Bull. Aust. Math. Soc. **91** (2015), 273–277
doi:10.1017/S0004972714001075

CONTINUOUS ON RAYS SOLUTIONS OF A GOŁĄB–SCHINZEL TYPE EQUATION

JACEK CHUDZIAK

(Received 11 October 2014; accepted 31 October 2014; first published online 15 December 2014)

Abstract

We show that if the pair (f, g) of functions mapping a linear space X over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ into $\mathbb{K}$ satisfies the composite equation

$$f(x + g(x)y) = f(x)f(y) \quad \text{for } x, y \in X$$

and f is nonconstant, then the continuity on rays of f implies the same property for g. Applying this result, we determine the solutions of the equation.

2010 *Mathematics subject classification*: primary 39B12; secondary 39B22, 39B32.

Keywords and phrases: Gołąb–Schinzel equation, continuity on rays, algebraically interior point, exponential function, multiplicative function.

1. Introduction

The functional equation

$$f(x + g(x)y) = f(x)f(y) \tag{1.1}$$

is a generalisation of the Gołąb–Schinzel equation

$$f(x + f(x)y) = f(x)f(y).$$

Equations of this type belong to the most intensively studied composite functional equations. They have their roots in the problems related to determination of substructures of various algebraic structures. Such equations are also closely connected with some problems in mathematical meteorology and fluid mechanics, for example evaporation of cloud droplets and water discharging from a reservoir (see [9]). More details concerning these Gołąb–Schinzel type functional equations and their applications can be found in [1] and in a survey paper [3].

In the recent papers [7] and [8], Jabłońska considered the solutions of (1.1) in the real case under the assumption that f is continuous and continuous on rays,

This work was partially supported by the Centre for Innovation and Transfer of Natural Sciences and Engineering Knowledge.

respectively. The main idea of the considerations presented in [7] and [8] is to prove that if the pair (f, g) satisfies (1.1) and f is nonconstant then the continuity or continuity on rays of f implies the same property for g. Then, in order to determine the solutions of (1.1), it is enough to apply the results of [4] and [6], respectively.

The aim of this paper is to present a significantly shorter and simpler proof of the main results in [7] and [8] which works also in the complex case.

2. Results

The crucial new idea in our considerations is the nontrivial fact that, in some classes of functions, the condition

$$g(x + g(x)y) = 0 \quad \text{if and only if} \quad g(x)g(y) = 0 \tag{2.1}$$

is equivalent to the Gołąb–Schinzel equation. It seems that the approach suggested in this section could be applied in several problems arising from Gołąb–Schinzel type equations.

In order to formulate the result precisely, we need to recall some notions. Let X be a linear space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A function $f : X \to \mathbb{K}$ is said to be *continuous on rays* provided that, for every $x \in X\setminus\{0\}$, the function $f_x : \mathbb{K} \to \mathbb{K}$ given by $f_x(t) = f(tx)$, for $t \in \mathbb{K}$, is continuous. Furthermore, given a nonempty subset A of X, a point $a \in A$ is said to be *an algebraically interior point of* A provided that, for every $x \in X\setminus\{0\}$, there is an $r_x > 0$ such that $\{a + tx : t \in \mathbb{K}, |t| < r_x\} \subset A$. By $\mathrm{int}_a A$, we will denote the set (possibly empty) of all algebraically interior points of A.

THEOREM 2.1. *Assume that X is a linear space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $g : X \to \mathbb{K}$, $0 \in g(X)$ and $\mathrm{int}_a\{x \in X \mid g(x) \neq 0\} \neq \emptyset$. Then the following three conditions are equivalent:*

(i) *(2.1) holds for every $x, y \in X$;*

(ii) *$g(x + g(x)y) = g(x)g(y)$ for $x, y \in X$;*

(iii) *there exists a nontrivial $\mathbb{K}$-linear functional $L : X \to \mathbb{K}$ such that $g(x) = L(x) + 1$ for $x \in X$, or there exists a nontrivial $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ such that $g(x) = \max\{L(x) + 1, 0\}$ for $x \in X$.*

The equivalence of (i) and (ii) has been proved in [5, Theorem 1] and the equivalence of (ii) and (iii) has been established in [2, Theorem 3].

The next theorem is the main result of the paper.

THEOREM 2.2. *Let X be a linear space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $f, g : X \to \mathbb{K}$. Assume that f is nonconstant and that the pair (f, g) satisfies the equation*

$$f(x + g(x)y) = f(x)f(y) \quad \textit{for } x, y \in X. \tag{2.2}$$

If f is continuous on rays, then so is g.

Proof. Assume that f is continuous on rays and put $F := \{x \in X \mid f(x) = 0\}$ and $G := \{x \in X \mid g(x) = 0\}$. Then

$$F = G. \tag{2.3}$$

In fact, if $x \in G$, then, in view of (2.2), $f(x) = f(x)f(y)$ for $y \in X$. As f is nonconstant, this means that $x \in F$. Conversely, if $x \in F$, then, making use of (2.2), we get $f(x + g(x)y) = 0$ for $y \in X$. Hence, $x \in G$, because otherwise f would be constant.

First consider the case where $G \neq \emptyset$. Then, taking into account (2.2) and (2.3), for every $x, y \in X$,

$$g(x + g(x)y) = 0 \Leftrightarrow f(x + g(x)y) = 0 \Leftrightarrow f(x)f(y) = 0 \Leftrightarrow g(x)g(y) = 0.$$

Therefore, for every $x, y \in X$, (2.1) holds. Furthermore, applying (2.2) with $x = y = 0$, we get $f(0) \in \{0, 1\}$. If $f(0) = 0$, then, substituting $y = 0$ in (2.2), we obtain $f = 0$, which yields a contradiction. Thus, $f(0) = 1$ and so $f_x(0) = 1$ for $x \in X$. Since f is continuous on rays, this means that $0 \in \mathrm{int}_a\{x \in X \mid f(x) \neq 0\}$. Hence, in view of (2.3), $\mathrm{int}_a\{x \in X \mid g(x) \neq 0\} \neq \emptyset$. Consequently, as $G \neq \emptyset$, by applying Theorem 2.1, we conclude that (iii) holds. Thus, g is continuous on rays.

Now assume that $G = \emptyset$. Then, in view of (2.3), $F = \emptyset$. Let $X_1 := \{x \in X \mid f_x = 1\}$. Since f is nonconstant and $f_x(0) = 1$ for $x \in X$, we have $X \backslash X_1 \neq \emptyset$. We claim that

$$g_x = 1 \quad \text{for } x \in X \backslash X_1. \tag{2.4}$$

To this end, fix an $x \in X \backslash X_1$. Since f_x is continuous, there exists a nonempty open set $U_x \subset \mathbb{K}$ such that $f(tx) = f_x(t) \neq 1$ for $t \in U_x$. Suppose that $g(t_0x) \neq 1$ for some $t_0 \in U_x$. Then, as $F = \emptyset$, applying (2.2) with x and y replaced by t_0x and $t_0x/(1 - g(t_0x))$, respectively, we get $f(t_0x) = 1$, which yields a contradiction. Therefore, $g(tx) = 1$ for $t \in U_x$, which together with (2.2) gives

$$f_x(t + s) = f(tx + sx) = f(tx + g(tx)sx) = f(tx)f(sx) = f_x(t)f_x(s) \quad \text{for } s \in \mathbb{K},\ t \in U_x.$$

Hence, as f_x maps $\mathbb{K}$ into $\mathbb{K}\backslash\{0\}$, according to [10, Lemma 18.5.1],

$$f_x(t + s) = f_x(t)f_x(s) \quad \text{for } s \in \mathbb{K},\ t \in [U_x],$$

where $[U_x]$ denotes the subgroup of the additive group of $\mathbb{K}$ generated by U_x. However, as U_x is open, we have $[U_x] = \mathbb{K}$. Thus, $f_x(t + s) = f_x(t)f_x(s)$ for $s, t \in \mathbb{K}$ and so f_x is the exponential function. Therefore, since $F = \emptyset$, taking into account (2.2), for every $s, t \in \mathbb{K}$,

$$f_x((g_x(t) - 1)s) = \frac{f_x(g_x(t)s)}{f_x(s)} = \frac{f_x(t)f_x(g_x(t)s)}{f_x(t)f_x(s)} = \frac{f_x(t + g_x(t)s)}{f_x(t)f_x(s)} = \frac{f(tx + g(tx)sx)}{f(tx)f(sx)} = 1.$$

Thus, $g_x(t) = 1$ for $t \in \mathbb{K}$, because $x \in X \backslash X_1$. In this way, we have proved (2.4).

Now note that f is exponential, that is, for every $x, y \in X$,

$$f(x + y) = f(x)f(y). \tag{2.5}$$

In fact, fix $x, y \in X$. If $x \in X \setminus X_1$, then (2.5) follows from (2.2) and (2.4). If $x \in X_1$, then, applying (2.2),

$$f(x+y) = f\left(y + g(y)\frac{x}{g(y)}\right) = f(y) f\left(\frac{x}{g(y)}\right) = f(y) f_x\left(\frac{1}{g(y)}\right) = f(y) = f(x) f(y).$$

Since f is exponential and $F = \emptyset$, taking into account (2.2), for every $x, y \in X$,

$$f((g(x)-1)y) = \frac{f(g(x)y)}{f(y)} = \frac{f(x)f(g(x)y)}{f(x)f(y)} = \frac{f(x+g(x)y)}{f(x)f(y)} = 1.$$

As f is nonconstant, this means that $g = 1$. Therefore, g is continuous on rays. □

We conclude the paper with the following result describing the solutions of (2.2).

Proposition 2.3. *Assume that X is a linear space over the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, $f : X \to \mathbb{K}$ is nonconstant and continuous on rays and $g : X \to \mathbb{K}$. Then the pair (f, g) satisfies (2.2) if and only if one of the following possibilities holds:*

(a) *$g = 1$ and f is exponential;*

(b) *there exist a nontrivial $\mathbb{K}$-linear functional $L : X \to \mathbb{K}$ and a nonconstant continuous multiplicative function $m : \mathbb{K} \to \mathbb{K}$ such that*

$$g(x) = L(x) + 1 \quad \text{for } x \in X \tag{2.6}$$

and

$$f(x) = m(L(x) + 1) \quad \text{for } x \in X; \tag{2.7}$$

(c) *there exist a nontrivial $\mathbb{R}$-linear functional $L : X \to \mathbb{R}$ and a nonconstant continuous multiplicative function $m : [0, \infty) \to \mathbb{K}$ such that, for $x \in X$, $g(x) = \max\{L(x) + 1, 0\}$ and*

$$f(x) = \begin{cases} m(L(x)+1) & \text{whenever } L(x) + 1 \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.8}$$

Proof. In view of Theorem 2.2 and [6, Proposition 3], in order to complete the proof, it is enough to prove the continuity of m. To this end note that in the case of (2.7), we have $m(t) = f((t-1)x/L(x)) = f_x((t-1)/L(x))$ for $t \in \mathbb{K}$, where $x \in X \setminus \ker L$ is fixed; and, in the case of (2.8), we get $m(t) = f_x((t-1)/L(x))$ for $t \in [0, \infty)$, where $x \in X \setminus \ker L$ is fixed. Since f is continuous on rays, in both cases we obtain that m is continuous. □

Remark 2.4. According to [10, Theorem 13.1.6], a nonconstant continuous function $m : \mathbb{R} \to \mathbb{R}$ is multiplicative if and only if there is an $\alpha \in (0, \infty)$ such that either $m(t) = |t|^\alpha$ for $t \in \mathbb{R}$, or $m(t) = |t|^\alpha \operatorname{sgn} t$ for $t \in \mathbb{R}$. A nonconstant continuous function $m : [0, \infty) \to \mathbb{R}$ is multiplicative if and only if there exists an $\alpha \in (0, \infty)$ such that $m(t) = t^\alpha$ for $t \in [0, \infty)$.

Continuous multiplicative complex-valued functions are closely related to the characters. From [11, Section 3.2], it can be shown that a nonconstant continuous function $m : [0, \infty) \to \mathbb{C}$ is multiplicative if and only if there exist an $\alpha \in (0, \infty)$ and a $\beta \in \mathbb{R}$ such that

$$m(t) = \begin{cases} 0 & \text{for } t = 0, \\ t^{\alpha} e^{i\beta \ln t} & \text{for } t \in (0, \infty). \end{cases}$$

Furthermore, a nonconstant continuous function $m : \mathbb{C} \to \mathbb{C}$ is multiplicative if and only if $m(0) = 0$ and

$$m(re^{i\phi}) = r^{\alpha} e^{i(\beta \ln r + n\phi)} \quad \text{for } r \in (0, \infty),\ \phi \in \mathbb{R}$$

with some $\alpha \in (0, \infty)$, $\beta \in \mathbb{R}$ and $n \in \mathbb{Z}$.

References

[1] J. Aczél and J. Dhombres, *Functional Equations in Several Variables*, Encyclopedia of Mathematics and its Applications, 31 (Cambridge University Press, Cambridge, 1989).

[2] J. Brzdęk, ‘Subgroups of the group Z_n and a generalization of the Gołąb–Schinzel functional equation’, *Aequationes Math.* **43** (1992), 59–71.

[3] J. Brzdęk, ‘The Gołąb–Schinzel equation and its generalizations’, *Aequationes Math.* **70** (2005), 14–24.

[4] J. Chudziak, ‘Semigroup-valued solutions of the Gołąb–Schinzel type functional equation’, *Abh. Math. Semin. Univ. Hambg.* **76** (2006), 91–98.

[5] J. Chudziak, ‘Stability problem for the Gołąb–Schinzel type functional equations’, *J. Math. Anal. Appl.* **339** (2008), 454–460.

[6] J. Chudziak, ‘Semigroup-valued solutions of the Gołąb–Schinzel type functional equation’, *Aequationes Math.* **88** (2014), 183–198.

[7] E. Jabłońska, ‘On continuous solutions of an equation of the Gołąb–Schinzel type’, *Bull. Aust. Math. Soc.* **87** (2013), 10–17.

[8] E. Jabłońska, ‘On continuous on rays solutions of a composite-type equation’, *Aequationes Math.*, doi:10.1007/s00010-013-0243-5.

[9] P. Kahlig and J. Matkowski, ‘A modified Gołąb–Schinzel equation on restricted domain (with applications to meteorology and fluid mechanics)’, *Österreich. Akad. Wiss. Math.-Natur. Kl. Sitzungsber. II* **211** (2002), 117–136.

[10] M. Kuczma, *An Introduction to the Theory of Functional Equations and Inequalities*, 2nd edn (ed. A. Gilányi) (Birkhäuser, Berlin, 2009).

[11] H. Stetkaer, *Functional Equations on Groups* (World Scientific, Hackensack, NJ, 2013).

JACEK CHUDZIAK, Faculty of Mathematics and Natural Sciences,
University of Rzeszów, Prof. St. Pigonia 1, 35-310 Rzeszów, Poland
e-mail: chudziak@ur.edu.pl

https://doi.org/10.1017/S0004972714001075 Published online by Cambridge University Press