now $\mathcal{D}\mathfrak{A}$ having the quotient topology the map $w : \mathcal{D}\mathfrak{A} \longrightarrow \mathcal{D}\mathcal{B}$ is continuous if and only if $w\circ P$ is continuous. But $w\circ P$ equals $P\circ v$ and so is continuous to $\mathcal{D}\mathcal{B}$. Similarly if U is a neighborhood of 0 in $\mathcal{D}\mathcal{B}$, $P^{-1}(\cap_i w_i^{-1}(U)) = \cap_i P^{-1} w_i^{-1}(U) = \cap_i v_i^{-1}(P^{-1}(U))$ is a neighborhood of 0 in $\mathfrak{A}\bar{\otimes}\mathfrak{A}$ and so $\cap_i w_i^{-1}(U)$ is a neighborhood of 0 in $\mathcal{D}\mathfrak{A}$. □

Proof of 4.3.1. (1) $axy = P(ax \otimes y)$. Thus if $L_a : \mathfrak{A} \longrightarrow \mathfrak{A}$ is the operator of left translation we have, $L_a P(x \otimes y) = P(L_a x \otimes y)$, and so by continuity $L_a P = P \circ (L_a \otimes I)$. It follows that the operator L_a leaves $\mathcal{D}\mathfrak{A}$ invariant and is continuous. Similarly R_a corresponds to $I \otimes R_a$.

(2) If $\mathfrak{A}$ is a Fréchet space the separately continuous linear maps on

$\mathfrak{A} \times \mathfrak{A}$ are continuous, and the separately equicontinuous sets of bilinear maps are equicontinuous. Thus $\mathfrak{A}\bar{\otimes}\mathfrak{A} = \mathfrak{A}\hat{\otimes}\mathfrak{A}$. Consequently $\mathfrak{A}\bar{\otimes}\mathfrak{A}$ and $\mathfrak{A}\bar{\otimes}\mathfrak{A}/\mathcal{N}$ are Banach, resp, Fréchet resp. nuclear Fréchet spaces.

(3) Let $I : x \longrightarrow x^*$ be the involution and $\sigma : \mathfrak{A}\bar{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}\bar{\otimes}\mathfrak{A}$ be the exchange map: $\sigma(x \otimes y) = y \otimes x$. Since $(xy)^* = y^*x^*$ we have $I \circ P = P \circ \sigma \circ (I \otimes I)$. It follows that the $\mathbf{R}$-linear operator I leaves $\mathcal{D}\mathfrak{A}$ invariant, and that it is continuous.

(4) We have $u(xy) = u(x)u(y)$. By continuity $u \circ P = P \circ (u \otimes u)$. This implies that $u(\mathcal{D}\mathfrak{A}) \subset \mathcal{D}\mathcal{B}$ and that the restriction-corestriction $\mathcal{D}u$ of u to $\mathcal{D}\mathfrak{A}$ and $\mathcal{D}\mathcal{B}$ is continuous.

(5) This is obvious because the involutions in $\mathcal{D}\mathfrak{A}$ and $\mathcal{D}\mathcal{B}$ are the restrictions to $\mathcal{D}\mathfrak{A}$ and $\mathcal{D}\mathcal{B}$ of the involutions in $\mathfrak{A}$ and $\mathcal{B}$ respectively.

(6) Similarly $\{\mathcal{D}u\}_{u\in U}$ is equicontinuous if and only if the set $\{\mathcal{D}u \circ P\}_{u\in U}$ is equicontinuous. But this set is $\{P \circ (u \otimes u)\}_{u\in U}$ and this is equicontinuous because the set $\{u \otimes u\}_{u\in U}$ is equicontinuous [**13**: Produits tensoriels topologiques, Prop. 13, p. 73].

(7) As in the proof of (1) we have $L_{e_\alpha}P = P \circ (L_{e_\alpha} \otimes I)$. The maps $L_{e_\alpha} \otimes I$ being equicontinuous in $\mathfrak{A}\bar{\otimes}\mathfrak{A}$, it follows that the operators L_{e_α} are equicontinuous on $\mathcal{D}\mathfrak{A}$. Since $L_{e_\alpha}(xy) = P(e_\alpha x \otimes y)$ and the canonical map $(x, y) \longmapsto x \otimes y$ is separately continuous, it follows that $L_{e_\alpha}(xy)$ tends to xy in $\mathcal{D}\mathfrak{A}$. The products being total in $\mathcal{D}\mathfrak{A}$ (because the elements $x \otimes y$ are total in $\mathfrak{A}\bar{\otimes}\mathfrak{A}$) the pointwise convergence (and even uniform convergence on compact sets) follows from the equicontinuity. To prove the last statement it is sufficient to show that the family $(e_\alpha^2)_\alpha$ is an equicontinuous approximate identity in $\mathfrak{A}$. The operators $L_{e_\alpha}^2 = (L_{e_\alpha}^2)$ are equicontinuous. It is sufficient to prove that they converge pointwise to the identity in $\mathcal{L}(\mathfrak{A})$. Let p be a continuous seminorm on $\mathfrak{A}$. Then there exists a continuous seminorm q on $\mathfrak{A}$ such that $p(e_\alpha x) \leqq q(x)$ for all $x \in \mathfrak{A}$. Hence $p(e_\alpha e_\beta x - x) \leqq p(e_\alpha e_\beta x - e_\alpha x) + p(e_\alpha x - x) \leqq q(e_\beta x - x) + p(e_\alpha x - x)$ which is arbitrary small for α, β larger than some index α_0, in particular for $\alpha = \beta \geqq \alpha_0$. □

As we first remarked in the introduction, we are particularly interested in algebras $\mathfrak{A}$ for which $\mathcal{D}\mathfrak{A} = \mathfrak{A}$.

Definition 4.3.3. A *self-derivative* algebra is a topological algebra $\mathfrak{A}$ such that $\mathcal{D}\mathfrak{A} = \mathfrak{A}$, as topological algebras.

Equivalently the map $\dot{P} : \mathfrak{A}\bar{\otimes}\mathfrak{A}/\mathcal{N} \longrightarrow \mathfrak{A}$ is a linear topological isomorphism, or $P : \mathfrak{A}\bar{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}$ is surjective and open.

Remark 4.3.4. Let $\mathfrak{A}$ be a Fréchet algebra. Then $\mathfrak{A}\bar{\otimes}\mathfrak{A} = \mathfrak{A}\hat{\otimes}\mathfrak{A}$, and every element $t \in \mathfrak{A}\hat{\otimes}\mathfrak{A}$ has a series expansion $t = \sum_{n\in\mathbf{N}} \lambda_n a_n \otimes b_n$ with $(\lambda_n) \in \ell^1$, and (a_n) and (b_n) bounded in $\mathfrak{A}$. [**13**: Produits tensoriels topologiques, Thm. 1, p. 51]. In this case $\mathcal{D}\mathfrak{A}$ is the set of elements $a \in \mathfrak{A}$ which can be written as a

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convergent sum:

$$(5) \qquad a = \sum_{n \in \mathbf{N}} \lambda_n a_n b_n$$

with $(a_n)_{n \in \mathbf{N}}$ and $(b_n)_{n \in \mathbf{N}}$ bounded in $\mathfrak{A}$ and $(\lambda_n)_{n \in \mathbf{N}} \in \ell^1$. If $\mathfrak{A}$ is a Banach algebra it is simpler to write (5) as:

$$(5') \qquad a = \sum_{n \in \mathbf{N}} a_n b_n$$

where $\sum_{n \in \mathbf{N}} \|a_n\| \, \|b_n\| < +\infty$.

Thus $\mathcal{D}\mathfrak{A} = \mathfrak{A}$ if and only if every element a in $\mathfrak{A}$ has a series expansion as in (5) or (5′). In that case, $\mathcal{D}\mathfrak{A}$ and $\mathfrak{A}$ being Fréchet spaces, the equality $\mathcal{D}\mathfrak{A} = \mathfrak{A}$ is a linear topological isomorphism by the open mapping theorem.

If $\mathfrak{A} = \lim\limits_{\vec{n}} E_n$ is a nuclear $\mathcal{LF}$ space, strict inductive limit of closed Fréchet subspaces E_n, $\mathfrak{A}\bar{\otimes}\mathfrak{A} = \lim\limits_{\vec{n}} E_n \hat{\otimes} E_n$ is a strict inductive limit of tensor products of Fréchet spaces and every $t \in \mathfrak{A}\bar{\otimes}\mathfrak{A} = \cup_n E_n \hat{\otimes} E_n$ has a series expansion as before. The open mapping theorem is also valid in this case [**13**: Produits tensoriels topologiques, Thm. B, p. 17] and so $\mathfrak{A}$ is self-derivative iff every element $a \in \mathfrak{A}$ has an expansion as in (5).

This shows that self-derivative algebras are algebras in which the product does not 'regularise': products and sums of products are not better than any other element in the algebra.

Examples.

(1) *Convolution group algebras.* Consider the convolution algebra $\mathcal{C}(\mathbf{T})$ of continuous functions on the circle equipped with the sup norm, it is a cummutative Banach algebra. Let $\mathfrak{A}(\mathbf{T})$ be the algebra of continuous functions with absolutely convergent Fourier series equipped with its usual topology: the topology for which the Fourier transform is an isomorphism with $\ell^1(\mathbf{Z})$. It is a convolution subalgebra of the convolution algebra $\mathcal{C}(\mathbf{T})$ (the pointwise product of two ℓ^1 sequences being in ℓ^1). The space $L^2(\mathbf{T})$ is also a convolution algebra, containing $\mathcal{C}(\mathbf{T})$ as a dense subalgebra: $\mathcal{C}(\mathbf{T}) \hookrightarrow L^2(\mathbf{T})$. The convolution product of two L^2-functions is continuous. In fact L^2 is not self-derivative. Even $\mathcal{C}(\mathbf{T})$ is not self-derivative. We have precisely:

$$\mathcal{D}L^2(\mathbf{T}) = \mathfrak{A}(\mathbf{T}), \; \mathcal{D}\mathcal{C}(\mathbf{T}) = \mathfrak{A}(\mathbf{T}), \; \mathcal{D}\mathfrak{A}(\mathbf{T}) = \mathfrak{A}(\mathbf{T}).$$

the equalities being linear topological isomorphism.

Similar results apply to more general compact abelian groups.

Proof. We restrict ourselves to the case of the torus $\mathbf{T}$. Then $L^1(\hat{\mathbf{T}}) = \ell^1(\mathbf{Z})$. We know that the product $(\varphi,\psi) \longmapsto \varphi * \psi$ is continuous from $L^2(\mathbf{T}) \times L^2(\mathbf{T})$ to $\mathfrak{A}(\mathbf{T})$, and so a fortiori it is continuous from $\mathcal{C}(\mathbf{T}) \times \mathcal{C}(\mathbf{T})$ to $\mathfrak{A}(\mathbf{T})$. Thus $\mathcal{D}\mathcal{C}(\mathbf{T}) \hookrightarrow \mathcal{D}L^2(\mathbf{T}) \hookrightarrow \mathfrak{A}(\mathbf{T})$ the inclusions being continuous. Now every element $\varphi \in \mathfrak{A}(\mathbf{T})$ can be written $\varphi = \sum_{n\in\mathbf{Z}} \lambda_n e_n$, where $(\lambda_n)_{n\in\mathbf{Z}} \in \ell^1(\mathbf{Z})$ and $e_n(\theta) = e^{in\theta}$. The series converges in the algebra $\mathfrak{A}(\mathbf{T})$. Since $e_n * e_n = e_n$ has norm 1 in $\mathfrak{A}(\mathbf{T})$ we can interpret this as $\varphi = P(\varphi)$ where $\varphi = \sum_{n\in\mathbf{Z}} \lambda_n e_n \otimes e_n \in \mathfrak{A}(\mathbf{T})\hat{\otimes}\mathfrak{A}(\mathbf{T})$. Thus $\mathcal{D}\,\mathfrak{A}(\mathbf{T}) = \mathfrak{A}(\mathbf{T})$. In particular $\mathfrak{A}(\mathbf{T}) = \mathcal{D}\,\mathfrak{A}(\mathbf{T}) \hookrightarrow \mathcal{D}\mathcal{C}(\mathbf{T}) \hookrightarrow \mathcal{D}L^2(\mathbf{T})$ which proves the equalities. These are topological isomorphisms by the open mapping theorem. □

(2) *Unital algebras.* Every topological algebra with a left or right unit is self-derivative.

If 1 is a right unit we have $a = a1 = P(a \otimes 1)$, so P is surjective. To show that P is open, or equivalently that $\dot{P}$ is an isomorphism, let $Q(a) = a \otimes 1$ and let $\pi : \mathfrak{A}\bar{\otimes}\mathfrak{A} \longrightarrow \mathfrak{A}\bar{\otimes}\mathfrak{A}/\mathcal{N}$ be the canonical map. Then $P(Q(a)) = a$ and so $P \circ Q = \dot{P} \circ \pi \circ Q = I$ the identity on $\mathfrak{A}$. Thus $\dot{P}$ has $\pi \circ Q$ as continuous inverse. A similar argument applies to a left unit.

(3) L^1 *convolution algebras.* Let G be a locally compact group. Then the convolution algebra $L^1(G)$ is self-derivative.

Proof. It is well known that $L^1(G)\hat{\otimes}L^1(G) = L^1(G \times G)$. We have to identify the map $P : L^1(G \times G) \longrightarrow L^1(G)$. For $\varphi, \psi \in L^1(G)$ we have $P(\varphi \otimes \psi) = \varphi * \psi$. Thus almost everywhere $P(\varphi \otimes \psi)(y) = \int \varphi(x)\psi(x^{-1}y)dx$. Generally therefore, we have, for every $K \in L^1(G \times G)$, $P(K)(y) = \int K(x, x^{-1}y)dx$ almost everywhere. In fact by the left invariance of the Haar measure $\int dy \int |K(x, x^{-1}y)|dx = \int\int |K(x,y)|dxdy$ so this map is well defined and continuous, and has the right value on $\varphi \otimes \psi$. To prove that $\mathcal{D}L^1 = L^1$ we have to show that P is surjective. Let $f \in L^1(G)$. Let $\varphi \in L^1(G)$ be a function with $\int \varphi(x)dx = 1$. Then, if $K(x,y) = \varphi(x)f(xy)$, K belongs to $L^1(G \times G)$ and $PK = f$. □

(4) *The algebra* $\mathcal{C}_0(T)$. Let T be a locally compact space. Then the algebra $\mathcal{C}_0(T)$ of continuous functions vanishing at infinity, with pointwise defined product, is self-derivative. In fact every non negative element is a square, and so every element is a sum of at most four squares.

(5) *C$*$-algebras.* More generally, if $\mathfrak{A}$ is a not necessarily unital C^*-algebra, every non-negative element is a square, and every element can be written as a sum of at most four squares. Thus $\mathfrak{A}$ is its own derivative algebra.

(6) *The group algebras.* Let G be a locally compact abelian group and $\hat{G}$ the dual group. Then the group algebra $\mathfrak{A}(B)$, with pointwise multiplication and with the topology making the Fourier transform from $\mathfrak{A}(G)$ to $L^1(\hat{G})$ an isomorphism, is self-derivative. This is an immediate consequence of example (3). If G is compact the result is obvious because $\mathfrak{A}(G)$ has a unit.

Question. If G is a non cummutative locally compact group one can still define the group algebra $\mathfrak{A}(G)$, for instance as the set of convolution products of L^2 functions (cf. [**10**: p. 218]). Is $\mathcal{D}\mathfrak{A}(G) = \mathfrak{A}(G)$?

(7) *The algebra* $\mathcal{C}_c^\infty(G)$. Let G be a Lie group. Then the convolution algebra $\mathcal{C}_c^\infty(C)$ composed of infinitely differentiable functions with compact support is self-derivative.

Since $\mathcal{C}_c^\infty(G)\bar{\otimes}\mathcal{C}_c^\infty(G) = \mathcal{C}_c^\infty(G \times G)$ this can be proved in the same way as in the case of $L^1(G)$. It suffices to choose the auxilary function φ in $\mathcal{C}_c^\infty(G)$. Since $\mathcal{C}_c^\infty(G)$ is a strict inductive limit of a sequence of the closed subspaces $\mathcal{C}_K^\infty(G)$ of functions with support in K, this shows that every $\varphi \in \mathcal{C}_c^\infty(G)$ has a series expansion:

$$\varphi = \sum_{n\in\mathbf{N}} \lambda_n \varphi_n * \psi_n \tag{6}$$

where the sequences (φ_n) and (ψ_n) are bounded in $\mathcal{C}_c^\infty(G)$ and $(\lambda_n) \in \ell^1$.

In fact, the theorem of Dixmier and Malliavin [**8**] shows that every $\varphi \in \mathcal{C}_c^\infty(G)$ has a finite expansion $\varphi = \sum_{n=1}^N \varphi_n * \psi_n$, with $N \leqq 2^{dim(G)}$.

(8) *Trace class and Hilbert Schmidt operators.* Let $\mathcal{H}$ be a Hilbert space, and let $\mathcal{L}_1(\mathcal{H})$ be the Banach algebra of trace class operators. Then $\mathcal{L}_1(\mathcal{H})$ is self-derivative.

Let $A \in \mathcal{L}_1(\mathcal{H})$ be self-adjoint. Then we have the spectral expansion

$$A = \sum_n \lambda_n P_n, \tag{7}$$

the P_n being finite dimensional orthogonal projections, and (λ_n) being a sequence such that $\sum_n |\lambda_n| rank(P_n) < +\infty$. Thus the series converges absolutely in the space $\mathcal{L}_1(\mathcal{H})$. Since the P_n are idempotent this is an expansion as in (5′). If A is not self-adjoint one can use a polar decomposition $A = VS$, with S a self-adjoint trace class operator, and we get a decomposition $A = \sum_n \lambda_n VP_nP_n = \sum_n A_nB_n$ as in (5′).

Now consider the algebra $\mathcal{L}_2(\mathcal{H})$ of Hilbert Schmidt operators. Then $\mathcal{D}\mathcal{L}_2(\mathcal{H}) = \mathcal{L}_1(\mathcal{H})$. In fact, if $A = \sum_n A_nB_n$ with $\sum_n \|A_n\|_2\|B_n\|_2 < +\infty$, the index indicating the Hilbert-Schmidt norm, we have $A \in \mathcal{L}_1(\mathcal{H})$. Thus $\mathcal{D}\mathcal{L}_2(\mathcal{H}) \hookrightarrow \mathcal{L}_1(\mathcal{H})$. Conversely every trace class operator factors as a product of Hilbert Schmidt operators, so this inclusion is a surjection.

Similarly, if we consider the spaces ℓ^1 and ℓ^2 with pointwise multiplication, we have $\mathcal{D}\ell^2 = \ell^1$, and $\mathcal{D}\ell^1 = \ell^1$.

In this case ℓ^1 and ℓ^2 can be identified with the closed subspaces of $\mathcal{L}_1(\mathcal{H})$ and $\mathcal{L}_2(\mathcal{H})$ composed of diagonal operators (with respect to some countable orthonormal basis).

Higher derivatives. Now let

$$\mathfrak{A}^{(0)} = \mathfrak{A},\ \mathfrak{A}^{(n)} = \mathcal{D}\mathfrak{A}^{(n-1)} \text{ if } n \geqq 1, \text{ and } \mathfrak{A}^{(\infty)} = \bigcap_n \mathfrak{A}^{(n)},$$

the last space being equipped with the weakest topology for which the inclusions in the algebras $\mathfrak{A}^{(n)}$ are continuous. In other words $\mathfrak{A}^{(\infty)}$ is the projective limit of the spaces $\mathfrak{A}^{(n)}$: $\mathfrak{A}^{(\infty)} = \varprojlim \mathfrak{A}^{(n)}$. One then has:

$$(1) \qquad \mathfrak{A}^{(\infty)} \hookrightarrow \mathfrak{A}^{(n)} \hookrightarrow \mathfrak{A}^{(n-1)} \hookrightarrow \ldots \hookrightarrow \mathfrak{A}^{(1)} \hookrightarrow \mathfrak{A}$$

Note that if $\mathfrak{A}^{(n)} = \mathfrak{A}^{(n+1)}$ one also has $\mathfrak{A}^{(n)} = \mathfrak{A}^{(k)}$ for all $k \geqq n$. Let us remark also that if $\mathfrak{A}$ is a Fréchet algebra so are the algebras $\mathfrak{A}^{(n)}$ and $\mathfrak{A}^{(\infty)}$.

In all the above examples we had either $\mathfrak{A}^{(1)} = \mathfrak{A}$, or $\mathfrak{A}^{(2)} = \mathfrak{A}^{(1)}$. It is easy however to give examples of algebras where all the derivatives are different.

(9) *The algebra* $\mathcal{C}^{(n)}(\mathbf{T})$. Let $\mathfrak{A} = \mathcal{C}^{(n)}(\mathbf{T})$ denote the convolution algebra of all n times continuously differentiable functions on $\mathbf{T}$. Then $\mathcal{D}(\mathcal{C}^{(n)}(\mathbf{T})) \subset \mathcal{C}^{(2n)}(\mathbf{T})$ and $\mathfrak{A}^{(\infty)} = \mathcal{C}^{(\infty)}(\mathbf{T})$ the space of Schwartz test functions. Note that $\mathfrak{A}^{(\infty)}$ is a nuclear space although all the $\mathfrak{A}^{(n)}$ are Banach spaces.

In fact, we have more generally:

PROPOSITION 4.3.5. *If $\mathfrak{A}$ is a Banach algebra then $\mathfrak{A}^{(\infty)} = \varprojlim \mathfrak{A}^{(n)}$ is a nuclear space if and only if $\forall n, \exists m > n$ such that the natural inclusion of $\mathfrak{A}^{(m)}$ into $\mathfrak{A}^{(n)}$ is absolutely summing* (see [**21**: p. 36] *for the definition of such maps).*

Since the composition of two absolutely summing operators gives a nuclear operator we have that this condition is equivalent to the existence for each n of an m such that the inclusion of $\mathfrak{A}^{(m)}$ into $\mathfrak{A}^{(n)}$ is nuclear.

(10) *The tensor algebra*. Let $T(E)$ be the tensor algebra over the locally convex space E (see 2.2). More generally let $T_n(E)$ be the locally convex direct sum of the completed tensor products $E^{\otimes k}$, with $k \geqq n$. Then $T(E) = \mathbf{C} \oplus T_1(E)$. Clearly the subspaces $T_n(E)$ are closed ideals in $T(E)$. Since $T(E)$ is unital $\mathcal{D}T(E) = T(E)$. On the other hand, $T_1(E)$ is not unital, and $\mathcal{D}T_1(E) = T_2(E)$. More generally: $\mathcal{D}T_n(E) \subset T_{2n}(E)$. Thus if $\mathfrak{A} = T_1(E)$ the inclusions (1) are all strict, and $\mathfrak{A}^{(\infty)} = (0)$.

In contrast to this we have the following:

PROPOSITION 4.3.6. *Let $\mathfrak{A}$ be a Fréchet algebra.*

(1) *If the products xy form a total subset of $\mathfrak{A}$ then $\mathfrak{A}^{(n)}$ is a dense subspace of $\mathfrak{A}^{(n-1)}$ for all n, and $\mathfrak{A}^{(\infty)}$ is dense in $\mathfrak{A}$.*

(2) *$\mathfrak{A}^{(n)}$ is an ideal in $\mathfrak{A}$ for all $n \leqq \infty$. If $a \in \mathfrak{A}$, L_a and R_a are continuous in $\mathfrak{A}^{(n)}$ for all n.*

(3) *If (e_α) is an equicontinuous approximate unit in $\mathfrak{A}$, L_{e_α} is an equicontinuous approximation of I in $\mathcal{L}(\mathfrak{A}^{(n)})$ for all $n \leqq \infty$.*

(4) *If moreover $(e_\alpha) \subset \mathfrak{A}^{(\infty)}$, (e_α) is an equicontinuous approximation of the identity in $\mathfrak{A}^{(n)}$ for all $n \leqq \infty$.*

(5) *If $\mathfrak{A}$ is a separable Fréchet algebra having an equicontinuous approximate unit, there exists a sequential approximate unit $(e_k)_{n \in \mathbf{N}}$, with $e_k \in \mathfrak{A}^{(\infty)}$ $\forall k$. Consequently this is an approximate unit for $\mathfrak{A}^{(n)}$ for all $n \leqq \infty$.*

Proof. (1) By hypothesis $\mathfrak{A}^{(1)} = \mathcal{D}\mathfrak{A}$ is dense in $\mathfrak{A}$. It follows that $\mathfrak{A}^{(1)}\bar{\otimes}\mathfrak{A}^{(1)}$ has dense image in $\mathfrak{A}\bar{\otimes}\mathfrak{A}$ and thus $\mathfrak{A}^{(2)} = \mathcal{D}\mathfrak{A}^{(1)}$ is dense in $\mathfrak{A}^{(1)}$. Similarly $\mathfrak{A}^{(n)}$ is dense in $\mathfrak{A}^{(n-1)}$. Now the fact that $\mathfrak{A}^{(\infty)}$ is dense in $\mathfrak{A}$, and in all the algebras $\mathfrak{A}^{(n)}$ is a consequence of the abstract Mittag Leffler theorem [**5**: Chap. 2, §3, no. 5].

(2) For finite n, we do an induction on n. Assume that L_a is continuous on $\mathfrak{A}^{(n)}$. Then the relation $L_a P = P(L_a \otimes I)$ and lemma 4.3.2 give the continuity of L_a on $\mathfrak{A}^{(n+1)}$. For $n = \infty$, note first that an intersection of ideals is an ideal. Now $L_a : \mathfrak{A}^{(\infty)} \longrightarrow \mathfrak{A}^{(\infty)}$ is continuous, because $L_a : \mathfrak{A}^{(\infty)} \longrightarrow \mathfrak{A}^{(n)}$ is continuous for all n since it is the composition of the injection $\mathfrak{A}^{(\infty)} \hookrightarrow \mathfrak{A}^{(n)}$ with $L_a : \mathfrak{A}^{(n)} \longrightarrow \mathfrak{A}^{(n)}$. The proof for right multiplication is entirely analogous.

(3) We prove it again by induction on n. Assume that L_{e_α} is an equicontinuous approximation of I in $\mathcal{L}(\mathfrak{A}^{(n)})$. Again using the relation $L_{e_\alpha}P = P(L_{e_\alpha} \otimes I)$ and lemma 4.3.2 we get the desired conclusion. The argument for $\mathfrak{A}^{(\infty)}$ parallels the argument used in 2.

(4) Obvious by 3.

(5) Let $\{p_n\}$ be an increasing sequence of seminorms defining the topology of $\mathfrak{A}$. Let $H = \{L_{e_\alpha}\}$ be an equicontinuous approximation of the identity in $\mathcal{L}(\mathfrak{A})$. $\mathfrak{A}$ being separable, H and its closure are metrizable. Hence there exists a sequence $e_n \in \mathfrak{A}$ such that L_{e_α} converges to I. Let $\tilde{e}_n \in \mathfrak{A}^{(\infty)}$ be such that $p_n(\bar{e}_n - e_n) \leqq 1/n$. By 4. $(\bar{e}_n)$ is an approximate unit in $\mathfrak{A}^{(\infty)}$ and in $\mathfrak{A}^{(n)}$ $\forall n \in \mathbf{N}$. □

It is well-known that C^*-algebras and $L^1(G)$ do have bounded approximate identities and that the convolution algebra $C_c^\infty(G)$, for a Lie group G, has an equicontinuous approximate identity. In ℓ_1, with pointwise multiplication, the sequence $\sum_{i=1}^n e_i$ does the trick. Similarly, $\mathcal{L}_1(\mathcal{H})$ has an equicontinuous approximate identity composed of finite rank projections.

THEOREM 4.3.7. *If $\mathfrak{A}$ is a nuclear Fréchet algebra with an equicontinuous approximate identity then $\mathfrak{A}^{(\infty)}$ has the following properties:*

(i) *$\mathfrak{A}^{(\infty)}$ is a nuclear Fréchet algebra.*

(ii) *$\mathcal{D}(\mathfrak{A}^{(\infty)}) = \mathfrak{A}^{(\infty)}$, i.e. $\mathfrak{A}^{(\infty)}$ is self-derivative.*

(iii) *$\mathfrak{A}^{(\infty)}$ has an equicontinuous approximate identity.*

Proof. (i) $\mathcal{D}(\mathfrak{A}) = \mathfrak{A}\hat{\otimes}\mathfrak{A}/\mathrm{Ker}(P)$ is a nuclear and Fréchet space. Hence $\mathfrak{A}^{(n)} = \mathcal{D}(\mathfrak{A}^{(n-1)})$ is also nuclear and Fréchet. Thus $\mathfrak{A}^{(\infty)}$ as the projective limit of nuclear and Fréchet spaces is itself nuclear and Fréchet by [**12** Espaces nucléaires; p. 48]. (iii) and the remainder of (i) are consequences of 4.3.6.

(ii) Let

$$\pi_{n,n+1} : \mathfrak{A}^{(n+1)}\hat{\otimes}\mathfrak{A}^{(n+1)} \longrightarrow \mathfrak{A}^{(n)}\hat{\otimes}\mathfrak{A}^{(n)}$$

and

$$\pi_{n+1} : \mathfrak{A}^{(\infty)}\hat{\otimes}\mathfrak{A}^{(\infty)} \longrightarrow \mathfrak{A}^{(n+1)}\hat{\otimes}\mathfrak{A}^{(n+1)}$$

be the maps induced by the inclusions j_n : $\mathfrak{A}^{(\infty)} \hookrightarrow \mathfrak{A}^{(n)}$ and $\mathfrak{A}^{(n)} \hookrightarrow \mathfrak{A}^{(n-1)}$. Then these form a projective system: $\pi_n = \pi_{n,n+1} \circ \pi_{n+1}$. It follows that there exists a continuous linear map:

$$\pi : \mathfrak{A}^{(\infty)} \hat{\otimes} \mathfrak{A}^{(\infty)} \longrightarrow \lim_{\overleftarrow{n}} \mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)} \qquad \square$$

LEMMA 4.3.8. *The map π is a linear topological isomorphism.*

Proof. Since $\mathfrak{A}^{(n)}$ is nuclear the ϵ and π tensor product topologies coincide on $\mathfrak{A}^{(n)} \otimes \mathfrak{A}^{(n)}$: $\mathfrak{A}^{(n)} \hat{\otimes}_\pi \mathfrak{A}^{(n)} = \mathfrak{A}^{(n)} \hat{\otimes}_\epsilon \mathfrak{A}^{(n)}$. Similarly $\mathfrak{A}^{(\infty)} \hat{\otimes}_\pi \mathfrak{A}^{(\infty)} = \mathfrak{A}^{(\infty)} \hat{\otimes}_\epsilon \mathfrak{A}^{(\infty)}$. To show that π is an isomorphism we show that a typical continuous seminorm p on $\mathfrak{A}^{(\infty)} \hat{\otimes} \mathfrak{A}^{(\infty)}$ is of the form $q \circ \pi$, where q is a continuous seminorm on the projective limit. Given the definition of the projective limit this means that there exists n such that p is equal to $q_n \circ \pi_n$ where q_n is a continuous seminorm on $\mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)}$. Now the topology of $\mathfrak{A}^{(\infty)} \hat{\otimes}_\epsilon \mathfrak{A}^{(\infty)}$ being the topology of bi-equicontinuous convergence, the typical seminorm is of the form $p_{A,B}(t) = \sup\{|t(\xi, \eta)| : \xi \in A \text{ and } \eta \in B\}$ where A and B are equicontinuous subsets of $\mathfrak{A}^{(\infty)'}$. Now $\mathfrak{A}^{(\infty)}$ being the projective limit of the spaces $\mathfrak{A}^{(n)}$ there exist an index n and a continuous seminorm p_n on $\mathfrak{A}^{(n)}$ such that the elements of A are in absolute value majorized by $p_n \circ j_n$, i.e. A is the image under the injection $\mathfrak{A}^{(n)'} \hookrightarrow \mathfrak{A}^{(\infty)'}$ of an equicontinuous set A_n in $\mathfrak{A}^{(n)'}$. Similarly B comes from an equicontinuous set B_m in a space $\mathfrak{A}^{(m)'}$. Replacing if need be n and m by the largest of the two, we may assume $m = n$. It then follows that $p_{A,B} = p_{A_n,B_n} \circ \pi_n$. This implies in particular that the map π is one to one and that it is an isomorphism onto its image. Next we observe that π has a dense image. In fact $\mathfrak{A}^{(\infty)}$ being dense in $\mathfrak{A}^{(n)}$, the image of $\mathfrak{A}^{(\infty)} \hat{\otimes} \mathfrak{A}^{(\infty)}$ is dense in $\mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)}$. This being the case for all n, $\mathfrak{A}^{(\infty)} \hat{\otimes} \mathfrak{A}^{(\infty)}$ has a dense image in the projective limit. Now the image being isomorphic to $\mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)}$, hence complete and so closed. This proves the lemma. $\square$

The proof of the theorem is complicated by the fact that we do not know whether the maps $\pi_{n,n+1}$ and π_n are injective. (Contrary to the situation in lemma 2.1.3 we are not dealing here with topological vector space homomorphisms).

To discuss this more easily consider generally a projective sequence $(E_n, \pi_{n,n+1})$ of Fréchet spaces such that the maps $\pi_{n,n+1}$ have dense range. Let $E = \lim_{\overrightarrow{n}} E_n$ be the projective limit, $\pi_n : E \longrightarrow E_n$ the canonical map. Let there be given continuous linear maps $u_n : E_n \longrightarrow F$ to a locally convex space F, such that $u_{n+1} = u_n \circ \pi_{n,n+1}$ for all n. Let $u : E \longrightarrow F$ be the continuous linear map such that $u \circ \pi_n = u_n$. Let N_n (resp. N) be the kernel of u_n (resp. u). Then $\pi_{n,n+1}(N_{n+1}) \subset N_n$, and $\pi_n(N) \subset N_n$. Thus there are maps $\dot{\pi}_{n,n+1} : E_{n+1}/N_{n+1} \longrightarrow E_n/N_n$ and $\dot{\pi}_n : E/N \longrightarrow E_n/N_n$ obtained by passing to the quotients. Obviously $(E_n/N_n, \dot{\pi}_{n,n+1})$ is a projective system and $\dot{\pi}_n = \dot{\pi}_{n,n+1} \circ \dot{\pi}_{n+1}$ for all n. Thus there exists a continuous linear map $\rho : E/N \longrightarrow \lim_{\overleftarrow{n}} E_n/N_n$ such that the composition of ρ with the canonical map $\lim_{\overleftarrow{n}} E_n/N_n \longrightarrow E_n/N_n$ is equal to $\dot{\pi}_n$.

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LEMMA 4.3.9. *If* $\mathrm{Im}(\pi_n) \cap N_n$ *is dense in* N_n, *then the map* ρ *is a linear topological isomorphism.*

Proof. Since E is a Fréchet space the space E/N is a Fréchet space (in particular, complete). Thus as in the previous lemma it suffices to prove that ρ is an isomorphism onto its image, and that the image is dense. Since the maps $\dot{\pi}_{n,n+1}$ have dense range, ρ has a dense range (cf. [**5**: Chap. 2, §3, no. 5]). Thus it suffices to prove that a typical continuous seminorm on E/N is for some n the composition of a continuous seminorm on E_n/N_n with the map π_n. Let $x \longmapsto \dot{x}$ denote the canonical map $E \longrightarrow E/N$ or from E_n to E_n/N_n. The topology of E/N is defined by the quotient seminorms $\dot{p}$ defined by $\dot{p}(\dot{x}) = \inf_{y \in N+x} p(y)$ where p describes a fundamental directed system of continuous seminorm on E. Then there exists $n \in \mathbf{N}$ such that $p = p_n \circ \pi_n$, where p_n is a continuous seminorm on E_n. Since $u(x) = u_n(\pi_n(x))$, we have: $x \in N \Leftrightarrow \pi_n(x) \in N_n$. Thus

$$\begin{aligned}\dot{p}(\dot{x}) = \inf\{p(y) : y \in N + x\} &= \inf\{p_n(\pi_n(y)) : \pi_n(y) \in N_n + \pi_n(x)\}\\ &= \inf\{p_n(z) : z \in N_n + \pi_n(x)\}\end{aligned}$$

where the last equality makes use of the assumption. But this means that

$$\dot{p}(\dot{x}) = \dot{p}_n([\pi_n(x)]) = \dot{p}_n(\dot{\pi}_n(\dot{x}))$$

Thus $\dot{p} = \dot{p}_n \circ \dot{\pi}_n$ and this ends the proof of this lemma (the assumption 'Fréchet' was only used to prove that E/N is complete). □

We continue with the same framework. Let $\dot{u}_n : E_n/N_n \longrightarrow F$ be the map obtained by passing to the quotient. Then $\mathrm{Im}(u_n) = \mathrm{Im}(\dot{u}_n)$. Since $u_{n+1} = u_n \circ \pi_{n,n+1}$ and $u = u_n \circ \pi_n$, we obviously have:

$$\mathrm{Im}(u) \subset \mathrm{Im}(u_{n+1}) \subset \mathrm{Im}(u_n) \subset F.$$

LEMMA 4.3.10. $\mathrm{Im}(u) = \bigcap_n \mathrm{Im}(u_n)$.

Proof. It is sufficient to prove that $\bigcap_n \mathrm{Im}(u_n)$ is included in $\mathrm{Im}(u)$. Let $y \in F$ belong to the intersection. Then there exist unique elements $\dot{x}_n \in E_n/N_n$ such that $y = \dot{u}_n(\dot{x}_n)$. Since $\dot{u}_n(\dot{\pi}_{n,n+1}(\dot{x}_{n+1})) = \dot{u}_{n+1}(\dot{x}_{n+1}) = y$ we have $\dot{x}_n = \dot{\pi}_{n,n+1}(\dot{x}_{n+1})$, i.e. $(\dot{x}_n)_{n \in \mathbf{N}}$ belongs to $\varprojlim_n E_n/N_n$. Thus by the previous lemma there exists an element $\dot{x} \in E/N$ such that $\dot{\pi}_n(\dot{x}) = x_n$ for all n. If $x \in E$ represents $\dot{x}$, we have $u(x) = u_n(\pi_n(x)) = \dot{u}_n(\dot{x}_n) = y$. Since $y \in \mathrm{Im}(u)$ this proves the lemma.

For $n \leqq \infty$ let us denote P_n the map $\mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)} \longrightarrow \mathfrak{A}$ characterized by the fact that $P_n(x \otimes y) = xy$. By lemma 1 we can apply the previous results to the spaces $E = \mathfrak{A}^{(\infty)} \hat{\otimes} \mathfrak{A}^{(\infty)}$, $E_n = \mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)}$, $F = \mathfrak{A}$ and to the maps $u_n = P_n$. Obviously $P_{n+1} = P_n \circ \pi_{n,n+1}$ and $P_\infty = P_n \circ \pi_n$ for all n. If $N_n = \ker(P_n) \subset \mathfrak{A}^{(n)} \hat{\otimes} \mathfrak{A}^{(n)}$ and $N = \mathrm{Ker}(P_\infty)$, there only remains to check that $\mathrm{Im}(\pi_n) \cap N_n$ is dense in N_n. In order to check this, a closer look at the elements of N and N_n will be needed. In the next lemma $\mathfrak{A}$ can be taken to be a general topological algebra.

LEMMA 4.3.11. *If $\mathfrak{A}$ is an algebra with an equicontinuous approximate identity then* $\ker(P) = \overline{span}\{xy \otimes z - x \otimes yz : x, y, z \in \mathfrak{A}\}$.

Proof. Let $\{e_\alpha\}$ be an equicontinuous approximate identity in $\mathfrak{A}$. Let us denote by K the closed subspace $\overline{span}\{xy \otimes z - x \otimes yz : x, y, z \in \mathfrak{A}\}$. If $t \in \ker(P)$ then let $t = \lim_k \sum_{i \in I(k)} x_i^{(k)} \otimes y_i^{(k)}$ with $I(k)$ finite. Then $z_k = \sum_{i \in I(k)} x_i^{(k)} y_i^{(k)} = P(t^{(k)})$ tends to zero as k goes to infinity. For each α, define

$$t_\alpha^{(k)} = \sum_{i \in I(k)} [e_\alpha x_i^{(k)} \otimes y_i^{(k)} - e_\alpha \otimes x_i^{(k)} y_i^{(k)}].$$

Then clearly $t_\alpha^{(k)} \in K$ for all k and α. Now let $L_\alpha : \mathfrak{A} \bar{\otimes} \mathfrak{A} \longrightarrow \mathfrak{A} \bar{\otimes} \mathfrak{A}$ be defined by $L_\alpha(a \otimes b) = e_\alpha a \otimes b$. L_α is well-defined, linear and equicontinuous in α by the universal property of $\bar{\otimes}$ [**13**: Produits tensoriels topologiques, p. 73]. But $t_\alpha^{(k)} = L_\alpha(t^{(k)}) - e_\alpha \otimes z_k$ where of course $t^{(k)} = \sum_{i \in I(k)} x_i^{(k)} \otimes y_i^{(k)}$ and $L_\alpha(x \otimes y) = e_\alpha x \otimes y$ tends to $x \otimes y$. By equicontinuity the maps L_α extends to the whole of $\mathfrak{A} \bar{\otimes} \mathfrak{A}$ and that we still have $t = \lim_\alpha L_\alpha(t)$. Thus $t \in K$. □

This preceding lemma will also be of use later but it enables us now to prove that we are in a situation where we can apply lemma 4.3.9 and 4.3.10.

LEMMA 4.3.12. *$\pi_n(N)$ is dense in N_n.*

Proof. Use theorem 4.3.1 part 7 in order to apply lemma 4.3.11 to $\mathfrak{A}^{(n)}$ and get that N_n is generated by the elements of the form $xy \otimes z - x \otimes yz$ for $x, y, z \in \mathfrak{A}^{(n)}$. $\mathfrak{A}^{(\infty)}$ being dense in $\mathfrak{A}^{(n)}$ it implies that any element of the form $xy \otimes z - x \otimes yz$ for x, y and $z \in \mathfrak{A}^{(n)}$ can be approximated by $x_\alpha y_\alpha \otimes z_\alpha - x_\alpha \otimes y_\alpha z_\alpha$ with $x_\alpha, y_\alpha, z_\alpha \in \mathfrak{A}^{(\infty)}$. But $x_\alpha y_\alpha \otimes z_\alpha - x_\alpha \otimes y_\alpha z_\alpha \in N$ for all α.

Now we can apply lemma 4.3.10 in order to get;

$$\mathrm{Im}(P_\infty) = \bigcap_n \mathrm{Im}(P_n) \text{ i.e. } \mathcal{D}\,\mathfrak{A}^{(\infty)} = \bigcap_n \mathcal{D}\,\mathfrak{A}^{(n)} = \bigcap_n \mathfrak{A}^{(n+1)} = \mathfrak{A}^{(\infty)}.$$

The identity $\mathcal{D}\,\mathfrak{A}^{(\infty)} = \mathfrak{A}^{(\infty)}$ is a linear topological isomorphism by the open mapping theorem. This ends the proof of the theorem. □

A degenerate example. Let $\mathfrak{A}$ be a Fréchet space (e.g. finite dimensional) and define $xy = 0$ for all $x, y \in \mathfrak{A}$.

(1) Then Ker $P = \mathfrak{A} \hat{\otimes} \mathfrak{A}$ while $xy \otimes z - x \otimes yz = 0$ for all x, y, z. Thus Ker P is not generated by the elementary expressions.

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(2) Any bilinear kernel H satisfies $H(zx, y) = H(x, zy)(= 0)$ but no kernel $\neq 0$ is of the form $H(x, y) = \omega(xy)$.

Taking the product of this algebra with a reasonable algebra (e.g. with unit) one obtains intermediate situations less drastically degenerate.

4.4. Correspondance theorem. Let $\mathfrak{A}$ be an $\mathcal{LF}$-algebra and let $P : \mathfrak{A}\bar{\otimes}\mathfrak{A} \longrightarrow \mathcal{D}(\mathfrak{A})$ be as in the last section. Let $f \in (\mathcal{D}\mathfrak{A})'$ and let H be defined by the equation:

$$(1) \qquad H(x, y) = f(xy).$$

Then H is a separately continuous bilinear form on $\mathfrak{A} \times \mathfrak{A}$. We may view H as an element of $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$. Then we have

$$(2) \qquad H = f \circ P = {}^{t}Pf$$

The map P being a surjective homomorphism the transpose ${}^{t}P : \mathcal{D}(\mathfrak{A})' \longrightarrow (\mathfrak{A}\bar{\otimes}\mathfrak{A})'$ is a homomorphism with respect to the weak* topologies. If $H \in (\mathfrak{A}\bar{\otimes}\mathfrak{A})'$ satisfies (2) we have $\mathrm{Ker}(H) \supset \mathrm{Ker}(P)$. Since the quotient map $\dot{P} : \mathfrak{A}\bar{\otimes}\mathfrak{A}/\mathrm{Ker}(P) \longrightarrow \mathcal{D}(\mathfrak{A})$ is an isomorphism onto, we get conversely that if H is 0 on $\mathrm{Ker}(P)$ there exists a unique $f \in \mathcal{D}(\mathfrak{A})'$ such that $H = f \circ P$.

PROPOSITION 4.4.1. *The map $f \longmapsto H = f \circ P$ is a bijective correspondance between $\mathcal{D}(\mathfrak{A})'$ and* $\mathrm{Ker}(P)^{\circ} = \{H \in (\mathfrak{A}\bar{\otimes}\mathfrak{A})' : \mathrm{Ker}(H) \supset \mathrm{Ker}(P)\}$. *This is an isomorphism with respect to the weak* topology on $\mathcal{D}(\mathfrak{A})'$ and the topology induced by the weak* topology of $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$.*

Let us recall that a bilinear map H on $\mathfrak{A} \times \mathfrak{A}$ is called invariant if

$$(3) \qquad H(ab, c) = H(a, bc) \quad \forall a, b, c \in \mathfrak{A}.$$

Let $B_{\mathrm{inv}}(\mathfrak{A})$ be the space of all separately continuous and invariant bilinear forms with the topology induced by the weak* topology of $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$. If $K = \overline{\mathrm{span}}\{xy\otimes z - x\otimes yz : x, y, z \in \mathfrak{A}\}$ as before, then obviously $H \in B_{\mathrm{inv}}(\mathfrak{A})$ iff H, as a continuous linear map on $\mathfrak{A}\bar{\otimes}\mathfrak{A}$, is such that $K \subset \mathrm{Ker}(H)$. Since $K \subset \mathrm{Ker}(P)$, $H = f \circ P \in B_{\mathrm{inv}}(\mathfrak{A})$ for all $f \in \mathcal{D}(\mathfrak{A})'$. Conversely, if each invariant H corresponds to an f in $\mathcal{D}(\mathfrak{A})'$ via (2), we have $\mathrm{Ker}(P) = K$ by the Hahn-Banach theorem. □

PROPOSITION 4.4.2. $\mathrm{Ker}(P) = \overline{\mathrm{span}}\{xy \otimes z - x \otimes yz : x, y, z \in \mathfrak{A}\}$ iff *the map $f \longmapsto H = f \circ P$ is an isomorphism of $\mathcal{D}(\mathfrak{A})'$ onto $B_{\mathrm{inv}}(\mathfrak{A})$.*

As we have seen in lemma 4.3.11, the condition $\mathrm{Ker}(P) = K$ is satisfied whenever $\mathfrak{A}$ has an equicontinuous approximate identity.

COROLLARY 4.4.3. *If $\mathfrak{A}$ is a self-derivative topological algebra with an equicontinuous approximate identity, then the correspondance $f \longmapsto f \circ P$ is an isomorphism between $\mathfrak{A}'$ and $B_{\text{inv}}(\mathfrak{A})$.*

Remarks 4.4.4. (1) The case of $\mathfrak{A}$ unital is of course trivial since the correspondance is given right away by the formula: $H(a, 1) = f(a)$.

(2) If $\mathfrak{A}$ is an $\mathcal{LF}$-algebra with an e.c. approximate identity, we get a converse to the corollary: $\mathfrak{A}'$ is isomorphic to $\mathcal{B}_{\text{inv}}(\mathfrak{A})$ iff $\mathcal{D}(\mathfrak{A}) = \mathfrak{A}$. Indeed, if ${}^{t}P : \mathfrak{A}' \longrightarrow (\mathfrak{A}\bar{\otimes}\mathfrak{A}/\operatorname{Ker}(P))'$ is an isomorphism onto then $P : \mathfrak{A}\bar{\otimes}\mathfrak{A}/\operatorname{Ker}(P) \longrightarrow \mathfrak{A}$ is a weak isomorphism onto by [**12**: Cor. de la Prop. 27, p. 109]. But $\mathfrak{A}\bar{\otimes}\mathfrak{A}/\operatorname{Ker}(P)$ and $\mathfrak{A}$ are both barrelled, hence they have their Mackey topology and P is a homorphism by [**12**: Cor. 3, p. 112]. This proves that in this case $\mathfrak{A}$ is self-derivative.

In order to use the results of this chapter in the context of $*$-algebras, it will be sufficient to adapt the notations.

Note first that if ω is a positive functional on $\mathfrak{A}$ then the map H_ω defined by:

(3) $$H_\omega(a, b) = \langle \omega, a^*b \rangle$$

for $a, b \in \mathfrak{A}$, is a non-negative, hermitian, separately continuous sesquilinear form on $\mathfrak{A} \times \mathfrak{A}$. The invariance of H_ω is now expressed by the following relation:

(4) $$H_\omega(ca, b) = H_\omega(a, c^*b) \quad \forall a, b, c \in \mathfrak{A}.$$

Such a map was called a left invariant kernel (3.1.1) and the space of all such maps, denoted $\operatorname{Herm}^+_{\text{inv}}(\mathfrak{A})$, is endowed with the topology induced by the weak* topology of $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$.

The analogue of the map P is now defined on $\mathfrak{A} \otimes \mathfrak{A}$ by $P'(a \otimes b) = b^*a$. P and P' correspond to each other via the $\mathbf{R}$-linear isomorphism $a \otimes b \longmapsto b^* \otimes a$ and hence they define the same "derivative". Note also that K becomes $\overline{span}\{ca \otimes b - a \otimes c^*b : a, b, c \in \mathfrak{A}\}$.

We can restate Corollary 4.4.3 as the first part of the following theorem:

THEOREM 4.4.5. *If $\mathfrak{A}$ is a self-derivative $\mathcal{LF}$ $*$-algebra with an equicontinuous approximate identity, then the map $\omega \longmapsto H_\omega$ is a homeomorphism between $\mathfrak{A}'_+$ and* $\operatorname{Herm}^+_{\text{inv}}(\mathfrak{A})$.

COROLLARY 4.4.6. *If $\mathfrak{A}$ is moreover a nuclear space a subset $\Omega \subset \mathfrak{A}'_+$ is bounded if and only if the corresponding kernels are bounded with respect to the topology of pointwise convergence on $\mathfrak{A} \times \mathfrak{A}$.*

Proof. Note that $\mathfrak{A}$ being barrelled it is equivalent for a set Ω in $\mathfrak{A}'$ to be bounded with respect to the weak* topology or with respect to the strong dual topology (uniform convergence on bounded sets). Given the isomorphism the boundedness of Ω is equivalent to the boundedness of $\{H_\omega\}_{\omega\in\Omega}$ in the space $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$ equipped with the weak* topology. This obviously implies:

$$(5) \qquad \sup_{\omega\in\Omega} |H_\omega(a,\ b)| < +\infty \quad \forall a,\ b \in \mathfrak{A}.$$

Conversely assume condition (5) is fulfilled. Then if $A_1,\ A_2 \subset \mathfrak{A}$ are absolutely convex bounded sets, the principle of uniform boundedness, applied to the Banach spaces $\mathfrak{A}_{A_i}$ whose unit ball is A_i, implies that

$$(6) \qquad \sup_{\omega\in\Omega} \sup\{H_\omega(a,\ b) : a \in A_1,\ b \in A_2\} < +\infty$$

LEMMA. *On* $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$ *the strong dual topology (uniform convergence on bounded sets) coincides with the topology of uniform convergence on sets* $A_1 \otimes A_2$*, with* $A_i \subset \mathfrak{A}$ *bounded.*

Proof. Let $(\mathfrak{A}_n)_{n\in\mathbf{N}}$ be a sequence of Fréchet spaces such that $\mathfrak{A} = \lim_{\vec{n}} \mathfrak{A}_n$. Then by the nuclearity of $\mathfrak{A}$, $\mathfrak{A}\bar{\otimes}\mathfrak{A}$ is the strict inductive limit of the spaces $\mathfrak{A}_n\hat{\otimes}\mathfrak{A}_n$ (2.1.2) and so every bounded subset A of $\mathfrak{A}\bar{\otimes}\mathfrak{A}$ is contained in one of the spaces $\mathfrak{A}_n\hat{\otimes}\mathfrak{A}_n$. Thus it is contained in the closed absolutely convex hull of a set $A_1 \otimes A_2$, with A_i bounded in $\mathfrak{A}_n$ and therefore in $\mathfrak{A}$ [**12**: Espaces Nucléaires, §3 no 1, prop. 12]. Conversely, if the sets $A_i \subset \mathfrak{A}$ are bounded, they are bounded in some space $\mathfrak{A}_n$, and so $A_1 \otimes A_2$ is bounded in $\mathfrak{A}_n\hat{\otimes}A_n$ and in $\mathfrak{A}\bar{\otimes}\mathfrak{A}$. This proves the lemma. □

As a consequence of the lemma we get that (6) and therefore (5) imply the boundedness of the kernels in $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'_b$, and so in $(\mathfrak{A}\bar{\otimes}\mathfrak{A})'$ equipped with the weak* topology. The map ${}^tP : \mathfrak{A}'_+ \longrightarrow \mathrm{Herm}^+_{\mathrm{inv}}(\mathfrak{A})$ being an isomorphism the corresponding set of linear forms in $\mathfrak{A}'$ is bounded. (The positivity of the kernels was not needed for this argument). □

Let us note finally that the result holds without the nuclearity assumption if $\mathfrak{A}$ is Fréchet algebra.

5. Entire elements.

5.1. Vector-valued holomorphic functions. Let E be a quasi-complete locally convex space over $\mathbf{C}$. Let $\Omega \subset \mathbf{C}$ be an open subset of $\mathbf{C}$. A continuous function $f : \Omega \longrightarrow E$ is holomorphic if for every disc D in Ω and $z \in D$ one has;

$$(1) \qquad f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z} d\zeta,$$

where $\Gamma = \partial D$.

Note that the quasi-completeness of E is required to make sure that the Cauchy integral makes sense a priori, for any continuous function with values in E.

It is well-known that this definition is equivalent to the existence of a derivative for f at each point of Ω or equivalently that there exists a neighborhood around each point of Ω on which f has an absolutely convergent Taylor series expansion.

PROPOSITION 5.1.1. *Let E and F be q.c. locally convex spaces with E continuously embedded in F. Let j be the injection*

(1) $\quad j : E \hookrightarrow F$

and such that E has a fundamental system of neighborhoods which are closed in the relative topology of F. If $f : \Omega \longrightarrow E$ is a locally bounded function such that $j \circ f$ is holomorphic, then f is holomorphic.

Proof. The condition on E and F is equivalent to the following: There exists a fundamental family of continuous seminorms p on E, which are lower semicontinuous for the topology of F, i.e.

(2) $$p(x) = \sup_{\eta} |\langle x, \eta \rangle|,$$

where η describes the set $\{\eta \in F' : |\langle x, \eta\rangle| \leqq p(x)\ \forall x \in E.\}$. Thus, in particular, we can view p as a lower semicontinuous function on F, which is finite on E.

Let $\tilde{f} = j \circ f$. Then we have in F:

(3) $$\tilde{f}'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\tilde{f}(\zeta)}{(\zeta - z)^2}\, d\zeta.$$

Thus

$$\langle \tilde{f}'(z), \eta \rangle = \frac{1}{2\pi i} \int_{\Gamma} \left\langle \frac{\tilde{f}(\zeta)}{(\zeta - z)^2}, \eta \right\rangle d\zeta.$$

Therefore, if $D = D(R)$ is a disc of radius R contained in Ω, and $M = \sup_{z \in D} p(f(z)) = \sup_{z \in D} p(\tilde{f}(z))$, we have for $z \in D(R/2)$, using (2):

(4) $$p(\tilde{f}'(z)) \leqq M_1 = \left(\frac{M}{2\pi}\right)\left(\frac{2}{R}\right)^2.$$

Now again in F we have, for $u, v \in D(R)$, $\tilde{f}(v) - \tilde{f}(u) = \int_u^v \tilde{f}'(z)dz$, the integral being taken along the line segment connecting u and v. Applying η and taking the supremum as before we get

(5) $$p(f(u) - f(v)) \leqq M_1 |u - v|,$$

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which proves that f is continuous. This being so, the Cauchy integral for f makes sense, and we have

$$jf(z) = \frac{1}{2\pi i}\int_\Gamma \frac{\tilde{f}(\zeta)}{\zeta - z}d\zeta = \frac{1}{2\pi i}j\int_\Gamma \frac{f(\zeta)}{\zeta - z}d\zeta$$

and so j being one-to-one we have (1). □

The following example shows that some assumption is needed on f for the proposition to be true.

Counterexample. Let $E = c(\mathbf{N})$ be the space of converging sequences with the sup norm. Let $F : \mathbf{C} \longrightarrow E$ be a function, $F(z) = (f_n(z))_{n\in\mathbf{N}}$ such that every f_n is holomorphic. Then F need not be holomorphic. In fact, lim being a continuous linear form on $c(\mathbf{N})$, the function $z \longmapsto \lim_{n\to+\infty} f_n(z)$ would then be holomorphic, whereas it is known that the limit of a pointwise convergent sequence of holomorphic functions need not be holomorphic.

5.2. Exponentially bounded one-parameter groups and entire elements. Let E be a locally convex space. The space of all continuous linear operators on E with values in E will be denoted $\mathcal{L}(E)$. If an element L in $\mathcal{L}(E)$ has a continuous inverse, then L is called an automorphism of E.

Definition 5.2.1. Let $\tau : \mathbf{R} \times E \longrightarrow E$ be a map such that;

(i) $\forall t \in \mathbf{R}$, $\tau_t = \tau(t, .) \in \mathcal{L}(E)$.

(ii) $\forall t_1, t_2 \in \mathbf{R}$, $\tau_{t_1+t_2} = \tau_{t_1} \circ \tau_{t_2}$ and $\tau_0 = \mathrm{Id}$.

(iii) τ is jointly continuous.

Then $\{\tau_t\}$ is called a continuous one-parameter group of automorphisms of E.

Note that whenever E is barrelled, we have by [**6**: chap. 8, §2, proposition 1] that iii) is equivalent to iii′) τ is separately continuous.

Now let us assume that $\{\tau_t\}$ is a continuous one-parameter group at automorphisms of E.

Definition 5.2.2. If $x \in E$ is such that the map $t \longmapsto \tau_t(x)$ is the restriction to $\mathbf{R}$ of an entire function (necessarily unique) then x is called an entire vector of E with respect to τ. The set of all entire vectors is denoted E_τ.

For $x \in E_\tau$, let $z \longmapsto \tau_z(x) \in E$ denote the entire function extending the map $t \longmapsto \tau_t(x)$. If $x, y \in E_\tau$ and $\lambda, \nu \in \mathbf{C}$ then by the principle of analytic continuation, $\tau_z(\lambda x + \nu y) = \lambda\tau_z(x) + \nu\tau_z(y)$. It is also immediate that if $x \in E_\tau$, $\tau_u(x) \in E_\tau$ for all $u \in \mathbf{C}$ and $\tau_{z+u}(x) = \tau_z \circ \tau_u(x)$. This proves that E_τ is a subspace of E which is invariant under the action of τ_z for all complex numbers z.

Now, given $x \in E_\tau$ let $f = f_x$ denote the entire function $f : \mathbf{C} \longrightarrow E; z \longmapsto \tau_z(x)$. This function f has the further property that;

(1) $\tau_t(f(z)) = f(z + t) \quad \forall t \in \mathbf{R}.$

Let Hol($\mathbf{C} : E$) denote the space of all entire functions with values in E endowed with the topology of uniform convergence on compact sets and let $\mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E)$ denote the closed subspace of Hol($\mathbf{C} : E$) composed of those functions f having the invariance property (1). Given $f \in \mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E)$, $f(0)$ is an entire element determining f completely on $\mathbf{C}$ by

(2) $\quad f(z) = \tau_z(f(0)).$

PROPOSITION 5.2.3. *The map $x \longmapsto f_x$ is a linear bijection between E_τ and* $\mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E)$.

Proof. Equip E_τ with the topology making this bijection a homeomorphism. Explicitely, E_τ is equipped with the seminorms $p_K(x) = \sup_{z \in K} p(\tau_z(x))$ where K is a compact subset of $\mathbf{C}$ and p is a continuous seminorm on E. Clearly, one has a continuous injection $E_\tau \hookrightarrow E$. □

Definition 5.2.4. Let $w \geqq 0$. $\{\tau_t\}$ is said to be exponentially bounded, of bound $\leqq w$, if for every continuous seminorm p on E, the map $x \longmapsto q(x) = \sup_{t \in \mathbf{R}} p(\tau_t x) e^{-w|t|}$ defines another continuous seminorm on E. Equivalently, if there exists a continuous seminorm q on E such that $p(\tau_t x) \leqq e^{w|t|} q(x)$ for all t and x.

Remarks 5.2.5. (1) Recall that if E is a Banach space, then any continuous one-parameter group is exponentially bounded [**9**: Cor. 5, p. 619].

(2) If E is barrelled, then $\{\tau_t\}$ is exponentially bounded of bound $\leqq w$ iff $\sup_{t \in \mathbf{R}} p(\tau_t x) e^{-w|t|} < +\infty$ for each continuous seminorm p and for all $x \in E$.

PROPOSITION 5.2.6. (1) *If E is a Fréchet space then so is E_τ.*

(2) *If E is an $\mathcal{LF}$-space and if $\{\tau_t\}$ is exponentially bounded then E_τ is also an $\mathcal{LF}$-space.*

(3) *If E is a nuclear space then so is E_τ.*

Proof. (1) If E is Fréchet then so is Hol($\mathbf{C} : E$) and its closed subspace E_τ.

(2) (Note that a closed subspace of an $\mathcal{LF}$-space need not be an $\mathcal{LF}$-space). For any $x \in E$, the boundedness of the function $e^{-w|t|} p(\tau_t(x))$ implies that $e^{-w|t|}\tau_t(x)$ takes all its values in E_n for a certain n which, in turn, implies that $\tau_t(x) \in E_n \ \forall t \in \mathbf{R}$. In particular, the same is true for $x \in E_\tau$. But if $\ell \in E_n^\circ$, the annihilator of E_n in E', then the function $z \longmapsto \langle \tau_z(x), \ \ell \rangle$ is entire and 0 on $\mathbf{R}$, hence identically 0 on $\mathbf{C}$ which means that $\tau_z(x) \in E_n \ \forall z \in \mathbf{C}$. Therefore if

$$\mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E_n) := \{f \in \mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E) : f(\mathbf{C}) \subset E_n\}$$

then $E_\tau \simeq \mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E)$ which is equal to $\varinjlim_n \mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E_n)$. But $\mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E_n)$ is a closed subspace of the Fréchet space $\mathrm{Hol}(\mathbf{C} : E_n) = \mathrm{Hol}(\mathbf{C}) \hat{\otimes} E_n$. Thus $\mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E_n)$ is a Fréchet space and E_τ is an $\mathcal{LF}$-space.

(3) If E is nuclear then each subspace E_n is nuclear. Then $\mathrm{Hol}(\mathbf{C})\hat{\otimes}E_n$ (the tensor product of two nuclear spaces) is nuclear and so is the inductive limit $\mathrm{Hol}(\mathbf{C} : E)$. Therefore its subspace E_τ is nuclear as well. □

Let us assume all throughout this section that E is the strict inductive limit of a sequence of Fréchet spaces $\{E_n\}_{n\in\mathbf{N}}$ and that $\{\tau_t\}_{t\in\mathbf{R}}$ is a continuous and exponentially bounded one-parameter group of automorphisms of E.

PROPOSITION 5.2.7. (1) *$\{\tau_t\}$ is a continuous and exponentially bounded one-parameter group of automorphisms of E_τ with the same bound.*
(2) *$(E_\tau)_\tau = E_\tau$ as topological vector spaces.*

Proof. (1) First note that τ_t acts on $E_\tau = \mathrm{Hol}_{\mathrm{inv}}(\mathbf{C} : E)$ as translation by t i.e. $\tau_t(f(z)) = f(z+t)$. If $\{f_\alpha\}$ is a net converging to f in E_τ, then the translates of f_α converge to the translate of f uniformly on compact sets thus proving that τ_t acts continuously on E_τ. τ is moreover continuous in t because continuous functions are equicontinuous on compact sets. Moreover, if $x \in E_\tau$, K is a compact subset of $\mathbf{C}$ and p is a continuous seminorm on E, then there exists q, a continuous seminorm on E such that $p(\tau_z(\tau_t x)) \leqq p(\tau_t(\tau_z x)) \leqq e^{w|t|}q(\tau_z(x))$ for all t in $\mathbf{R}$ and z in K. Taking the sup on each side we get $p_K(\tau_t x) \leqq e^{w|t|}q_K(x)$.

(2) By definition $(E_\tau)_\tau \hookrightarrow E_\tau$. If $x \in E_\tau$ then $f_x : \mathbf{C} \longrightarrow E_\tau \hookrightarrow E$ is holomorphic in E. f_x is locally bounded since we have for every $u \in H$, a compact subset of $\mathbf{C}$, $p_K(\tau_u x) \leqq p_{K+H}(x)$. Applying proposition 5.1.1 we get that $x \in (E_\tau)_\tau$ and the equality between the two spaces is an isomorphism by the open mapping theorem.

THEOREM 5.2.8. (1) *If $\{\tau_t\}$ is exponentially bounded then E_τ is dense in E.*
(2) *If $F \subset E_\tau$ is τ-invariant and dense in E then F is dense in E_τ.*

Without the assumption in 1. E_τ may reduce to (0).

Example. Let $\mathcal{D}(\mathbf{R})$ be the space of Schwartz test functions on $\mathbf{R}$. Let $\tau_t(\psi)(x) = \psi(x+t)$ for all x, $t \in \mathbf{R}$ and for all test functions ψ. Then ψ is an entire vector for τ iff ψ is the restriction to $\mathbf{R}$ of an entire function. But ψ has compact support hence 0 is the only entire vector of $\mathcal{D}(\mathbf{R})$. One can check of course that in this case, τ is not exponentially bounded. Thus some growth condition has to be assumed in order to get enough entire vectors.

To prove this theorem, we introduce the following space. Define $\mathcal{S}$ to be the space of all holomorphic functions $\varphi : \mathbf{C} \longrightarrow \mathbf{C}$ having the property that

$$p_{\lambda,b}(\varphi) = \sup_{\substack{t\in\mathbf{R} \\ |\gamma|\leqq b}} \{e^{\lambda|t|}|\varphi(t+i\gamma)|\} < +\infty$$

for all λ and b positive.

PROPOSITION 5.2.9. (1) *The space $\mathcal{S}$ with the seminorms $\{p_{\lambda,b}\}_{\lambda,b\geqq 0}$ is a nuclear Fréchet space containing the Gaussian functions $g_n(t) = (n/\pi)^{1/2}e^{-nt^2}$.*

(2) *For every* $\varphi \in \mathcal{S}$ *and* $z \in \mathbf{C}$, *the function* $\varphi_z : u \longmapsto \varphi(u-z)$ *belongs to* $\mathcal{S}$. *Moreover* $z \longmapsto \varphi_z$ *is holomorphic i.e. every vector in* $\mathcal{S}$ *is holomorphic with respect to the group of real-translations.*

Proof. (1) The only non-trivial thing to prove is the nuclearity of $\mathcal{S}$ and since we make no use of this property here we will omit it.

(2) Let $\varphi \in \mathcal{S}$ and $z \in \mathbf{C}$. Define $\varphi_z := \varphi(u-z)$ for every $u \in \mathbf{C}$. If $|z| \leqq r$ then $p_{\lambda,b}(\varphi_z) \leqq e^{\lambda r} p_{\lambda,b+r}(\varphi)$ which shows that $\varphi_z \in \mathcal{S}$ and that the map $z \longmapsto \varphi_z$ from $\mathbf{C}$ to $\mathcal{S}$ is locally bounded. Applying 5.1.1 we get that $z \longmapsto \varphi_z$ is an entire functions with values in $\mathcal{S}$. □

PROPOSITION 5.2.10. (1) *For every* $\varphi \in \mathcal{S}$ *and* $x \in E$, $\tau_\varphi x := \int_{\mathbf{R}} \varphi(t)\tau_t(x)dt$ *exists in* E. *Moreover for all* $x \in E$, $\tau_\varphi(x) \in E_\tau$ *with* $\tau_z\tau_\varphi(x) = \tau_{\varphi_z}(x)$ *and if* $x \in E_\tau$, $\tau_z\tau_\varphi(x) = \tau_\varphi\tau_z(x)$.

(2) *The map* $(\varphi, x) \longmapsto \tau_\varphi(x)$ *is continuous from* $\mathcal{S}xE$ *to* E_τ.

(3) *For* $x \in E$ *(resp.* $x \in E_\tau$*)* $\tau_{g_n}x \longmapsto x$ *in* E *(resp. in* E_τ*).*

Proof. (1) Since $\{\tau_t\}$ has an exponential bound less than $w \geqq 0$, there exists for each continuous seminorm p on E a continuous seminorm q such that $p(\tau_t(x)) \leqq e^{w|t|}q(x)$ for every $t \in \mathbf{R}$ and $x \in E$. If $\varphi \in \mathcal{S}$, we get that $p(\varphi(t)\tau_t(x)) \leqq |\varphi(t)|e^{w|t|}q(x) = e^{-|t|}p_{w+1,0}(\varphi)q(x)$. Thus

$$p(\tau_\varphi(x)) \leqq 2p_{w+1,0}(\varphi)q(x).$$

This proves the first statement and the continuity of the map $(\varphi, x) \longmapsto \tau_\varphi(x)$ from $\mathcal{S}xE$ to E. Note that $\tau_z\tau_\varphi(x) = \tau_{\varphi_z}(x)$ for all $z \in \mathbf{R}$. But the map $z \longmapsto \tau_{\varphi_z}(x)$ from $\mathbf{C}$ to E is holomorphic since it is the composition of a continuous map and a holomorphic one (5.2.9). Thus $\tau_\varphi(x) \in E_\tau$ and $\tau_z\tau_\varphi(x) = \tau_{\varphi_z}(x)$ for all $z \in \mathbf{C}$.

(2) Let $\varphi \in \mathcal{S}$, $x \in E$, $z \in \mathbf{C}$ and p be a continuous seminorm on E. Note that

$$p(\tau_z\tau_\varphi(x)) = p(\tau_{\varphi_z}(x)) \leqq 2p_{w+1,0}(\varphi_z)q(x).$$

If $p_r(x) := \sup_{|z|\leqq r} p(\tau_z(x))$ then $p_r(\tau_\varphi(x)) \leqq 2e^{r(w+1)}p_{w+1,r}(\varphi)q(x)$. Which proves the continuity of the map $(\varphi, x) \longmapsto \tau_\varphi(x)$ from $\mathcal{S}xE$ to E_τ.

(3) The proof is based on the two well-known facts:

(i) $\int_{\mathbf{R}} g_n(t)dt = 1$ for all $n \in \mathbf{N}$ and

(ii) $\lim_{n\to\infty} \int_{|t|>\delta} g_n(t)e^{\lambda|t|}dt = 0$ for any $\delta > 0$ and $\lambda > 0$.

First, let us take an x in E_τ. Let K be a compact subset on $\mathbf{C}$. Note that

$$p(\tau_z(\tau_{\varphi_n}(x) - x)) \leqq \int_{\mathbf{R}} p(\tau_z(\tau_t(x) - x))\varphi_n(t)dt$$

for all z in K and for all $n \in \mathbf{N}$. Given $\epsilon > 0$, use the continuity of τ_t on E_τ (5.2.7) to get $\delta > 0$ such that $|t| < \delta$ implies $\sup_{z \in K} p(\tau_z(\tau_t(x) - x)) \leqq \epsilon/2$. Using i), we then get

$$\int_{-\delta}^{\delta} p(\tau_z(\tau_t(x) - x))\varphi_n(t)dt \leqq \epsilon/2.$$

Now since $\{\tau_t\}$ is exponentially bounded with bound $\leqq w$, there exists a semi-norm q on E such that $p(\tau_t(x) \leqq e^{w|t|}q(x)$. Let $M = \sup_{z \in K}(p+q)(\tau_z x)$ then for all $z \in K$,

$$\begin{aligned} \int_{|t|>\delta} p(\tau_z(\tau_t(x) - x))\varphi_n(t)dt &\leqq \int_{|t|>\delta} p(\tau_t\tau_z(x))\varphi_n(t)dt \\ &+ \int_{|t|>\delta} p(\tau_z(x))\varphi_n(t)dt \\ &\leqq \int_{|t|>\delta} q(\tau_z(x))e^{w|t|}\varphi_n(t)dt \\ &+ \int_{|t|>\delta} p(\tau_z(x))\varphi_n(t)dt \\ &\leqq 2M\left(\int_{|t|>\delta} e^{w|t|}\varphi_n(t)dt\right). \end{aligned}$$

Using (ii) we get $N \in \mathbf{N}$ such that for all $n \geqq N$ and for all $z \in K$,

$$\int_{|t|>\delta} p(\tau_z(\tau_t(x) - x))\varphi_n(t)dt \leqq \epsilon/2$$

Thus given $\epsilon > 0$ there exists $n \geqq N$, such that $p_K(\tau_{\varphi_n}(x) - x) \leqq \epsilon$. This proves the case $x \in E_\tau$ while the case $x \in E$ is proved applying the same reasoning with $K = \{0\}$. □

Proof of 5.2.8. (1) Follows from 5.2.10 part 3.

(2) Replacing F by its closure in E_τ, we may and do assume that F is closed in E_τ. If $x \in E_\tau$ then $\tau_t x \in E_\tau$ for every $t \in \mathbf{R}$. Since τ is exponentially bounded on E_τ (5.2.7), $\int \tau_t(x)\varphi(t)dt$ exists as a vector integral in E_τ for each $\varphi \in \mathcal{S}$ by 5.2.10. If $x \in F$, we see that $\tau_\varphi(x) = \int \tau_t(x)\varphi(t)dt \in F$ since F is assumed closed and τ-invariant. If $x \in E_\tau$ let $\{x_\alpha\} \subset F$ be a net such that $x_\alpha \longrightarrow x$ in E. By 5.2.10 part 2 this implies that $\tau_\varphi(x_\alpha) \longrightarrow \tau_\varphi(x)$ in E_τ for all $\varphi \in \mathcal{S}$. In particular, it means that $\tau_\varphi(x) \in F$ and since $\tau_{\varphi_n}(x) \longrightarrow x$ in E_τ when $n \longrightarrow \infty$, this implies that $x \in F$. Thus $F = E_\tau$ as was to be proved. □

5.3. One-parameter groups of $*$-automorphisms. Let us return to the context of $\mathcal{LF}$ $*$-algebras. Let $\mathfrak{A}$ be an $\mathcal{LF}$ $*$-algebra. Now we further assume that

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each τ_t is a $*$-automorphism of $\mathfrak{A}$ (as a $*$-algebra). We denote $\mathfrak{A}_\tau$ the set of all the entire elements of $\mathfrak{A}$.

PROPOSITION 5.3.1. (1) *$\mathfrak{A}_\tau$ is a τ-invariant $*$-subalgebra of $\mathfrak{A}$. Moreover, $\tau_z(ab) = \tau_z(a)\tau_z(b)$ and $\tau_z(a^*) = (\tau_{\bar{z}}(a))^*$.*

(2) *$\mathfrak{A}_\tau$ is an $\mathcal{LF}$ $*$-algebra.*

(3) *$\{\tau_t\}$ is a continuous one-parameter group of $*$-automorphisms on $\mathfrak{A}_\tau$. Moreover τ acting on $\mathfrak{A}_\tau$ is exponentially bounded with the same bound as τ acting on $\mathfrak{A}$.*

(4) *$(\mathfrak{A}_\tau)_\tau = \mathfrak{A}_\tau$ as topological $*$-algebras.*

Proof. (1) Given a and b in $\mathfrak{A}_\tau$, the functions $z \longmapsto \tau_z(a)\tau_z(b)$ and $z \longmapsto (\tau_{\bar{z}}(a))^*$ are holomorphic and equal to $\tau_z(ab)$ and $\tau_z(a^*)$ resp. for all $z \in \mathbf{R}$. Thus they are equal for all z and (1) follows.

(2) This is a particular case of 5.2.6.

(3) and (4) follow from (1) and 5.2.7. □

Example 1. Let $\mathcal{H}$ be a finite dimensional Hilbert space. Let $\{\tau_t\}$ be a continuous one-parameter group of $*$-automorphisms of $\mathcal{L}(\mathcal{H})$. Then by [**7**: p. 243] we get that $\{\tau_t\}$ is implemented by a continuous group of unitary operators on $\mathcal{H}$, hence there exists a self-adjoint operator h on $\mathcal{H}$ such that

$$\tau_t(a) = e^{ith} a e^{-ith} \quad \forall a \in \mathcal{L}(\mathcal{H}).$$

But then all elements are entire and $\tau_z(a) = e^{izh} a e^{-izh}\ \forall z \in \mathbf{C}$.

Example 2. Let $\mathfrak{A} = \mathcal{C}_{2\pi}$ be the 2π periodic functions on $\mathbf{R}$ with the usual product and the sup norm. Let $\tau_t f(x) = f(x+t)$ i.e. τ is the group of translations on $\mathbf{R}$. Then $\mathfrak{A}_\tau$ is the space of all holomorphic functions on $\mathbf{C}$ with the property that $f(x+iy) \in \mathcal{C}_{2\pi}$ in x for all $y \in \mathbf{R}$. Note that $\mathfrak{A}_\tau$ with its natural topology is nuclear even though $\mathfrak{A}$ is a Banach space.

Example 3. Let $\mathfrak{A} = \mathcal{S}(\mathbf{R})$ the Schwartz space of rapidly decreasing functions on $\mathbf{R}$. Let τ_t be translation as in example 2. Then τ has polynomial growth i.e. for each continuous seminorm p on $\mathfrak{A}$, there exists a continuous seminorm q and $n \in \mathbf{N}$ such that

$$p(\tau_t(a)) \leqq (1+|t|)^n q(a) \quad \forall t \in \mathbf{R},\ a \in \mathfrak{A}.$$

a fortiori τ has exponential bound $\leqq w$ for every $w > 0$.

Example 4. Let $\mathcal{H}$ be a Hilbert space and let H be a self-adjoint operator defined on a dense domain in $\mathcal{H}$. Let D be the space of the analytic vectors for H, i.e.

$$D = \left\{ v \in \mathcal{H} : v \in D(H^n)\ \forall n \in \mathbf{N} \text{ and } \sum_{n=1}^{\infty} \frac{\|H^n v\| r^n}{n!} < +\infty\ \forall r > 0 \right\}.$$

D is a core for H and hence $H|_D$ is also self-adjoint. Thus we will let D be the domain of H. Let $U_t = e^{itH}$ $\forall t \in \mathbf{R}$. $\{U_t\}$ defines a one-parameter group of isometries on $\mathcal{H}$. If $v \in D$ then

$$U_z v = e^{izH}(v) = \sum_{n=1}^{\infty} \frac{H^n v(iz)^n}{n!}(iz)^n$$

makes sense for any $z \in \mathbf{C}$. If $A \in \mathcal{L}(D)$ then define the following group action: $A_z = \alpha_z(A) = U_z A U_{-z}$. Let $\mathfrak{A}$ be the set of all those A's such that the seminorms $p_K(A) = \sup_{z \in K} \|A_z\| < +\infty$ for all K compact in $\mathbf{C}$. It can be shown that $\mathfrak{A}$ is stable under the product and the $*$ operation and moreover that $\mathfrak{A}$ is an algebra for which these two operations are continuous. $\mathfrak{A}$ is actually a Fréchet algebra continuously embedded in $\mathcal{L}(D)$. It is possible moreover to show that all $A \in \mathfrak{A}$ are entire. If $\mathfrak{A}_\circ = \{A \in \mathcal{L}(\mathcal{H})$: the map $t \longmapsto \alpha_t(A)$ is $\| \quad \|$-continuous$\}$ then $\mathfrak{A}$ is a $*$-subalgebra of the closed algebra $\mathfrak{A}_\circ$ in $\mathcal{L}(\mathcal{H})$. In fact $\mathfrak{A} = \mathfrak{A}_{\circ,\tau}$.

Let us end this section with the following result about the relation between the derived algebra and the entire elements.

Proposition 5.3.2. *Let $\mathfrak{A}$ be a topological algebra (resp. $*$-algebra). Let $(\tau_t)_{t\in\mathbf{R}}$ be a one-parameter group of automorphisms (resp. $*$-automorphisms) of $\mathfrak{A}$ of exponential bound $\leqq w$. Then $\mathcal{D}\tau_t$ form a one-parameter group of automorphisms of exponential bound $\leqq w$ on $\mathcal{D}(\mathfrak{A})$.*

Proof. In fact by the parts 3. and 4. of 4.2.1 $\{\mathcal{D}\tau_t\}$ forms a one-parameter group of automorphisms, resp. $*$ automorphisms of $\mathcal{D}\mathfrak{A}$. Now the exponential bound means that the linear operators $e^{-w|t|}\tau_t$ are equicontinuous. But we have $e^{-w|t|}\mathcal{D}\tau_t = \mathcal{D}e^{-w|t|}\tau_t$ and so these operators are again equicontinuous in $\mathcal{D}\mathfrak{A}$. □

6. Central and KMS functionals. Throughout this section, $\mathfrak{A}$ will denote an $\mathcal{LF}$ $*$-algebra with an equicontinuous approximate identity. Let $\{\tau_t\}$ be a continuous and exponentially bounded group of $*$-automorphisms of $\mathfrak{A}$. The second definition below is due to Kubo [**16**], Martin and Schwinger [**19**].

Definitions 6.1.1. Let ω be a positive functional on $\mathfrak{A}$ invariant under the action of a continuous one-parameter group of $*$-automorphisms $\{\tau_t\}$ i.e. $\omega(\tau_t(a)) = \omega(a)$ for all $t \in \mathbf{R}$ and $a \in \mathfrak{A}$.

(1) ω is called a central functional (also called abelian or tracial functional) if

$$(1) \qquad \langle \omega, ab \rangle = \langle \omega, ba \rangle \quad \forall a,\ b \in \mathfrak{A}.$$

(2) Let $\beta \in \mathbf{R}$. Then ω is said to satisfy the (τ, β)-KMS condition (for short, ω is a (τ, β)-KMS functional) if

$$(2) \qquad \langle \omega, a\tau_{i\beta}(b) \rangle = \langle \omega, ba \rangle \quad \forall b \in \mathfrak{A}_\tau \text{ and } \forall a \in \mathfrak{A}.$$

Remark. If τ_t equals the identity on $\mathfrak{A}$ for all $t \in \mathbf{R}$ then all the elements of the algebra are entire and (1) and (2) are identical. Then the KMS functionals with respect to the trivial group are the central ones. This observation motivates our treatment of central functionals as a particular case of the more general KMS functionals.

If $\beta \neq 0$ and the action of the group τ is bounded then it is known that condition (2) implies the invariance of the functional. Similarly,

PROPOSITION 6.1.2. *If $\beta \neq 0$ and suppose that the group τ has polynomial growth (see example 3, §5.2) i.e. for each continuous seminorm p on $\mathfrak{A}$, there exists a continuous seminorm q and $n \in \mathbf{N}$ such that*

$$(3) \qquad p(\tau_t(a)) \leqq (1 + |t|)^n q(a) \quad \forall t \in \mathbf{R},\ a \in \mathfrak{A}.$$

If ω is a functional on $\mathfrak{A}$ satisfying (2) then ω is τ-invariant.

Proof. Since $\mathfrak{A}_\tau$ is dense (5.2.8) and τ_t and ω are continuous it is sufficient to prove that $\omega(\tau_t a) = \omega(a)$ for all $t \in \mathbf{R}$ and $a \in \mathfrak{A}_\tau$. Let $a \in \mathfrak{A}_\tau$ and define the entire function $F(z) := \omega(\tau_z a)$. Let p be a continuous seminorm on $\mathfrak{A}$ such that $|\omega(a)| \leqq p(a)$. Let q be as in (3) and let $M := \sup_{0 \leqq \gamma \leqq \beta} q(\tau_{i\gamma} a)$. Then $\forall t \in \mathbf{R}$ and $\forall \gamma \in [0, \beta]$, we have

$$(4) \qquad |F(t + i\gamma)| = |\omega(\tau_t(\tau_{i\gamma} a))| \leqq p(\tau_t(\tau_{i\gamma} a)) \leqq (1 + |t|)^n q(\tau_{i\gamma} a) \leqq M(1 + |t|)^n.$$

Let us also define the functions $F_\alpha(z) := \omega(\tau_z(a)e_\alpha)$ and $G_\alpha(z) := \omega(e_\alpha \tau_z a)$. Since $\{e_\alpha\}$ is an approximate identity they both converge pointwise to F. The KMS condition implies that $G_\alpha(z + i\beta) := \omega(e_\alpha \tau_{i\beta}(\tau_z a) = \omega(\tau_z(a)e_\alpha) = F_\alpha(z)$. Passing to the limit we get $F(z + i\beta) = f(z)\ \forall z \in \mathbf{C}$. But (4) and the periodicity of F now gives us that $|F(z)|| \leqq M(1 + |Re(z)|)^n$ for all $z \in \mathbf{C}$. Thus the function F is a polynomial and since it is periodic it is constant. □

Clearly the KMS functionals for a fixed a fixed τ and a fixed β form a weak* closed and convex subcone of the positive cone in $\mathfrak{A}'$.

Example 1. Let K be a compact set. Let $\{\tau_t\}$ be a group of ∗-automorphisms of $C(K)$. Then τ_t is implemented by a group $\{h_t\}$ of homeomorphisms of K i.e. $\tau_t(f(x)) = f(h_t(x))$. Suppose that there exists an $x \in K$ such that $h_t(x) = x$ for all $t \in \mathbf{R}$. Then the evaluation at $x, \delta_x \in C(K)'$ is a $(\tau,\ \beta)$-KMS for every $\beta \in \mathbf{R}$.

Example 2. Let $\mathcal{H}_n$ denote then n-dimensional Hilbert space, and $\mathcal{L}(\mathcal{H}_n)$, the von Neumann algebra of all linear operators on $\mathcal{H}_n$. It is well-known that the trace is the unique (up to a constant) abelian functional on $\mathcal{L}(\mathcal{H}_n)$. Recall that if $\{\tau_t\}$ is a continuous group of *-automorphisms then

$$\tau_z(a) = e^{izh} a e^{-izh} \quad \forall z \in \mathbf{C} \text{ and } \forall a \in \mathcal{L}(\mathcal{H}_n).$$

where $h \in \mathcal{L}(\mathcal{H}_n)$ is self-adjoint. Suppose then that ω is a (τ, β)-KMS functional for a certain $\beta \in \mathbf{R}$. Again, a simple calculation shows that $\omega'(a) = \omega(e^{\beta h}a)$ now defines an abelian functional. Therefore $\omega(a) = tr(e^{-\beta h}a)$ is the unique (τ, β)-KMS functional on $\mathcal{L}(\mathcal{H}_n)$. Now if $\mathfrak{A} \subset \mathcal{L}(\mathcal{H}_n)$ is a $*$-subalgebra with a non-trivial projection p in its center and if e^{ith} belongs to $\mathfrak{A}$ for all $t \in \mathbf{R}$ (i.e. $\{\tau_t\}$ is inner) then we get two other KMS functionals ω_1 and ω_2 defined by $\omega_1(a) = tr(e^{-\beta h}ap)$ and $\omega_2(a) = tr(e^{-\beta h}a(1-p))$.

Example 3. We can also combine the two last examples as follows. Recall that if $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are two Banach $*$-algebras then $\mathfrak{A}_1 \hat{\otimes}_\pi \mathfrak{A}_2$ is also a Banach $*$-algebra with product defined as follows: If $u = \sum_{i=1}^n a_i \otimes b_i$ and $v = \sum_{j=1}^n c_j \otimes d_j$ then $uv = \sum_{i,j} a_i c_j \otimes b_i d_j$. (Actually, it is the unique Banach algebra which extends the product $(a \otimes b)(c \otimes d) = ac \otimes bd$. For more details see [**2**: p. 235]). Let $\mathfrak{A}_1 = C(K)$, τ_t^1 and $\omega_1 = \delta_x$ be as in the first example, and let $\mathfrak{A}_2 = \mathcal{L}(\mathcal{H}_n)$ with τ_t^2 and $\omega_2(a) = tr(e^{-\beta h}a)$ be as in the second. Then one checks easily that $\omega_1 \otimes \omega_2$ is a $(\tau^1 \otimes \tau^2, \beta)$-KMS functional on $C(K)\hat{\otimes}_\pi \mathcal{L}(\mathcal{H}_n)$.

Let us now define the action of τ_t on $\mathfrak{A}'$ simply by the adjoint action;

$$(5) \qquad \langle \tau_t(\omega), a \rangle = \langle \omega, \tau_{-t}(a) \rangle \quad \forall t \in \mathbf{R}.$$

PROPOSITION 6.1.3. *For every $\omega \in \mathfrak{A}'$ and every $a \in \mathfrak{A}$,*

$$(6) \qquad \tau_t(a\omega) = \tau_t(a)\tau_t(\omega),$$

and

$$(6') \qquad \tau_t(\omega a) = \tau_t(\omega)\tau_t(a).$$

If ω is τ-invariant then $\tau_t(\omega) = \omega$ for all $t \in \mathbf{R}$ and (6) *becomes; $\tau_t(a\omega) = \tau_t(a)\omega$ while* (6′) *can now be read; $\tau_t(\omega a) = \omega\tau_t(a)$.*

We will now assume for simplicity that $\mathfrak{A} = \mathfrak{A}_\tau$ in the following propositions. Later we will drop this assumption and apply these results to the subalgebra $\mathfrak{A}_\tau$ (which equals $(\mathfrak{A}_\tau)_\tau$) and to the restriction of $\omega \in \mathfrak{A}'$ to $\mathfrak{A}_\tau$.

Then

$$(7) \qquad \langle \tau_z(\omega), a \rangle = \langle \omega, \tau_{-\bar{z}}(a) \rangle \quad \forall z \in \mathbf{C}$$

is the analytic extension of the dual action defined above and each $\omega \in \mathfrak{A}'$ becomes an entire element for this group action.

LEMMA 6.1.4. *If $\omega \in \mathfrak{A}'$ and $a \in \mathfrak{A}$ then*

$$(8) \qquad \tau_z(a\omega) = \tau_z(a)\tau_z(\omega)$$

and

(8′) $\tau_z(\omega a) = \tau_z(\omega)\tau_z(a)$.

Thus, if ω is invariant $\tau_z(\omega) = \omega$ for all $z \in \mathbf{C}$ and (8) *and* (8′) *become; $\tau_z(a\omega) = \tau_z(a)\omega$ and $\tau_z(\omega a) = \omega\tau_z(a)$.*

PROPOSITION 6.1.5. *The following statements are equivalent:*
(1) *ω is a (τ, β)-*KMS *functional.*
(2) $\omega\tau_{-i\beta}(a) = a\omega \quad \forall a \in \mathfrak{A}$.
(3) $\tau_{i\beta/2}(a)\omega = \omega\tau_{-i\beta/2}(a) \quad \forall a \in \mathfrak{A}$.

Proof. ω is a (τ, β)-KMS functional iff $\langle\omega, b\tau_{i\beta}(a^*)\rangle = \langle\omega, a^*b\rangle \quad \forall a, b \in \mathfrak{A}$. But $\langle\omega, b\tau_{i\beta}(a^*)\rangle = \langle\omega\tau_{-i\beta}(a), b\rangle$ and $\langle\omega, a^*b\rangle = \langle a\omega, b\rangle$. While (2) and (3) correspond replacing a by $\tau_{\pm i\beta/2}(a)$. ☐

In the case of central functionals it reduces to:

COROLLARY 6.1.6. *ω is abelian if and only if $a\omega = \omega a \; \forall a \in \mathfrak{A}$.*

For $\omega \in \mathfrak{A}'$, let $\omega^* \in \mathfrak{A}'$ be defined as before by the formula; $\langle\omega^*, a\rangle := \overline{\langle\omega, a^*\rangle}$.

LEMMA 6.1.7. (1) *If $\omega \in \mathfrak{A}'$ then $\tau_z(\omega)^* = \tau_{\bar{z}}(\omega^*)$.*
(2) *If $\omega \in \mathfrak{A}'$ and $a \in \mathfrak{A}$ then $(a\omega)^* = \omega^*a^*$. Thus, if ω is hermitian, we have $(a\omega)^* = \omega a^*$.*

Define the following anti-automorphism on $\mathfrak{A}$,

(9) $\kappa(a) := \tau_{i\beta/2}(a^*) \quad \forall a \in \mathfrak{A}$.

In fact κ is an involution on $\mathfrak{A}$ i.e. κ^2 equals the identity on $\mathfrak{A}$. Recall that the adjoint of an antilinear map is defined by $\langle\kappa^*\omega, a\rangle = \overline{\langle\omega, \kappa a\rangle}$. Let us denote J the adjoint of κ. One checks directly that

(10) $\kappa^*\omega = J\omega = \tau_{-i\beta/2}(\omega^*) \quad \forall\omega \in \mathfrak{A}'$.

J is also involutive and anti-linear. Note that in the case where ω is central, $J\omega = \omega^*$. The last two propositions can now be restated as follows;

PROPOSITION 6.1.8. *If H_ω is the reproducing operator of ω then ω is a (τ, β)-*KMS *functional if and only if $JH_\omega\kappa = H_\omega$. Equivalently, ω is a (τ, β)-*KMS *functional if and only if $J\mathcal{H}_\omega = \mathcal{H}_\omega$ i.e. J restricted to $\mathcal{H}_\omega$ defines an anti-unitary operator.*

Proof. In view of proposition 2.3.4, we only need to prove the first assertion. For all $a \in \mathfrak{A}$,

$$\begin{aligned} JH_\omega\kappa(a) = JH_\omega(\tau_{i\beta/2}(a^*)) &= J(\tau_{i\beta/2}(a^*)\omega) \\ &= \tau_{-i\beta/2}\{(\tau_{i\beta/2}(a^*)\omega)^*\} \\ &= \tau_{-i\beta/2}\{\omega\tau_{-i\beta/2}(a)\} \\ &= \omega\tau_{-i\beta}(a) \end{aligned}$$

This equals $H_\omega a = a\omega$ if and only if ω is a (τ, β)-KMS by proposition 6.1.5.

Note that J leaves D_ω invariant. Let us denote by $\underline{J}$, the anti-unitary operator obtained by restricting J to $\mathcal{H}_\omega$. As usual, let $\pi_\omega(a)f = af$ for $f \in D_\omega$ be the GNS representation of $\mathfrak{A}$, associated to ω.

Let us define the following representation, denoted ρ_ω, of $\bar{\mathfrak{A}}$ in $\mathcal{L}(D_\omega)$;

(11) $\quad \rho_\omega(\bar{a})f := f\tau_{-i\beta/2}(a^*) \quad \forall \bar{a} \in \bar{\mathfrak{A}}.$

In the central case, ρ_ω becomes simply;

(11′) $\quad \rho_\omega(\bar{a})f = fa^* \quad \forall \bar{a} \in \bar{\mathfrak{A}}.$

Note that the action of ρ_ω on $\mathfrak{A}$ defined by (3) in 3.1 gives;

(11″) $\quad \rho_\omega(\bar{a})b = b\tau_{i\beta/2}(a^*) = b\kappa(a)$

for every a and b belonging to $\mathfrak{A}$. □

PROPOSITION 6.1.9. $\rho_\omega(\bar{a}) = \underline{J}\pi_\omega(a)\underline{J} \quad \forall a \in \mathfrak{A}.$

The proof is again straightforward. Now recall from section 4.1 that if ω is a self-adjoint functional on $\mathfrak{A}$ then $\pi_\omega(\mathfrak{A})'$ and hence

(12) $\quad \rho_\omega(\bar{\mathfrak{A}})' = \underline{J}\pi_\omega(\mathfrak{A})'\underline{J}$

are both von Neumann algebras.
Define

(13) $\quad \mathfrak{M} := \pi_\omega(\mathfrak{A})' \cap \rho_\omega(\bar{\mathfrak{A}})'.$

Of course, $\mathfrak{M}$ is another von Neumann algebra. Denote the positive cone of $\mathfrak{M}$ by $\mathfrak{M}^+$ i.e. $\mathfrak{M}^+ = \{T \in \mathfrak{M} : T \geqq 0\}$. □

Now let us return to the general situation and drop the assumption $\mathfrak{A} = \mathfrak{A}_\tau$.

LEMMA 6.1.10 *Let* $\omega \in \mathfrak{A}'$. *Let* $D_\tau := \{a\omega : a \in \mathfrak{A}_\tau\}$ *and let* $\pi_\tau := \pi_\omega|_{D_\tau}$ *be the representation* π_ω *restricted to* D_τ. *Then*

(14) $$\pi_\tau(\mathfrak{A}_\tau)' = \pi_\omega(\mathfrak{A})'.$$

Similarly, we have

(15) $$\rho_\tau(\bar{\mathfrak{A}}_\tau)' = \rho_\omega(\bar{\mathfrak{A}})'.$$

Thus

(16) $$\mathfrak{M} = \pi_\tau(\mathfrak{A}_\tau)' \cap \rho_\tau(\bar{\mathfrak{A}}_\tau)' = \pi_\omega(\mathfrak{A})' \cap \rho_\omega(\bar{\mathfrak{A}})'.$$

Proof. Suppose (14) verified, then (15) and (16) follow since

$$\rho_\omega(\bar{\mathfrak{A}})' = \underline{J}\pi_\omega(\mathfrak{A})'\underline{J} = \underline{J}\pi_\tau(\mathfrak{A}_\tau)'\underline{J} = \rho_\tau(\bar{\mathfrak{A}}_\tau)'.$$

Let us prove (14). $S \in \pi_\tau(\mathfrak{A}_\tau)'$ if and only if

(17) $$(S\pi_\omega(a)\varphi|\psi) = (S\varphi|\pi_\omega(a^*)\psi)$$

for all $\varphi, \psi \in D_\tau$ and all $a \in \mathfrak{A}_\tau$. Let $b, c \in \mathfrak{A}_\tau$ be such that $\varphi = j^*(b)$ and $\psi = j^*(c)$. Then (17) becomes

(18) $$\langle jSj^*ab, c\rangle = \langle jSj^*b, a^*c\rangle$$

for all a, b and c in $\mathfrak{A}_\tau$. But each side of the equality in (18) is separately continuous in a, b and c. Thus the equality extend to every a, b and c in $\mathfrak{A}$. This implies that (17) holds for every $\varphi, \psi \in D_\omega$ and every $a \in \mathfrak{A}$ and it proves (14). □

Let us recall that $\beta \in \mathbf{R}$ is fixed and that $\{\tau_t\}_{t\in\mathbf{R}}$ is a continuous one-parameter group of $*$-automorphisms of $\mathfrak{A}$ with exponential bound. Let Γ denote the cone of all positive (τ, β)-KMS functionals. Given $\omega \in \Gamma$ let $\Gamma(\omega) = \{\omega' \in \Gamma : \exists\lambda \in \mathbf{R}^+$ such that $\omega' \leqq \lambda\omega\}$. $\Gamma(\omega)$ is called the *face* generated by ω in the cone Γ.

THEOREM 6.1.11. *Let* $\mathfrak{A}$ *be a self-derivative* $\mathcal{LF}$ $*$-*algebra having an equicontinuous approximate identity. Let* $\omega \in \mathfrak{A}'$ *be a self-adjoint* (τ, β)-KMS *functional. Then* $\Gamma(\omega)$ *is linearly isomorphic to* $\mathfrak{M}^+$. *Moreover, if* $T \in \mathfrak{M}^+$ *one has* $\underline{J}T\underline{J} = T$.

Proof. Let ω and ω' be two functionals on $\mathfrak{A}$ such that $0 \leqq \omega' \leqq \omega$. Then $\mathcal{H}_{\omega'} \hookrightarrow \mathcal{H}_\omega \hookrightarrow (\mathfrak{A}_\tau)'$. Let us denote by T the reproducing operator of $\mathcal{H}_{\omega'}$ as a Hilbert subspace of $\mathcal{H}_\omega$. $\mathcal{H}_{\omega'}$ being invariant we get by proposition 3.1.10 that $T \in \pi_\omega(\mathfrak{A}_\tau)'$. Now let us assume that ω and ω' are KMS functionals. Then

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$J\mathcal{H}_\omega = \mathcal{H}_\omega$ and $J\mathcal{H}_{\omega'} = \mathcal{H}_{\omega'}$. Thus $\underline{J}\mathcal{H}_{\omega'} = \mathcal{H}_{\omega'}$. But this is equivalent by 2.3.4 to $\underline{J}T\underline{J}^* = T$ i.e. $\underline{J}T\underline{J} = T$. Hence $T \in \underline{J}\pi_\omega(\mathfrak{A})'\underline{J} = \rho_\omega(\bar{\mathfrak{A}})'$. Thus $T \in \mathfrak{M}^+$.

Conversely let $T \in \mathfrak{M}^+$. Then in particular $T \in \pi_\omega(\mathfrak{A}_\tau)'_+ = \pi_\omega(\mathfrak{A})'_+$. Let $\mathcal{K} \hookrightarrow \mathcal{H}_\omega \hookrightarrow \mathfrak{A}'$ be the π_ω-invariant Hilbert subspace of $\mathcal{H}$ having T as a reproducing operator in $\mathcal{H}_\omega$. Let K denote the reproducing operator of $\mathcal{K}$ in $\mathfrak{A}'$. Since $\mathfrak{A} = \mathcal{D}(\mathfrak{A})$ and $\mathfrak{A}$ has an equicontinuous approximate identity, we can apply theorem 4.4.5 in order to get an element $\omega' \in \mathfrak{A}'_+$ such that $Ka = a\omega'$ for all $a \in \mathfrak{A}$. Now $T \in \rho_\omega(\bar{\mathfrak{A}})'$. Applying 3.1.10 with the representation ρ_ω this time we get that $\mathcal{H}_{\omega'}$ is invariant under ρ_ω. Thus $\rho_\omega(\bar{a})H_{\omega'}(b) = H_{\omega'}(\rho_\omega(\bar{a})b)$. Hence $\rho_\omega(\bar{a})(b\omega') = (\rho_\omega(\bar{a})b)\omega'$ i.e., $b\omega'\tau_{-i\beta/2}(a^*) = b\tau_{i\beta/2}(a^*)\omega'$. Since the products are dense in $\mathfrak{A}$, this implies $\omega'\tau_{-i\beta/2}(a^*) = \tau_{i\beta/2}(a^*)\omega'$ for all $a \in \mathfrak{A}$. Therefore ω' is a KMS functional by proposition 6.1.5. □

THEOREM 6.1.12. *Let $\mathfrak{A}$ be a self-derivative $\mathcal{LF}$ $*$-algebra with an equicontinuous approximate identity. Then the face generated by a self-adjoint* (τ, β)-KMS *functional in the cone of all* (τ, β)-KMS *functionals is a lattice.*

Proof. From the last assertion of 6.1.11, we have $\underline{J}A\underline{J} = A$ also for $A \in \mathfrak{M}$ self-adjoint. Thus for a general $A = A_1 + iA_2 \in \mathfrak{M}$ we have by the antilinearity of $\underline{J}$ that $\underline{J}A\underline{J} = A_1 - iA_2 = A^*$. Then $A^*B^* = JABJ = B^*A^*$ which implies that $\mathfrak{M}$ is commutative and hence a lattice (actually, $\mathfrak{M}$ is a lattice iff $\mathfrak{M}$ is commutative [**27**: Theorem 1, p. 227]). □

COROLLARY 6.1.13. *If $\mathfrak{A}$ is a self-derivative Banach $*$-algebra then the cone of all* (τ, β)-KMS *functionals is a lattice cone. In particular, if $\mathfrak{A}$ has a unit then the* (τ, β)-KMS *states form a simplex.*

Proof. Since positive functionals on Banach $*$-algebras are bounded (4.1.2), the first statement is a particular case of 6.1.12. The second assertion comes from the fact that the evaluation of a functional at the unit defines a hyperplane section of the positive cone. Thus the (τ, β)-KMS states form a base of the lattice cone of all (τ, β)-KMS functionals. □

This last result is due to Ruelle [**24**: Theorem 5.1.6] in the case of C^*-algebras with a unit.

7. Existence and uniqueness of an integral representation for self-adjoint KMS functionals on nuclear $*$-algebras. We will prove in this chapter the theorem announced in the introduction on the existence and uniqueness of an integral representation for KMS functionals on nuclear $*$-algebras.

Let $\mathfrak{A}$ be a nuclear $\mathcal{LF}$ $*$-algebra. Let $\mathfrak{A}'_+$ be the closed convex cone of positive functionals on $\mathfrak{A}$.

Let Ω be a Suslin section of $\mathfrak{A}'_+$ i.e. a Suslin subset which meets each ray in precisely one point $\neq 0$. If $\mathfrak{A}$ has a unit 1, we naturally define Ω to be the set of states:

(1) $\quad \Omega = \{\omega \in \mathfrak{A}'_+ : \omega(1) = 1\}.$

In the general case one may be obliged to choose Ω in some arbitrary fashion. Since it is known that $\mathfrak{A}'$ is a Suslin space [**26**: Cor. 1. p. 115], the cone $\mathfrak{A}'_+$ is a Suslin cone and as such has a Suslin section Ω [**28**: Thm 1.19]. None of the results essentially depend on this choice however. Using conical integrals instead of Radon measures, one can avoid any choice ([**28**]).

Let $\Gamma \subset \mathfrak{A}'_+$ be any closed convex subcone. If ω belongs to Γ, the set $\Gamma \cap (\omega - \Gamma)$ is the interval between 0 and ω with respect to the order relation defined by Γ, and $\Gamma(\omega) = \cup_{\lambda \geqq 0} \Gamma \cap (\lambda\omega - \Gamma)$ is the face generated by ω in Γ. The set $\text{ext}(\Gamma)$ of extreme generators of Γ is the subset of those elements ω for which $\Gamma(\omega)$ is the half line $\mathbf{R}^+ \omega$.

PROPOSITION 7.1.1. *Let $\mathfrak{A}$ be a self-derivative nuclear $\mathcal{LF}$ $*$-algebra having an equicontinuous approximate identity. Let Γ be a closed convex cone in $\mathfrak{A}'_+$.*

(1) *The cone Γ is generated by its extreme rays i.e.*

(2) $\quad \Gamma = \overline{co}\ \text{ext}(\Gamma).$

Let $S = \Omega \cap \text{ext}(\Gamma)$.

(2) *For every element $\underline{\omega} \in \Gamma$ there exists a Radon measure m on S such that*

(3) $\quad \underline{\omega} = \int_S \omega dm(\omega).$

The measure is uniquely determined by $\underline{\omega}$ if and only if the face $\Gamma(\underline{\omega})$ is a lattice with respect to its proper order (i.e. the order induced by Γ).

Proof. Corollary 4.4.6 shows that the order intervals $\Gamma \cap (\omega - \Gamma)$, $\omega \in \Gamma$, are bounded subsets of the topological vector space $\mathfrak{A}'$. The theorem now results from theorems 5.5, 5.8, 1.18 and 1.19 of [**28**].

Let $\{\tau_t\}_{t \in \mathbf{R}}$ be a continuous and exponentially bounded one-parameter group of $*$-automorphisms of $\mathfrak{A}$. Let β be any real number. Let Γ_β be the cone of positive and invariant KMS functionals and let

(4) $\quad S = \text{ext}(\Gamma_\beta) \cap \Omega$

be the corresponding normalized set of extreme KMS functionals. If $\mathfrak{A}$ has a unit and Ω is defined by (1) S is the set of extreme β-KMS states.

Recall that a functional ω is said to be self-adjoint if the corresponding GNS representation π_ω is essentially self-adjoint in the sense of Powers (4.1.1). Also ω is said to be bounded if the operators $\pi_\omega(a)$ are bounded for all $a \in \mathfrak{A}$.

THEOREM 7.1.2. *Let $\mathfrak{A}$ be a self-derivative nuclear $\mathcal{LF}$ $*$-algebra with an equicontinuous approximate identity. Let $\{\tau_t\}_{t\in\mathbf{R}}$ be a continuous and exponentially bounded one-parameter group of $*$-automorphisms of $\mathfrak{A}$. Let β be any real number, and let Γ_β be the corresponding cone of* KMS *functionals. If $\underline{\omega} \in \Gamma_\beta$ is self-adjoint there exists a unique Radon measure m on S with the property (3).*

This decomposition is orthogonal in as much as the integral

$$(5) \qquad \mathcal{H}_{\underline{\omega}} = \int_S^{\oplus} \mathcal{H}_\omega dm(\omega)$$

is direct. The GNS representation $\pi_{\underline{\omega}}$ is the integral of the representations π_ω:

$$(6) \qquad \pi_{\underline{\omega}} = \int_S \pi_\omega dm(\omega)$$

i.e. the vector

$$(7) \qquad f = \int_S f(\omega) dm(\omega); \qquad \int_S \|f(\omega)\|^2 dm(\omega) < +\infty$$

belongs to the maximal domain of $\pi_{\underline{\omega}}$ if and only if m-almost all the components $f(\omega)$ belong to the maximal domains of the corresponding representations and

$$(8) \qquad \int_S \|af(\omega)\|^2 dm(\omega) < +\infty \quad \forall a \in \mathfrak{A}.$$

If the maximal domain of $\pi_{\underline{\omega}}$ is metrizable in the graph topology, almost all the representations π_ω are essentially self-adjoint. If $\underline{\omega}$ is bounded the extreme functionals ω are almost all bounded.

Proof. Let $\Gamma = \Gamma_\beta$. By theorem 6.1.12 the face $\Gamma(\underline{\omega})$ is lattice isomorphic to the positive part of the commutative von Neumann algebra:

$$(9) \qquad \mathfrak{M} = \pi_{\underline{\omega}}(\mathfrak{A})' \cap \rho_{\underline{\omega}}(\bar{\mathfrak{A}})'.$$

The order intervals of Γ being bounded (4.3.6) the existence and uniqueness of the measure m representing $\underline{\omega}$ is a consequence of theorems 5.1 and 5.8 in [**28**], S being an admissible section of ext(Γ) (cf. 1.18, 1.19 loc. cit.).

Next we prove that the integral $\mathcal{H} = \int_S \mathcal{H}_\omega dm(\omega)$ is direct. We have to prove that if A_1 and A_2 are disjoint Borel subsets of S, the spaces $\mathcal{H}_{A_i} = \int_{A_i} \mathcal{H}_\omega dm(\omega)$ are disjoint, i.e. $\mathcal{H}_{A_1} \cap \mathcal{H}_{A_2} = (0)$. Let T_i be the reproducing operator of $\mathcal{H}_{A_i}$ in $\mathcal{H} = \mathcal{H}_{\underline{\omega}}$. Let $M^+(m)$ be the face generated by m in the cone of non negative Radon measures on S. The correspondance between $M^+(m)$ and $\mathfrak{M}^+$ being a linear bijection, the fact that the measures $1_{A_1}m$ and $1_{A_2}m$ are mutually singular implies that $\inf(T_1, T_2) = 0$ in the lattice $\mathfrak{M}^+$. Let E_1 and E_2 be the spectral measures associated to T_1 and T_2. Let $P_i = E_i(0, +\infty)$. Then since $P_i = f(T_i)$, with $f = 1_{(0,+\infty)}$ vanishing at 0, $\inf(P_1, P_2) = 0$ in $\mathfrak{M}^+$. Thus $P_1 + P_2 = \sup(P_1, P_2) \leqq I$, i.e. $P_1 \leqq I - P_2$, which implies that P_1 and P_2 project on orthogonal subspaces. Since $\mathcal{H}_{A_i} = \mathrm{Im}(T_i^{1/2}) \subset \mathrm{Im}(P_i)$, $i = 1, 2$, the spaces $\mathcal{H}_{A_1}$ and $\mathcal{H}_{A_2}$ are disjoint.

Let $\mathcal{M}$ be the maximal domain for the representation $\pi_{\underline{\omega}}$ and let $\mathcal{M}_\omega$ be the maximal domains for the π_ω. Let $f = \int f(\omega)dm(\omega)$ belong to $\mathcal{M}$. Then by lemma 3.2.4 we have $f(\omega) \in \mathcal{M}_\omega$ for m-almost all ω. Moreover $\|af\|^2 = \int \|af(\omega)\|^2 dm(\omega) < +\infty$ for all $a \in \mathfrak{A}$. Conversely, if $f(\omega) \in \mathcal{M}_\omega$ for m-almost all ω and $\int \|af(\omega)\|^2 dm(\omega) < +\infty$ for all $a \in \mathfrak{A}$, $af = \int af(\omega)dm(\omega)$ belongs to $\mathcal{H}$ for all $a \in \mathfrak{A}$, i.e. $f \in \mathcal{M}$.

If $\mathcal{M}$ is metrizable, theorem 3.2.2 implies that π_ω essentially self-adjoint for almost all ω. The last assertion follows from proposition 3.2.5. □

COROLLARY 7.1.3. *The decomposition (5) is precisely the diagonalization of the commutative von Neumann algebra $\mathfrak{M}$, defined in (9). Explicitly: For $\varphi \in L^\infty(m)$ let T_φ be the operator defined as follows:*

$$(10) \qquad T_\varphi f = \int \varphi(\omega) f(\omega) dm(\omega); \qquad \int_S \|f(\omega)\|^2 dm(\omega) < +\infty$$

f being defined by (7). Then the map $\varphi \longmapsto T_\varphi$ is a $*$-isomorphism between $L^\infty(m)$ and $\mathfrak{M}$.

Proof. We have established linear bijections between the following cones: $\mathfrak{M}^+$ and $\Gamma(\underline{\omega})$, $\Gamma(\underline{\omega})$ and $M^+(m)$, and obviously $M^+(m)$ and $L^\infty(m)^+$. It suffices to prove that the resulting correspondance between $L^\infty(m)^+$ and $\mathfrak{M}^+$ is the map $\varphi \longmapsto T_\varphi$. Let $j : \mathcal{H}_{\underline{\omega}} \hookrightarrow \mathfrak{A}'$ and $j_\omega : \mathcal{H}_\omega \hookrightarrow \mathfrak{A}'$ be the canonical injections. Let $\varphi \in L^\infty(m)^+$, and let $\underline{\omega}'$ be the KMS functional corresponding to φm. Then we have

$$\langle H_{\underline{\omega}'} a, b\rangle = \int \langle H_\omega a, b\rangle \varphi(\omega) dm(\omega) = \int (j_\omega^* a | j_\omega^* b) \varphi(\omega) dm(\omega)$$
$$= \int (T_\varphi j_\omega^* a | j_\omega^* b) dm(\omega) = (T_\varphi j^* a | j^* b) = \langle j T_\varphi j^* a, b\rangle$$

Thus T_φ is the reproducing operator of $\mathcal{H}_{\underline{\omega}'}$ in $\mathcal{H}_{\underline{\omega}}$. In particular by 6.1.11 T_φ belongs to $\mathfrak{M}^+$. This proves that the map $\varphi \longmapsto T_\varphi$ is a linear bijection between $L^\infty(m)$ and $\mathfrak{M}$. The integral being direct, formula (10) shows that it is a $*$-isomorphism. □

COROLLARY 7.1.4. Let $\mathfrak{A}$ be a self-derivative nuclear $\mathcal{LF}$ $*$-algebra with an equicontinuous approximate identity. Let Γ be the cone of positive central functionals on $\mathfrak{A}$, let Ω be a Suslin section of Γ and let $S = \mathrm{ext}(\Gamma) \cap \Omega$. If $\underline{\omega} \in \Gamma$ is self-adjoint there exists a unique Radon measure m on S such that

$$(11) \qquad \underline{\omega} = \int_S \omega dm(\omega)$$

Moreover this decomposition is orthogonal. The other statements in the theorem also remain valid.

In fact this is a particular case of the preceding theorem, where τ_t is the identity on $\mathfrak{A}$ for all $t \in \mathbf{R}$.

Remark 7.1.5. It may happen that $\mathfrak{A}$ is neither self-derivative nor nuclear, but that the infinite derivative $\mathfrak{A}^{(\infty)}$ is both. In that case one may be able to apply the theorem or its corollary to $\mathfrak{A}^{(\infty)}$. For instance if $\mathfrak{A} = \mathcal{C}_c^1(G)$ is the convolution algebra of functions of class C^1 with compact support on a unimodular Lie group G, the algebra $\mathfrak{A}^{(\infty)}$ is equal (at least algebraically) to the space of Schwartz test functions $\mathcal{C}_c^\infty(G)$, which is self-derivative, nuclear, and has an approximate identity. If we apply corollary 7.1.4 to the convolution algebra $\mathcal{C}_c^\infty(G)$, taking into account remark 4.1.3, we recover the Bochner-Schwartz theorem for unimodular Lie groups (cf. [**30**]).

References

1. H. Araki, *Introduction to Operator Algebras*, in Stat. Mech. & Field theory, R. N. Sen and C. Weil editors, Halsted Press, New York 1972.

2. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer-Verlag, New York-Heidelberg-Berlin 1973.

3. H. J. Borchers, *Algebraic aspects of Wightman field theory*, in Stat. Mechanics & Field theory, edited by R. N. Sein and C. Weil, Halsted Press, New York 1972.

4. H. J. Borchers and J. Yngvason, *On the Algebra of Field operators. The Weak commutant and Integral Decompositions of states*, Comm. Math. Phys. *42* (1975) 231–252.

5. N. Bourbaki, N, *Topologie générale* chap. 1 à 4, Hermann, Paris 1971.

6. ———, *Intégration* livre IV, Hermann, Paris 1963.

7. O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics*, vol. I and II, Springer-Verlag, New York-Heidelberg-Berlin 1979 and 1981.

8. J. Dixmier and P. Malliavin, *Factorisations de fonctions et de vecteurs indéfiniment différentiables*, Bull. Soc. Math. France 2e série *102* (1978) 321–344.

9. N. Dunford and J. Schwartz, *Linear Operators*, Part I, Wiley, New York, 1966.

10. P. Eymard, *L'algèbre de Fourier d'un groupe localement compact*, Bull. Soc. Math. France *92* (1964) 181–236.

11. S. A. Fulling and S. N. M. Ruijsenaars, *Temperature, Periodicity and Horizons*, Physics Reports 152 no. 3 (1987) 135–176.

12. A. Grothendieck, *Espaces vectoriels topologiques*, Sao Paulo 1954.

13. ———, *Produits tensoriels topologiques et espaces nucléaires*, Memoirs of AMS 16, 1955.

14. R. Haag, N. M. Hugenholtz and M. Winnink, *On the equilibrium states in quantum statistical mechanics*, Comm. in Math. Phys. *5* (1967) 215– 236.

15. G. C. Hegerfeldt, *Extremal decomposition of Whightman functions and of states on nuclear ∗-algebras by Choquet theory*, Comm. in Math. Phys. *45* (1975) 133–135.

16. R. Kubo, *Statistical-Mechanical Theory of irreversible Processes* I. General Theory and simple Applications to magnetic and Conduction Problems, J. Phys. Soc. Japan, *12* (1967) 570–586.

17. O. E. Lanford and D. Ruelle, *Integral representations of invariant states on B∗-algebras*, J. Math. Phys. *8* (1967) 1460–1463.

18. K. Maurin, *General eigenfunction expansions and unitary representations of topological groups*, Polish Scientific Publishers, Warsaw 1968.

19. P. C. Martin and J. Schwinger, *Theory of Many Particles Systems* I, Phys. Rev. *115* (1959) 1342–1373.

20. N. A. Naimark, *Normed Rings*, Wolters-Noordhoff, Groningen 1970.
21. A, Pietsch, *Nuclear locally convex spaces*, Springer-Verlag, New York-Heidelberg-Berlin 1972.
22. R. T. Powers, *Self-adjoint algebras of unbounded operators*, Comm. in Math. Phys. *21* (1971) 85–124.
23. ———, *Self-adjoint algebras of unbounded operators* II, Trans. of AMS *187* (1974) 261–293.
24. D. Ruelle, *Symmetry breakdown in statistical mechanics*, Cargèse lectures in Physics, Vol. 4, ed. by D. Kastler, Gordon and Breach, New-York 1970.
25. L. Schwartz, *Sous-espaces hilbertiens d'espaces vectoriels topologiques et noyaux associés*, Journal d'Anal. Math. *13* (1964) 115–256.
26. ———, *Radon measures on arbitrary topological spaces and cylindrical measures*, Oxford University Press 1973.
27. S. Shermann, *Order in operator algebras*, Amer. Jour. of Math. vol. *73* (1951), 227–232.
28. E. G. F. Thomas, *Integral representations in conuclear cones*, Preprint, University of Groningen 1988.
29. ———, *Analysis in embedded Hilbert spaces*, Lecture Notes, Preprint University of Groningen 1988.
30. ———, The theorem of Bochner-Schwartz-Godement for generalized Gelfand pairs, Proc. 3rd Paderborn conference in Functional Analysis, North-Holland Publ. Co. 1983.

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