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WEAK POTENTIAL CONDITIONS FOR SCHRÖDINGER EQUATIONS WITH CRITICAL NONLINEARITIES

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Abstract

In this paper, we prove the existence of nontrivial solutions to the following Schrödinger equation with critical Sobolev exponent:

$$\begin{cases} -\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases}$$

under assumptions that (i) $V(x_0) < 0$ for some $x_0 \in \mathbb{R}^N$ and (ii) there exists $b > 0$ such that the set $\mathcal{V}_b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite measure, in addition to some common assumptions on K and f, where $N \geq 3$, $2^* = 2N/(N-2)$.

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1. Introduction

Consider the subcritical semilinear Schrödinger equation

$$-\Delta u + V(x)u = a(x)|u|^{p-2}u, \quad u \in H^1(\mathbb{R}^N), \tag{1.1}$$

where $N \geq 3$, $2 < p < 2^* = 2N/(N-2)$ (the critical Sobolev exponent), $V \in C(\mathbb{R}^N)$, $V(x)$ is bounded from below and $a \in C(\mathbb{R}^N, \mathbb{R}^+) \cap L^\infty(\mathbb{R}^N)$. To the authors' best knowledge, except for the three cases of (i) $a(x)$ is periodic or (ii) $a(x)$ is asymptotically constant: $a(x) \geq \lim_{|x|\to\infty} a(x) > 0$ or (iii) $a(x)$ is vanishing: $\lim_{|x|\to\infty} a(x) = 0$, results on existence of solutions to (1.1) in other cases all require the following coercivity condition on V (see, for example, [1–3, 12, 13, 15, 17–20, 22]):

(V1) there exists $d_0 > 0$ such that

$$\lim_{|y|\to+\infty} \text{meas}\{x \in \mathbb{R}^N : |x-y| \leq d_0,\ V(x) < M\} = 0, \quad \forall M > 0$$

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or the stronger condition

(V1′) for any $M > 0$, $\operatorname{meas}\{x \in \mathbb{R}^N : V(x) \le M\} < \infty$.

This is mainly due to the lack of compactness of the Sobolev embedding.

When it comes to the critical case, things become even more involving [5–9, 11] and complicated. For example, it is easy to see that if $V \ge 0$ and $V \ne 0$, the mountain pass value corresponding to the semilinear Schrödinger equation

$$-\Delta u + V(x)u = K(x)|u|^{2^*-2}u, \quad u \in H^1(\mathbb{R}^N) \tag{1.2}$$

is not a critical value, where:

(K) $K \in C(\mathbb{R}^N)$, $0 < K(x) \le K(x_0) := K_0$ and $K(x) - K(x_0) = o(|x - x_0|^2)$ for some $x_0 \in \mathbb{R}^N$.

Furthermore, it follows from the Pohozaev identity that (1.2) with $K(x) \equiv 1$ has no nontrivial solution if $V(x) + \nabla V(x) \cdot x > 0$ for all $x \in \mathbb{R}^N$. This shows that (V1) or (V1′), or even the stronger condition $\lim_{|x|\to\infty} V(x) = \infty$, is not sufficient to guarantee that (1.2) has a nontrivial solution.

Benci and Cerami [4] first studied (1.2) and proved the existence of one nontrivial solution if $V \ge 0$ and $\|V\|_{N/2}$ is sufficiently small. Here and in the sequel, by $\|\cdot\|_s$ we denote the usual norm in the space $L^s(\mathbb{R}^N)$. As far as the authors know, this is the only existence result available for general V in the *critical exponent case*. In the case that $\|V\|_{N/2}$ is not sufficiently small, it seems natural to assume that $\{x \in \mathbb{R}^N : V(x) < 0\} \ne \emptyset$. When (K) holds and $V(x) = q(x) - \lambda$ with $\lim_{|x|\to\infty} q(x) = \infty$, Chabrowski and Yang [8] proved that (1.2) has a nontrivial solution provided that $N \ge 5$, $q(x_0) = 0$ and $\lambda_k < \lambda < \lambda_{k+1}$ for $k = 1, 2, \ldots$, where $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ is the sequence of all eigenvalues of $-\Delta + q(x)$. Obviously, these conditions imply that $V(x_0) < 0$ and $\lim_{|x|\to\infty} V(x) = \infty$.

We point out that the coercivity condition $\lim_{|x|\to\infty} q(x) = \infty$ is very crucial to show that the energy functional associated with (1.2) satisfies the local (PS) condition in Chabrowski and Yang [8]. In addition, under the above coercivity condition, the spectrum of the operator $-\Delta + q$ is discrete, which is the sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$. A sequence of eigenfunctions corresponding to the eigenvalues $\{\lambda_n\}$ is complete in $L^2(\mathbb{R}^N)$. This fact is also very important in the arguments in [8].

In this paper, we shall show that (1.2) has a nontrivial solution if V satisfies the following weaker assumptions:

(V0) $V \in C(\mathbb{R}^N)$, $V(x_0) < 0$ and $V(x)$ is bounded from below;

(V2) there exists $b > 0$ such that the set $\mathcal{V}_b := \{x \in \mathbb{R}^N : V(x) < b\}$ has finite measure.

In addition, if V satisfies (V1), regardless of whether $V(x_0) < 0$ or not, we shall demonstrate that the following semilinear Schrödinger equation with critical Sobolev exponent

$$\begin{cases} -\Delta u + V(x)u = K(x)|u|^{2^*-2}u + f(x,u), & x \in \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N) \end{cases} \tag{1.3}$$

also has a nontrivial solution.

To state our results precisely, we first make the following assumptions:

(V0′) $V \in C(\mathbb{R}^N)$ and $V(x)$ is bounded from below;

(V3) $-\Delta u + V(x)u = 0$ has only one solution $u = 0$ in $H^1(\mathbb{R}^N)$;

(F1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$, $F(x,t) := \int_0^t f(x,s)\,ds \geq 0$ and there exist constants $p \in (2, 2^*)$ and $c_0 > 0$ such that

$$|f(x,t)| \leq c_0 K(x)(1 + |t|^{p-1}), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(F2) $f(x,t) = o(|t|)$, as $|t| \to 0$, uniformly in $x \in \mathbb{R}^N$;

(F3) $\mathcal{F}(x,t) := \frac{1}{2} t f(x,t) - F(x,t) \geq 0$, for all $(x,t) \in \mathbb{R}^N \times \mathbb{R}$;

(F4) there exists $a \in C(\mathbb{R}^N, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} a(x) = 0$ such that

$$0 \leq t f(x,t) \leq a(x)(|t|^2 + |t|^p), \quad \forall (x,t) \in \mathbb{R}^N \times \mathbb{R};$$

(F5) there exist $a_0, r_0 > 0$ and $\kappa \in (2, 2^*)$ such that $F(x,t) \geq a_0 |t|^\kappa$, for all $(x,t) \in B_{r_0}(x_0) \times \mathbb{R}$.

We are now in a position to state the main results of this paper.

Theorem 1.1. *Assume that $N \geq 5$ and that V, K and f satisfy (V0), (V2), (K) and (F1)–(F4). Then problem (1.3) has a nontrivial solution.*

Theorem 1.2. *Assume that $N = 4$ and that V, K and f satisfy (V0), (V2), (V3), (K) and (F1)–(F4). Then problem (1.3) has a nontrivial solution.*

Theorem 1.3. *Assume that $N \geq 5$ and that V, K and f satisfy (V0), (V1), (K) and (F1)–(F3). Then problem (1.3) has a nontrivial solution.*

Theorem 1.4. *Assume that $N \geq 4$ and that V, K and f satisfy (V0′), (V1), (K) and (F1)–(F3) and (F5). Then problem (1.3) has a nontrivial solution.*

The remainder of this paper is organized as follows. In Section 2, we establish some useful lemmas. In Section 3, we give an estimate for critical levels. In the last section, we give the proofs of Theorems 1.1–1.4.

2. Preliminaries

In this section, we present some useful lemmas.

By (V0′), $V(x)$ is bounded from below and so there is an $\alpha_0 > 0$ such that

$$V(x) + \alpha_0 \geq 1, \quad \forall x \in \mathbb{R}^N. \tag{2.1}$$

Set

$$E_* = \left\{ u \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} [|\nabla u|^2 + (V(x) + \alpha_0)u^2]\,dx < +\infty \right\},$$

$$(u,v)_* = \int_{\mathbb{R}^N} [\nabla u \nabla v + (V(x) + \alpha_0)uv]\,dx, \quad \forall u, v \in E_*$$

and

$$\|u\|_* = \left\{ \int_{\mathbb{R}^N} [|\nabla u|^2 + (V(x) + \alpha_0) u^2] \, dx \right\}^{1/2}, \quad \forall u \in E_*.$$

Obviously, E_* is a Hilbert space with the inner product $(\cdot, \cdot)_*$ given above; moreover, $\|u\|_{H^1(\mathbb{R}^N)} \leq \|u\|_*$ for all $u \in E_*$.

LEMMA 2.1 [2, Lemma 3.1]. *Suppose that (V0′) and (V1) is satisfied. Then the embedding from E_* into $L^s(\mathbb{R}^N)$ is compact for $2 \leq s < 2^*$.*

Let $\mathcal{A} := -\Delta + V$. Then $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}) \subseteq H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\mu) : -\infty < \mu < +\infty\}$ be the spectral family of $\mathcal{A}$ and $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = id - \mathcal{E}(0) - \mathcal{E}(0-)$. Then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [10, Theorem 4.3.3]). Let $E = \mathfrak{D}(|\mathcal{A}|^{1/2})$ and

$$E^- = \mathcal{E}(0-)E, \quad E^0 = [\mathcal{E}(0) - \mathcal{E}(0-)]E, \quad E^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0)]E.$$

Then $E = E^- \oplus E^0 \oplus E^+$. For any $u \in E$, let

$$u^- = \mathcal{E}(0-)u, \quad u^0 = [\mathcal{E}(0) - \mathcal{E}(0-)]u, \quad u^+ = [\mathcal{E}(+\infty) - \mathcal{E}(0)]u.$$

Then

$$u = u^- + u^0 + u^+ \in E^- \oplus E^0 \oplus E^+ = E.$$

Note that $E^0 = \mathrm{Ker}(\mathcal{A})$; on E, we can define another inner product

$$(u, v) = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_{L^2} + (u^0, v^0)_{L^2}, \quad \forall u, v \in E$$

and the norm

$$\|u\| = \sqrt{(u, u)}, \quad \forall u \in E,$$

where, as usual, $(\cdot, \cdot)_{L^2}$ denotes the inner product of $L^2(\mathbb{R}^N)$. Then E is a Hilbert space with the inner product $(\cdot, \cdot)$ given above. Clearly, $C_0^\infty(\mathbb{R}^N)$ is dense in E.

LEMMA 2.2. *Suppose that V satisfies (V0′). Then*

$$\mathcal{A}u^- = -|\mathcal{A}|u^-, \quad \mathcal{A}u^+ = |\mathcal{A}|u^+, \quad \forall u \in E \cap \mathfrak{D}(\mathcal{A})$$

and, for the inner products $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$,

$$E^- \perp E^0, \quad E^- \perp E^+, \quad E^0 \perp E^+.$$

The proof of Lemma 2.2 is standard, so we omit it.

LEMMA 2.3. *Suppose that V satisfies (V0′) and (V2). Then*

$$\dim[\mathcal{E}(b/2)E] < +\infty.$$

This lemma should be a well-known result, but we could not find its proof. For the reader's convenience we give a proof in detail.

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Proof. In fact, if $\dim[\mathcal{E}(b/2)E] = +\infty$, then there exists a $\lambda_0 \in \sigma_e(\mathcal{A}) \cap (-\infty, b/2]$, where $\sigma_e(\mathcal{A})$ is the essential spectrum of $\mathcal{A}$. By [10, Theorem IX 1.3] or [16, Theorem 4.5.2], there exists a sequence $\{u_n\} \subset \mathfrak{D}(\mathcal{A})$ such that

$$u_n \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^N), \quad \|u_n\|_2 = 1, \quad \|(\mathcal{A} - \lambda_0)u_n\|_2 \to 0. \tag{2.2}$$

For $u \in \mathfrak{D}(\mathcal{A}) \setminus \{0\}$,

$$(\mathcal{A}u, u)_{L^2} = \int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\,dx > -\alpha_0\|u\|_2^2.$$

It follows that the numerical range $\Theta(\mathcal{A})$ of $\mathcal{A}$ is

$$\Theta(\mathcal{A}) = \{(\mathcal{A}u, u)_{L^2} : u \in \mathfrak{D}(\mathcal{A}), \|u\|_2 = 1\} \subseteq (-\alpha_0, +\infty) \tag{2.3}$$

and so the spectral set $\sigma(\mathcal{A}) \subset (-\alpha_0, +\infty)$; see [10, Theorem III 4.4]. Choose $\alpha_1 < -\alpha_0$; then $\alpha_1 \in \rho(\mathcal{A})$ (the resolvent set of $\mathcal{A}$). Let $v_n = (\mathcal{A} - \lambda_0)u_n$ and $w_n = (\mathcal{A} - \alpha_1)^{-1}u_n$. Then, by (2.2) and (2.3),

$$1 = \|u_n\|_2 \ge \frac{(u_n, w_n)_{L^2}}{\|w_n\|_2} = \frac{((\mathcal{A} - \alpha_1)w_n, w_n)_{L^2}}{\|w_n\|_2} \ge -(\alpha_0 + \alpha_1)\|w_n\|_2,$$

which, together with (2.1), implies that

$$\begin{aligned}
\|w_n\|_{H^1(\mathbb{R}^N)}^2 &= \int_{\mathbb{R}^N} (|\nabla w_n|^2 + w_n^2)\,dx \\
&\le \int_{\mathbb{R}^N} [|\nabla w_n|^2 + (V(x) + \alpha_0)w_n^2]\,dx \\
&= ((\mathcal{A} + \alpha_0)w_n, w_n)_{L^2} = (u_n + (\alpha_0 + \alpha_1)w_n, w_n)_{L^2} \\
&\le \|u_n\|_2\|w_n\|_2 + |\alpha_0 + \alpha_1|\,\|w_n\|_2^2 \\
&\le \frac{2}{-(\alpha_0 + \alpha_1)}.
\end{aligned} \tag{2.4}$$

Equation (2.4) shows that $\{\|w_n\|_{H^1(\mathbb{R}^N)}\}$ is bounded. Passing to a subsequence if necessary, it can be assumed that $w_n \rightharpoonup w_0$ in $H^1(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$ and $(\mathcal{A} - \alpha_1)^{-1}$ is a bounded linear operator in $L^2(\mathbb{R}^N)$, $w_n \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$; consequently, $w_0 = 0$. It follows that $w_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$. Hence,

$$\begin{aligned}
(\mathcal{A}w_n, w_n)_{L^2} &= \int_{\mathbb{R}^N} (|\nabla w_n|^2 + V(x)w_n^2)\,dx \\
&\ge b\int_{\mathbb{R}^N} w_n^2\,dx + \int_{\mathcal{V}_b} (V(x) - b)w_n^2\,dx \\
&= b\|w_n\|_2^2 + o(1).
\end{aligned} \tag{2.5}$$

There are two possible cases: (1) $\liminf_{n\to\infty} \|w_n\|_2 > 0$; (2) $\liminf_{n\to\infty} \|w_n\|_2 = 0$.

Case 1. From (2.5),

$$1 = \|u_n\|_2 \ge \frac{(u_n, w_n)_{L^2}}{\|w_n\|_2} = \frac{((\mathcal{A} - \alpha_1)w_n, w_n)_{L^2}}{\|w_n\|_2} \ge (b - \alpha_1)\|w_n\|_2 + o(1). \tag{2.6}$$

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Since $v_n = (\mathcal{A} - \lambda_0)u_n$,

$$(\mathcal{A} - \alpha_1)^{-1}v_n = (\mathcal{A} - \alpha_1)^{-1}(\mathcal{A} - \lambda_0)u_n = u_n + (\alpha_1 - \lambda_0)(\mathcal{A} - \alpha_1)^{-1}u_n$$
$$= u_n + (\alpha_1 - \lambda_0)w_n. \tag{2.7}$$

Hence, from (2.2), (2.6) and (2.7),

$$1 = \|u_n\|_2 \le \|(\mathcal{A} - \alpha_1)^{-1}v_n\|_2 + |\alpha_1 - \lambda_0|\,\|w_n\|_2 \le \frac{\lambda_0 - \alpha_1}{b - \alpha_1} + o(1)$$
$$\le \frac{\frac{b}{2} - \alpha_1}{b - \alpha_1} + o(1),$$

since $(\mathcal{A} - \alpha_1)^{-1}$ is a bounded operator in $L^2(\mathbb{R}^N)$; we deduce a contradiction.

Case 2. Passing to a subsequence if necessary, it can be assumed that $\|w_n\|_2 \to 0$. Hence, from (2.2) and (2.7), one can get a contradictory inequality

$$1 = \|u_n\|_2 \le \|(\mathcal{A} - \alpha_1)^{-1}v_n\|_2 + |\alpha_1 - \lambda_0|\,\|w_n\|_2 = o(1).$$

Cases 1 and 2 show that $\dim[\mathcal{E}(b/2)E] < +\infty$. □

Lemma 2.4. *Suppose that V satisfies (V0′) and (V1). Then, for any $M > 0$,*

$$\dim[\mathcal{E}(M)E] < +\infty.$$

This lemma is a well-known result; here we give a simple proof.

Proof. If $\dim[\mathcal{E}(M)E] = +\infty$, then there exists a $\lambda_0 \in \sigma_e(\mathcal{A}) \cap (-\infty, M]$. By [10, Theorem IX 1.3] or [16, Theorem 4.5.2], there exists a sequence $\{u_n\} \subset \mathfrak{D}(\mathcal{A})$ such that (2.2) holds. Let $v_n = (\mathcal{A} - \lambda_0)u_n$. Then, by (2.1) and (2.2),

$$\|u_n\|_*^2 = \int_{\mathbb{R}^N} [|\nabla u_n|^2 + (V(x) + \alpha_0)u_n^2]\,dx$$
$$= ((\mathcal{A} + \alpha_0)u_n, u_n)_{L^2} = (v_n + (\alpha_0 + \lambda_0)u_n, u_n)_{L^2}$$
$$\le \|u_n\|_2\|v_n\|_2 + (\alpha_0 + \lambda_0)\|u_n\|_2^2 = \alpha_0 + \lambda_0 + o(1). \tag{2.8}$$

Equation (2.8) shows that $\{\|u_n\|_*\}$ is bounded. Passing to a subsequence if necessary, it can be assumed that $u_n \rightharpoonup u_0$ in $H^1(\mathbb{R}^N)$. Since $u_n \rightharpoonup 0$ in $L^2(\mathbb{R}^N)$, $u_0 = 0$. It follows from Lemma 2.1 that $u_n \to 0$ in $L^2(\mathbb{R}^N)$, which is a contradiction. □

Lemma 2.5. *Suppose that V satisfies (V0′) and (V1) or (V2). Then there exists a constant $\beta > 0$ such that*

$$\|u\|_2 \le \beta\|u\|, \quad \forall u \in E. \tag{2.9}$$

Proof. Since $\dim[\mathcal{E}(b/2)E] < +\infty$, there exists a constant $\beta_1 > 0$ such that

$$\|u\|_2 \le \beta_1\|u\|, \quad \forall u \in \mathcal{E}(b/2)E. \tag{2.10}$$

On the other hand,

$$\|u\|^2 = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u)_{L^2} + \|u^0\|_2^2 = (|\mathcal{A}|u, u)_{L^2}$$
$$= \int_{b/2}^{+\infty} |\lambda|\,d(\mathcal{E}(\lambda)u, u)_{L^2} \ge \frac{b}{2}\|u\|_2^2, \quad \forall u \in [\mathcal{E}(+\infty) - \mathcal{E}(b/2)]E. \tag{2.11}$$

The conclusion of Lemma 2.5 follows by the combination of (2.10) with (2.11). □

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Lemma 2.6. *Suppose that V satisfies (V0′) and (V1) or (V2). Then*

$$\frac{1}{\sqrt{2+\alpha_0}}\|u\| \le \|u\|_* \le \sqrt{1+\alpha_0\beta^2}\|u\|, \quad \forall u \in E = E_*. \tag{2.12}$$

Proof. For $u \in C_0^\infty(\mathbb{R}^N)$, it follows from (2.1) and (2.9) that

$$\begin{aligned}\|u\|_*^2 &= \int_{\mathbb{R}^N} [|\nabla u|^2 + (V(x)+\alpha_0)u^2]\,dx \\ &= (|\mathcal{A}|\mathcal{U}u, u)_{L^2} + \alpha_0\|u\|_2^2 = (\mathcal{U}|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u)_{L^2} + \alpha_0\|u\|_2^2 \\ &\le \|\mathcal{U}|\mathcal{A}|^{1/2}u\|_2\||\mathcal{A}|^{1/2}u\|_2 + \alpha_0\|u\|_2^2 \\ &= \||\mathcal{A}|^{1/2}u\|_2^2 + \alpha_0\|u\|_2^2 \le (1+\alpha_0\beta^2)\|u\|^2 \end{aligned} \tag{2.13}$$

and

$$\begin{aligned}\|u\|^2 &= (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}u)_{L^2} + \|u^0\|_2^2 \le (|\mathcal{A}|u, u)_{L^2} + \|u\|_2^2 \\ &= ((\mathcal{A}+\alpha_0)\mathcal{U}u, u)_{L^2} - \alpha_0(\mathcal{U}u, u)_{L^2} + \|u\|_2^2 \\ &= (\mathcal{U}(\mathcal{A}+\alpha_0)^{1/2}u, (\mathcal{A}+\alpha_0)^{1/2}u)_{L^2} - \alpha_0(\mathcal{U}u, u)_{L^2} + \|u\|_2^2 \\ &\le \|\mathcal{U}(\mathcal{A}+\alpha_0)^{1/2}u\|_2\|(\mathcal{A}+\alpha_0)^{1/2}u\|_2 + (1+\alpha_0)\|u\|_2^2 \\ &= \|(\mathcal{A}+\alpha_0)^{1/2}u\|_2^2 + (1+\alpha_0)\|u\|_2^2 \\ &= ((\mathcal{A}+\alpha_0)u, u)_{L^2} + (1+\alpha_0)\|u\|_2^2 \\ &= \|u\|_*^2 + (1+\alpha_0)\|u\|_2^2 \le (2+\alpha_0)\|u\|_*^2. \end{aligned} \tag{2.14}$$

Combining (2.13) with (2.14),

$$\frac{1}{\sqrt{2+\alpha_0}}\|u\| \le \|u\|_* \le \sqrt{1+\alpha_0\beta^2}\|u\|, \quad \forall u \in C_0^\infty(\mathbb{R}^N). \tag{2.15}$$

Since $C_0^\infty(\mathbb{R}^N)$ is dense in E and E_*, it follows from (2.15) that (2.12) holds. □

Remark 2.7. In view of Lemma 2.6, for any $s \in [2, 2^*]$, there exists a constant $\gamma_s > 0$ such that

$$\|u\|_s \le \gamma_s\|u\|, \quad \forall u \in E,\ 2 \le s \le 2^*. \tag{2.16}$$

Let $m = \dim(E^- \oplus E^0)$. In view of [16, Theorem 4.5.4], there exist m eigenfunctions $e_1, e_2, \ldots, e_m \in E^- \oplus E^0$ of $\mathcal{A}$ such that

$$E^- \oplus E^0 = \bigoplus_{i=1}^m \mathbb{R}e_i, \quad \mathcal{A}e_i = \lambda_i e_i,\ i = 1, 2, \ldots, m,$$

where $-\alpha_0 < \lambda_m \le \lambda_{m-1} \le \cdots \le \lambda_1 \le 0$. Hence, in view of [14, Theorem C.3.4], we can check easily the following lemma.

Lemma 2.8. *Suppose that V satisfies (V0′) and (V2). Then there exists a constant $C_0 > 0$ such that*

$$\|\mathcal{A}u\|_\infty + \|u\|_\infty \le C_0\|u\|_2, \quad \forall u \in \mathcal{E}(0)E = E^- \oplus E^0.$$

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Set

$$b(u, v) = \int_{\mathbb{R}^N} (\nabla u \cdot \nabla v + V(x)uv)\, dx, \quad \forall u, v \in E. \tag{2.17}$$

Then it is easy to check the following lemma.

LEMMA 2.9. *Suppose that V satisfies (V0′). Then $b(u, v)$ is a bilinear functional on E and*

$$b(u^+, u^+) = \|u^+\|^2, \quad b(u^-, u^-) = -\|u^-\|^2, \quad b(u^+, u^- + u^0) = 0, \quad \forall u \in E,$$
$$b(u, u) = \|u^+\|^2 - \|u^-\|^2, \quad \forall u \in E \tag{2.18}$$

and

$$b(u^+, v) = (u^+, v), \quad b(u^-, v) = -(u^-, v), \quad \forall u, v \in E. \tag{2.19}$$

Define a functional Φ on E as follows:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + V(x)u^2)\, dx - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*}\, dx - \int_{\mathbb{R}^N} F(x, u)\, dx.$$

Under assumptions (V0′), (V1) (or (V2)), (F1) and (F2), Φ is of class $C^1(E, \mathbb{R})$ and

$$\Phi(u) = \frac{1}{2} b(u, u) - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|u|^{2^*}\, dx - \int_{\mathbb{R}^N} F(x, u)\, dx, \quad \forall u \in E \tag{2.20}$$

and

$$\langle \Phi'(u), v\rangle = b(u, v) - \int_{\mathbb{R}^N} K(x)|u|^{2^*-2}uv\, dx - \int_{\mathbb{R}^N} f(x, u)v\, dx, \quad \forall u, v \in E. \tag{2.21}$$

3. Estimates for critical levels

In this section, we estimate the critical levels of Φ.

Without loss of generality, we may assume that $x_0 = 0$. By virtue of (V0), we can choose constants $r_1 \in (0, r_0/2)$ and $b_1 > 0$ such that

$$V(x) \le -b_1, \quad |x| \le 2r_1. \tag{3.1}$$

Set $\mathcal{D}^{1,2}(\mathbb{R}^N) = \{u \in L^{2^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$. Let

$$S := \inf_{u \in \mathcal{D}^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\|\nabla u\|_2^2}{\|u\|_{2^*}^2} \tag{3.2}$$

and

$$\vartheta_\varepsilon(x) := \frac{C(N)\eta(x)\varepsilon^{(N-2)/2}}{(\varepsilon^2 + |x|^2)^{(N-2)/2}}, \tag{3.3}$$

where $C(N) = [N(N-2)]^{(N-2)/4}$, $\varepsilon > 0$ and $\eta \in C_0^\infty(\mathbb{R}^N, [0, 1])$ with $\eta(x) = 1$ if $|x| \le r_1$ and $\eta(x) = 0$ if $|x| \ge 2r_1$.

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LEMMA 3.1 ([7], [21, Lemma 1.46]). *The following estimates are true:*

$$\|\vartheta_\varepsilon\|_{2^*}^{2^*} = S^{N/2} + O(\varepsilon^N), \quad \|\nabla\vartheta_\varepsilon\|_2^2 = S^{N/2} + O(\varepsilon^{N-2}),$$

$$\|\vartheta_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}), \quad \|\vartheta_\varepsilon\|_{2^*-1}^{2^*-1} = O(\varepsilon^{(N-2)/2}),$$

$$\|\nabla\vartheta_\varepsilon\|_1 = O(\varepsilon^{(N-2)/2}),$$

$$\|\vartheta_\varepsilon\|_2^2 = \begin{cases} \rho_1\varepsilon^2 + O(\varepsilon^{N-2}) & \text{if } N > 4, \\ \rho_2\varepsilon^2|\ln\varepsilon| + O(\varepsilon^2) & \text{if } N = 4 \end{cases}$$

and

$$\|\vartheta_\varepsilon\|_s^s = \begin{cases} \rho_3\varepsilon^{N-(N-2)s/2} + O(\varepsilon^{(N-2)s/2}) & \text{if } (N-2)s > N, \\ O(\varepsilon^{N/2}|\ln\varepsilon|) & \text{if } (N-2)s = N, \\ O(\varepsilon^{(N-2)s/2}) & \text{if } N-2 \le (N-2)s < N, \end{cases}$$

where ρ_1, ρ_2 and ρ_3 are positive constants.

LEMMA 3.2. *Suppose that $N \ge 5$ and that V, K and F satisfy (V0), (V2), (K), (F1) and (F2), respectively. Then there exist $\varepsilon_0 > 0$ and $\theta_0 > 0$ such that*

$$\|\vartheta_{\varepsilon_0}^+\|^2 - \|\vartheta_{\varepsilon_0}^-\|^2 > \frac{S^{N/2}}{2} \tag{3.4}$$

and

$$\sup\{\Phi(w + s\vartheta_{\varepsilon_0}) : w \in E^- \oplus E^0, s \ge 0\} \le \frac{S^{N/2}}{NK_0^{(N-2)/2}} - \theta_0. \tag{3.5}$$

PROOF. By virtue of (2.17), (2.18), (3.1), (3.3) and Lemma 3.1,

$$\|\vartheta_\varepsilon^+\|^2 - \|\vartheta_\varepsilon^-\|^2 = \int_{\mathbb{R}^N} (|\nabla\vartheta_\varepsilon|^2 + V(x)\vartheta_\varepsilon^2)\,dx \ge S^{N/2} - O(\varepsilon^{N-2}) - \alpha_0\|\vartheta_\varepsilon\|_2^2 \tag{3.6}$$

and

$$\|\vartheta_\varepsilon^+\|^2 - \|\vartheta_\varepsilon^-\|^2 = \int_{\mathbb{R}^N} (|\nabla\vartheta_\varepsilon|^2 + V(x)\vartheta_\varepsilon^2)\,dx \le S^{N/2} + O(\varepsilon^{N-2}) - b_1\|\vartheta_\varepsilon\|_2^2. \tag{3.7}$$

Applying Lemma 2.8 and taking note of Remark 2.7,

$$\|\mathcal{A}w\|_\infty + \|w\|_\infty \le C_1\|w\|, \quad \forall w \in E^- \oplus E^0,$$

which, together with (2.17), (2.19), (K) and Lemma 3.1, yields

$$\begin{aligned} |(\vartheta_\varepsilon^-, w^-)| = |(\vartheta_\varepsilon, w^-)| &= \left|\int_{\mathbb{R}^N} [\nabla\vartheta_\varepsilon\nabla w^- + V(x)\vartheta_\varepsilon w^-]\,dx\right| \\ &= \left|\int_{\mathbb{R}^N} \vartheta_\varepsilon(\mathcal{A}w^-)\,dx\right| \le C_2\|w^-\|\,\|\vartheta_\varepsilon\|_1 \\ &= O(\varepsilon^{(N-2)/2})\|w^-\|, \quad \forall w \in E^- \oplus E^0, \end{aligned} \tag{3.8}$$

$$\begin{aligned} \int_{\mathbb{R}^N} K(x)|\vartheta_\varepsilon|^{2^*-1}|w|\,dx &\le K_0\|w\|_\infty\|\vartheta_\varepsilon\|_{2^*-1}^{2^*-1} \\ &= O(\varepsilon^{(N-2)/2})\|w\|, \quad \forall w \in E^- \oplus E^0 \end{aligned} \tag{3.9}$$

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and

$$\begin{aligned}\int_{\mathbb{R}^N} K(x)|\vartheta_\varepsilon||w|^{2^*-1}\,dx &\le K_0\|w\|_\infty^{2^*-1}\|\vartheta_\varepsilon\|_1\\ &= O(\varepsilon^{(N-2)/2})\|w\|^{2^*-1}, \quad \forall w \in E^- \oplus E^0.\end{aligned} \tag{3.10}$$

Since $\dim(E^- \oplus E^0) < \infty$, there exists a constant $c_1 > 0$ such that

$$\int_{\mathbb{R}^N} K(x)|w|^{2^*}\,dx \ge c_1\|w\|^{2^*}, \quad \forall w \in E^- \oplus E^0. \tag{3.11}$$

Making use of the inequality

$$|s+t|^\varrho \ge |s|^\varrho - \varrho|s|^{\varrho-1}|t| - \varrho|s||t|^{\varrho-1} + |t|^\varrho, \quad \forall s, t \in \mathbb{R},\ \varrho > 2$$

and using (K) with $x_0 = 0$, (3.3), (3.9), (3.10) and (3.11),

$$\begin{aligned}&\int_{\mathbb{R}^N} K(x)|w+s\vartheta_\varepsilon|^{2^*}\,dx\\ &\quad\ge s^{2^*}\int_{\mathbb{R}^N} K(x)|\vartheta_\varepsilon|^{2^*}\,dx - 2^* s^{2^*-1}\int_{\mathbb{R}^N} K(x)|\vartheta_\varepsilon|^{2^*-1}|w|\,dx\\ &\qquad - 2^* s\int_{\mathbb{R}^N} K(x)|\vartheta_\varepsilon||w|^{2^*-1}\,dx + \int_{\mathbb{R}^N} K(x)|w|^{2^*}\,dx\\ &\quad\ge s^{2^*}[K_0 S^{N/2} - o(\varepsilon^2)] - s^{2^*-1}O(\varepsilon^{(N-2)/2})\|w\|\\ &\qquad - sO(\varepsilon^{(N-2)/2})\|w\|^{2^*-1} + c_1\|w\|^{2^*}, \quad \forall s \ge 0,\ w \in E^- \oplus E^0.\end{aligned} \tag{3.12}$$

Hence, from (2.18), (2.20), (3.7), (3.8), (3.12) and the the fact that $F(x,t) \ge 0$,

$$\begin{aligned}&\Phi(w+s\vartheta_\varepsilon)\\ &\quad= \frac{1}{2}(s^2\|\vartheta_\varepsilon^+\|^2 - \|w^- + s\vartheta_\varepsilon^-\|^2) - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|w+s\vartheta_\varepsilon|^{2^*}\,dx\\ &\qquad - \int_{\mathbb{R}^N} F(x, w+s\vartheta_\varepsilon)\,dx\\ &\quad\le \frac{1}{2}[s^2(\|\vartheta_\varepsilon^+\|^2 - \|\vartheta_\varepsilon^-\|^2) - \|w^-\|^2] - s(w^-, \vartheta_\varepsilon^-)\\ &\qquad - \frac{1}{2^*}\int_{\mathbb{R}^N} K(x)|w+s\vartheta_\varepsilon|^{2^*}\,dx\\ &\quad\le \frac{s^2}{2}[S^{N/2} + O(\varepsilon^{N-2}) - b_1\|\vartheta_\varepsilon\|_2^2] - \frac{1}{2}\|w^-\|^2 + sO(\varepsilon^{(N-2)/2})\|w^-\|\\ &\qquad - \frac{s^{2^*}}{2^*}[K_0 S^{N/2} - o(\varepsilon^2)] + s^{2^*-1}O(\varepsilon^{(N-2)/2})\|w\|\\ &\qquad + sO(\varepsilon^{(N-2)/2})\|w\|^{2^*-1} - \frac{c_1}{2^*}\|w\|^{2^*}\\ &\quad\le \frac{S^{N/2}}{NK_0^{(N-2)/2}} - b_2\|\vartheta_\varepsilon\|_2^2 - \frac{1}{2}\|w^-\|^2 + O(\varepsilon^{(N-2)/2})\|w^-\| + O(\varepsilon^{N-2})\end{aligned}$$

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$$
\begin{aligned}
&\quad + o(\varepsilon^2) + O(\varepsilon^{(N-2)/2})\|w\| + O(\varepsilon^{(N-2)/2})\|w\|^{2^*-1} - \frac{c_1}{2^*}\|w\|^{2^*} \\
&\leq \frac{S^{N/2}}{NK_0^{(N-2)/2}} - b_2\|\vartheta_\varepsilon\|_2^2 + O(\varepsilon^{N(N-2)/(N+2)}) + o(\varepsilon^2), \\
&\quad \forall s \geq 0,\ w \in E^- \oplus E^0,
\end{aligned}
\tag{3.13}
$$

where $b_2 > 0$ is a constant. Employing Lemma 3.1, due to the fact that $N \geq 5$, we can choose $\varepsilon_0 > 0$ such that

$$
\frac{S^{N/2}}{2} - O(\varepsilon^{N-2}) - \alpha_0\|\vartheta_\varepsilon\|_2^2 > 0, \quad 0 < \varepsilon \leq \varepsilon_0 \tag{3.14}
$$

and

$$
-\frac{b_2}{2}\|\vartheta_\varepsilon\|_2^2 + O(\varepsilon^{N(N-2)/(N+2)}) + o(\varepsilon^2) < 0, \quad 0 < \varepsilon \leq \varepsilon_0. \tag{3.15}
$$

Combining (3.13) with (3.15),

$$
\begin{aligned}
\Phi(w + s\vartheta_{\varepsilon_0}) &\leq \frac{S^{N/2}}{NK_0^{(N-2)/2}} - \frac{b_2}{2}\|\vartheta_{\varepsilon_0}\|_2^2 \\
&:= \frac{S^{N/2}}{NK_0^{(N-2)/2}} - \theta_0, \quad \forall s \geq 0,\ w \in E^- \oplus E^0.
\end{aligned}
\tag{3.16}
$$

Now the conclusion of Lemma 3.3 follows by (3.6), (3.14) and (3.16). □

Lemma 3.3. *Suppose that $N = 4$ and that V, K and F satisfy (V0), (V2), (V3), (K), (F1) and (F2), respectively. Then there exist $\varepsilon_0 > 0$ and $\theta_0 > 0$ such that (3.4) and (3.5) hold.*

Proof. (V3) yields $E^0 = \{0\}$ and so $w^- = w$ for all $w \in E^- \oplus E^0$. By virtue of (3.13),

$$
\begin{aligned}
\Phi(w + s\vartheta_\varepsilon) &\leq \frac{S^{N/2}}{NK_0^{(N-2)/2}} - b_2\|\vartheta_\varepsilon\|_2^2 - \frac{1}{2}\|w^-\|^2 + O(\varepsilon^{(N-2)/2})\|w^-\| + O(\varepsilon^{N-2}) \\
&\quad + o(\varepsilon^2) + O(\varepsilon^{(N-2)/2})\|w\| + O(\varepsilon^{(N-2)/2})\|w\|^{2^*-1} - \frac{c_1}{2^*}\|w\|^{2^*} \\
&= \frac{S^2}{4K_0} - b_2\|\vartheta_\varepsilon\|_2^2 - \frac{1}{2}\|w\|^2 + O(\varepsilon)\|w\| + O(\varepsilon^2) + O(\varepsilon)\|w\|^3 - \frac{c_1}{4}\|w\|^4 \\
&\leq \frac{S^2}{4K_0} - b_2\|\vartheta_\varepsilon\|_2^2 + O(\varepsilon^2), \quad \forall s \geq 0,\ w \in E^- \oplus E^0.
\end{aligned}
$$

The rest of the proof is the same as that of Lemma 3.2. □

Lemma 3.4. *Suppose that $N \geq 4$ and that V, K and F satisfy (V0′), (V1), (K), (F1), (F2) and (F5), respectively. Then there exist $\varepsilon_0 > 0$ and $\theta_0 > 0$ such that (3.4) and (3.5) hold.*

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Proof. We only prove that (3.5) holds. From (2.17), (2.18) and Lemma 3.1,

$$\begin{aligned}\|\vartheta_\varepsilon^+\|^2 - \|\vartheta_\varepsilon^-\|^2 &= \int_{\mathbb{R}^N} (|\nabla \vartheta_\varepsilon|^2 + V(x)\vartheta_\varepsilon^2)\,dx \\ &\le S^{N/2} + O(\varepsilon^{N-2}) + b_3\|\vartheta_\varepsilon\|_2^2, \end{aligned}\tag{3.17}$$

where $b_3 = 1 + \max_{|x|\le 2r_1} |V(x)|$. Since $\dim(E^- \oplus E^0) < \infty$, there exist two constants $c_2 > 0$ and $c_3 > 0$ such that

$$\int_{B_{r_0}(0)} K(x)|w|^{2^*}\,dx \ge c_2\|w\|_{\infty,B_{r_0}}^{2^*}, \quad \forall w \in E^- \oplus E^0$$

and

$$\int_{B_{r_0}(0)} |w|^\kappa\,dx \ge c_3\|w\|_{\infty,B_{r_0}}^{\kappa}, \quad \forall w \in E^- \oplus E^0,$$

where $\|w\|_{\infty,B_{r_0}} = \operatorname{ess\,sup}_{|x|\le r_0} |w(x)|$. Since $\kappa \in (2, 2^*)$, analogous to the proof of (3.12),

$$\begin{aligned}&\int_{B_{r_0}(0)} K(x)|w + s\vartheta_\varepsilon|^{2^*}\,dx \\ &\quad\ge s^{2^*}\int_{B_{r_0}(0)} K(x)|\vartheta_\varepsilon|^{2^*}\,dx - 2^* s^{2^*-1}\int_{B_{r_0}(0)} K(x)|\vartheta_\varepsilon|^{2^*-1}|w|\,dx \\ &\qquad - 2^* s\int_{B_{r_0}(0)} K(x)|\vartheta_\varepsilon||w|^{2^*-1}\,dx + \int_{B_{r_0}(0)} K(x)|w|^{2^*}\,dx \\ &\quad\ge s^{2^*}[K_0 S^{N/2} - o(\varepsilon^2)] - s^{2^*-1}O(\varepsilon^{(N-2)/2})\|w\|_{\infty,B_{r_0}} \\ &\qquad - sO(\varepsilon^{(N-2)/2})\|w\|_{\infty,B_{r_0}}^{2^*-1} + c_2\|w\|_{\infty,B_{r_0}}^{2^*}, \quad \forall s \ge 0,\ w \in E^- \oplus E^0 \end{aligned}\tag{3.18}$$

and

$$\begin{aligned}\int_{B_{r_0}(0)} |w + s\vartheta_\varepsilon|^\kappa\,dx &\ge s^\kappa\int_{B_{r_0}(0)} |\vartheta_\varepsilon|^\kappa\,dx - \kappa s^{\kappa-1}\int_{B_{r_0}(0)} |\vartheta_\varepsilon|^{\kappa-1}|w|\,dx \\ &\qquad - \kappa s\int_{B_{r_0}(0)} |\vartheta_\varepsilon||w|^{\kappa-1}\,dx + \int_{B_{r_0}(0)} |w|^\kappa\,dx \\ &\ge s^\kappa\|\vartheta_\varepsilon\|_\kappa^\kappa - O(\varepsilon^{(N-2)/2})[s^{\kappa-1}\|w\|_{\infty,B_{r_0}} + s\|w\|_{\infty,B_{r_0}}^{\kappa-1}] \\ &\qquad + c_3\|w\|_{\infty,B_{r_0}}^\kappa, \quad \forall s \ge 0,\ w \in E^- \oplus E^0. \end{aligned}\tag{3.19}$$

Hence, from (F5), (2.18), (2.20), (3.8), (3.17), (3.18) and (3.19),

$$\begin{aligned}&\Phi(w + s\vartheta_\varepsilon) \\ &\quad\le \frac{1}{2}[s^2(\|\vartheta_\varepsilon^+\|^2 - \|\vartheta_\varepsilon^-\|^2) - \|w^-\|^2] - s(w^-, \vartheta_\varepsilon^-) \\ &\qquad - \frac{1}{2^*}\int_{B_{r_0}(0)} K(x)|w + s\vartheta_\varepsilon|^{2^*}\,dx - a_0\int_{B_{r_0}(0)} |w + s\vartheta_\varepsilon|^\kappa\,dx \end{aligned}$$

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$$
\begin{aligned}
&\le \frac{s^2}{2}[S^{N/2} + O(\varepsilon^{N-2}) + b_3\|\vartheta_\varepsilon\|_2^2] - \frac{1}{2}\|w^-\|^2 + sO(\varepsilon^{(N-2)/2})\|w^-\| \\
&\quad - \frac{s^{2^*}}{2^*}[K_0 S^{N/2} - o(\varepsilon^2)] + s^{2^*-1}O(\varepsilon^{(N-2)/2})\|w\|_{\infty,B_{r_0}} \\
&\quad + sO(\varepsilon^{(N-2)/2})\|w\|_{\infty,B_{r_0}}^{2^*-1} - \frac{c_2}{2^*}\|w\|_{\infty,B_{r_0}}^{2^*} - a_0 s^\kappa\|\vartheta_\varepsilon\|_\kappa^\kappa \\
&\quad + O(\varepsilon^{(N-2)/2})[s^{\kappa-1}\|w\|_{\infty,B_{r_0}} + s\|w\|_{\infty,B_{r_0}}^{\kappa-1}] - a_0c_3\|w\|_{\infty,B_{r_0}}^\kappa \\
&\le \frac{S^{N/2}}{NK_0^{(N-2)/2}} + b_4\|\vartheta_\varepsilon\|_2^2 - \frac{1}{2}\|w^-\|^2 + O(\varepsilon^{(N-2)/2})\|w^-\| \\
&\quad + O(\varepsilon^{(N-2)/2})(\|w\|_{\infty,B_{r_0}}^{2^*-1} + \|w\|_{\infty,B_{r_0}}^{\kappa-1} + \|w\|_{\infty,B_{r_0}}) \\
&\quad - \frac{c_2}{2^*}\|w\|_{\infty,B_{r_0}}^{2^*} - a_1\|\vartheta_\varepsilon\|_\kappa^\kappa - a_0c_3\|w\|_{\infty,B_{r_0}}^\kappa \\
&\le \frac{S^{N/2}}{NK_0^{(N-2)/2}} + 2b_4\|\vartheta_\varepsilon\|_2^2 + O(\varepsilon^{(N-2)/2})\|w\|_{\infty,B_{r_0}} \\
&\quad - a_1\|\vartheta_\varepsilon\|_\kappa^\kappa - \frac{a_0c_3}{2}\|w\|_{\infty,B_{r_0}}^\kappa \\
&\le \frac{S^{N/2}}{NK_0^{(N-2)/2}} + 2b_4\|\vartheta_\varepsilon\|_2^2 + O(\varepsilon^{\kappa(N-2)/2(\kappa-1)}) - a_1\|\vartheta_\varepsilon\|_\kappa^\kappa, \\
&\quad \forall s \ge 0, \quad w \in E^- \oplus E^0, \qquad (3.20)
\end{aligned}
$$

where $a_1, b_4 > 0$ are constants. Employing Lemma 3.1, due to the fact that $N \ge 4$ and $2 < \kappa < 2N/(N-2)$, we can choose $\varepsilon_0 > 0$ such that

$$
2b_4\|\vartheta_\varepsilon\|_2^2 + O(\varepsilon^{\kappa(N-2)/2(\kappa-1)}) - \frac{a_1}{2}\|\vartheta_\varepsilon\|_\kappa^\kappa < 0, \quad 0 < \varepsilon \le \varepsilon_0. \qquad (3.21)
$$

Combining (3.20) with (3.21),

$$
\begin{aligned}
\Phi(w + s\vartheta_{\varepsilon_0}) \le \frac{S^{N/2}}{NK_0^{(N-2)/2}} - \frac{a_1}{2}\|\vartheta_{\varepsilon_0}\|_\kappa^\kappa := \frac{S^{N/2}}{NK_0^{(N-2)/2}} - \theta_0, \\
\forall s \ge 0, \; w \in E^- \oplus E^0. \qquad (3.22)
\end{aligned}
$$

Now the conclusion of Lemma 3.4 follows by (3.22). □

4. Existence of nontrivial solutions

In this section, we give the proofs of Theorems 1.1–1.4.

Applying the link theorem without the (PS) condition, by standard arguments, we can prove the following lemma.

LEMMA 4.1. *Suppose that V, K and F satisfy (V0′), (V1) (or (V2)), (K), (F1) and (F2), respectively. Then there exist a sequence $\{u_n\} \subset E$ and a constant $c_* \in (0, \sup\{\Phi(w + s\vartheta_{\varepsilon_0}) : w \in E^- \oplus E^0, \; s \ge 0\}]$ satisfying*

$$
\Phi(u_n) \to c_*, \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0.
$$

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LEMMA 4.2. *Suppose that V, K and F satisfy (V0′), (V1) (or (V2)), (K), (F1), (F2) and (F3), respectively. Then any sequence $\{u_n\} \subset E$ satisfying*

$$\Phi(u_n) \to c \geq 0, \quad \langle \Phi'(u_n), u_n \rangle \to 0, \quad \langle \Phi'(u_n), u_n^+ \rangle \to 0 \tag{4.1}$$

is bounded in E.

PROOF. To prove the boundedness of $\{u_n\}$, arguing by contradiction, suppose that $\|u_n\| \to \infty$. Let $v_n = u_n/\|u_n\|$; then $\|v_n\| = 1$. In view of (K) and (F2), we can choose $r_2 \in (0, 1)$ such that

$$|K(x)|t|^{2^*-2}t + f(x,t)| \leq \frac{1}{4\gamma_2^2}|t|, \quad \forall |t| \leq r_2, \tag{4.2}$$

where γ_2 is given by (2.16). Hence, from (2.16), (4.2) and the Hölder inequality,

$$\begin{aligned}\frac{1}{\|u_n\|}\int_{|u_n|\leq r_2} |[K(x)|u_n|^{2^*-2}u_n + f(x,u_n)]v_n^+|\,dx &\leq \frac{1}{4\gamma_2^2\|u_n\|}\int_{|u_n|\leq r_2} |u_n||v_n^+|\,dx \\ &\leq \frac{1}{4\gamma_2^2\|u_n\|}\|u_n\|_2\|v_n^+\|_2 \leq \frac{1}{4}. \end{aligned}\tag{4.3}$$

From (2.20), (2.21), (4.1) and (F3),

$$\begin{aligned} c + o(1) &= \int_{\mathbb{R}^N} \Big[\frac{1}{N}K(x)|u_n|^{2^*} + \frac{1}{2}f(x,u_n)u_n - F(x,u_n)\Big]\,dx \\ &\geq \frac{1}{N}\int_{|u_n|\geq r_2} K(x)|u_n|^{2^*}\,dx. \end{aligned}\tag{4.4}$$

By (F1), (4.4) and the Hölder inequality,

$$\begin{aligned} &\frac{1}{\|u_n\|}\int_{|u_n|\geq r_2} |[K(x)|u_n|^{2^*-2}u_n + f(x,u_n)]v_n^+|\,dx \\ &\quad\leq \frac{C_1}{\|u_n\|}\int_{|u_n|\geq r_2} K(x)|u_n|^{2^*-1}|v_n^+|\,dx \\ &\quad\leq \frac{C_2}{\|u_n\|}\|v_n^+\|_{2^*}\Big(\int_{|u_n|\geq r_2} K(x)|u_n|^{2^*}\,dx\Big)^{(2^*-1)/2^*} = o(1). \end{aligned}\tag{4.5}$$

Combining (4.3) with (4.5) and using (2.20), (2.21) and (4.1),

$$\begin{aligned} \frac{1}{2} + o(1) &= \frac{\|u_n^+\|^2 - \frac{1}{2}(\|u_n^+\|^2 - \|u_n^-\|^2) + \Phi(u_n) - \langle \Phi'(u_n), u_n^+\rangle}{\|u_n\|^2} \\ &\leq \frac{\|u_n^+\|^2 - \langle \Phi'(u_n), u_n^+\rangle}{\|u_n\|^2} = \frac{1}{\|u_n\|}\int_{\mathbb{R}^N}[K(x)|u_n|^{2^*-2}u_n + f(x,u_n)]v_n^+\,dx \\ &\leq \frac{1}{\|u_n\|}\int_{|u_n|<r_2} |[K(x)|u_n|^{2^*-2}u_n + f(x,u_n)]v_n^+|\,dx \\ &\qquad + \frac{1}{\|u_n\|}\int_{|u_n|\geq r_2} |[K(x)|u_n|^{2^*-2}u_n + f(x,u_n)]v_n^+|\,dx \\ &\leq \frac{1}{4} + o(1), \end{aligned}$$

which is a contradiction. Thus, the sequence $\{u_n\}$ is bounded in E. □

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Proof of Theorem 1.1. Applying Lemmas 3.2, 4.1 and 4.2, we deduce that there exists a bounded sequence $\{u_n\} \subset E$ satisfying

$$\Phi(u_n) \to c_* \in (0, S^{N/2}/NK_0^{N/(N-2)}), \quad \|\Phi'(u_n)\|(1 + \|u_n\|) \to 0. \tag{4.6}$$

Passing to a subsequence, we have $u_n \rightharpoonup \bar{u}$ in E and so $u_n \to \bar{u}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$, and $u_n \to \bar{u}$ almost everywhere on $\mathbb{R}^N$. Next, we prove that $\bar{u} \neq 0$.

Arguing by contradiction, suppose that $\bar{u} = 0$, that is, $u_n \rightharpoonup 0$ in E and so $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^N)$, $2 \le s < 2^*$, and $u_n \to 0$ almost everywhere on $\mathbb{R}^N$. Hence, by virtue of (V0), (V2) and (F4),

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2)\,dx = \int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N \setminus \mathcal{V}_b} V(x)u_n^2\,dx + o(1) \tag{4.7}$$

and

$$\int_{\mathbb{R}^N} f(x,u_n)u_n\,dx = o(1), \quad \int_{\mathbb{R}^N} F(x,u_n)\,dx = o(1). \tag{4.8}$$

From (2.20), (2.21), (4.6) and (4.8),

$$\begin{aligned}\frac{1}{N}\int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx &= \Phi(u_n) - \frac{1}{2}\langle \Phi'(u_n), u_n\rangle - \int_{\mathbb{R}^N}\Big[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\Big]\,dx \\ &= c_* + o(1), \end{aligned}\tag{4.9}$$

which, together with (2.21), (4.6) and (4.8), yields that

$$\begin{aligned}\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2)\,dx &= \langle \Phi'(u_n), u_n\rangle + \int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx \\ &\quad + \int_{\mathbb{R}^N} f(x,u_n)u_n\,dx = Nc_* + o(1). \end{aligned}\tag{4.10}$$

By virtue of (K), (3.2), (4.7), (4.9) and (4.10),

$$\begin{aligned} Nc_* &= \int_{\mathbb{R}^N} K(x)|u_n|^{2^*}\,dx + o(1) \le K_0\|u_n\|_{2^*}^{2^*} + o(1) \\ &\le K_0 S^{-N/(N-2)}\|\nabla u_n\|_2^{2^*} + o(1) \\ &\le K_0 S^{-N/(N-2)}\Big(\int_{\mathbb{R}^N} |\nabla u_n|^2\,dx + \int_{\mathbb{R}^N\setminus\mathcal{V}_b} V(x)u_n^2\,dx\Big)^{N/(N-2)} + o(1) \\ &= K_0 S^{-N/(N-2)}\Big[\int_{\mathbb{R}^N} (|\nabla u_n|^2 + V(x)u_n^2)\,dx\Big]^{N/(N-2)} + o(1) \\ &= K_0 S^{-N/(N-2)}(Nc_*)^{N/(N-2)} + o(1). \end{aligned}$$

Consequently, $c_* \ge S^{N/2}/NK_0^{N/(N-2)}$ and we deduce a contradiction. Thus, $\bar{u} \neq 0$. By a standard argument, it is easy to see that $\bar{u}$ is a nontrivial solution of (1.3). □

Theorem 1.2 can be proved in the same way as Theorem 1.1 by using Lemma 3.3 instead of Lemma 3.2.

In the proof of Theorem 1.1, if (V2) is replaced by (V1), the measure of the set $\mathcal{V}_b$ may be infinite, but Lemma 2.1 implies that $\int_{\mathcal{V}_b} V(x)u_n^2\,dx = o(1)$ is still true and thus (4.7) still holds. In addition, (4.8) also holds because of (F1), (F2) and Lemma 2.1. Hence, Theorems 1.3 and 1.4 can be proved in the same way as Theorem 1.1.

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