Ergod. Th. & Dynam. Sys. (1990), **10**, 247–262
Printed in Great Britain

Marker automorphisms of the one-sided *d*-shift

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(*Received* 18 *April* 1988)

Abstract. We identify a set of generators for the automorphism group of the one-sided d-shift. For the 3-shift, this set of generators has an application to the dynamics of cubic polynomials.

1. *Introduction*

The *one-sided d-shift*, X_d, is defined to be the set

$$X_d = \prod_{i=0}^{\infty} \{0, 1, \ldots, d-1\},$$

with the topology given by the product of the discrete topologies on the coordinate spaces. The shift map $\sigma: X_d \to X_d$ defined by

$$(\sigma(x))_i = x_{i+1}$$

is a continuous d-to-1 map. In this paper we study the group of homeomorphisms $\psi: X_d \to X_d$ that commute with the shift σ. We denote this group by aut (X_3, σ); it is the group of automorphisms of the dynamical system (X_d, σ).

The system (X_d, σ) is *isomorphic* to the system (J_p, p), where p is a degree d complex polynomial all of whose critical points escape to infinity and J_p is the Julia set of p [**B**]. For (X_d, σ) and (J_p, p) to be *isomorphic* or *conjugate* as dynamical systems means that there is a homeomorphism $\psi: J_p \to X_d$ with $\psi \circ p = \sigma \circ \psi$.

Blanchard et al. [**BDK**] have constructed automorphisms of (J_p, p) where p is a cubic polynomial all of whose critical points escape to infinity. These automorphisms are given by traversing loops in a parameter space for cubic polynomials. In conversation with me, Linda Keen and Robert Devaney posed the question: Does this construction give all of the automorphisms of (J_p, p)? The answer is yes. We prove this by identifying the automorphisms of $(J_p, p) \cong (X_3, \sigma)$ arising from their construction as those given by a simple combinatorial algorithm; and, using the algorithm, prove that these automorphisms generate the automorphism group of (X_3, σ), aut (X_3, σ).

In § 2, we state a result of Boyle et al. [**BFK**] giving a certain set of generators for aut (X_d, σ), called *marker* automorphisms.

In §§ 3 and 4 we show a way to factor a marker automorphism into a composition of *minimal marker* automorphisms; these are the automorphisms arising from the construction of Blanchard, Devaney and Keen (as shown in § 6).

In § 5 we present a simplified algorithm for constructing minimal marker automorphisms.

In § 6 we show that the minimal marker automorphisms of (X_3, σ) are exactly the automorphisms constructed by Blanchard, Devaney and Keen.

2. *Marker automorphisms and state splitting*

Let $G_{st} \subseteq \text{aut}(X_d, \sigma)$ be that subgroup of automorphisms g such that

$$g(x)_i = x_i \quad \text{if} \quad x_i \neq s \quad \text{and} \quad x_i \neq t,\ x \in X_d.$$

Thus $g \in G_{st}$ fixes all symbols except perhaps s and t. In [**BFK**], Boyle, Franks and Kitchens show that $\{G_{st}: 0 \leq s, t \leq d-1\}$ generate aut (X_d, σ) and that G_{st} is generated by *marker automorphisms*. To describe the construction of marker automorphisms we must first explain the state splitting algorithm.

State splitting

Let G_0 be the directed graph with one state, ε, and d directed edges $e_0, \ldots, e_{d-1}$ from state ε to itself. Edge e_i is labeled with symbol i; we denote the labeling function as L_{G_0}, or L if no confusion is possible. Thus $L(e_i) = i$. The system (X_d, σ) is obviously conjugate to the symbolic system (Σ_{G_0}, σ), where for a directed graph G, we define

$$\Sigma_G = \{e_{i_0} e_{i_1} e_{i_2} \ldots : \text{edge } e_{i_{j+1}} \text{ follows } e_{i_j} \text{ in } G\}.$$

The conjugacy is given by extending the map L to Σ_{G_0} by setting

$$L(e_{i_0} e_{i_1} \cdots) = L(e_{i_0}) L(e_{i_1}) \cdots = i_0 i_1 \cdots.$$

We say that a labeled graph G *presents* X_d if $L_G : \Sigma_G \to X_d$ is a conjugacy from (Σ_G, σ) to (X_d, σ).

Given any labeled directed graph G presenting X_d we may define a new graph G' as follows. Denote by $\mathcal{F}(S)$ the set of edges in G whose initial state is state S. For each state S_i in G, choose a partition $\{S_i^{(1)}, \ldots, S_i^{(r_i)}\}$ of the set of edges $\mathcal{F}(S_i)$. The states of G' are defined to be $\{S_i^{(j)}: S_i \text{ is a state of } G \text{ and } 1 \leq j \leq r_i\}$. For each edge $e \in S_i^{(j)}$ whose terminal state is S_k, G' has r_k edges:

$$S_i^{(j)} \xrightarrow{a} S_k^{(l)}, \quad 1 \leq l \leq r_k$$

each labeled with $a = L_G(e)$. The graph G' is said to be obtained from G by *one round of (forward) state splitting*. We show in Corollary (2.2) that G' presents X_d.

Denote the set $\{x \in X_d : x_0 = b_0, \ldots, x_{k-1} = b_{k-1}\}$ by the string $b = b_0 b_1 \ldots b_{k-1}$. The set b is called a *k-block*.

LEMMA 2.1. *Let G be a graph obtained from G_0 by a finite number $k \geq 0$ of rounds of state splitting. The states $\mathcal{S}$ of G partition X_d where we identify state $S \in \mathcal{S}$ with the set*

$$S = \{x \in X_d : L(p) = x \quad \text{where } p \text{ is a path in } G \text{ starting at state } S\}.$$

The edges $\mathcal{E}$ of G also partition X_d, where we identify $e = \mathcal{E}$ with the set

$$e = \{x \in X_d : L(p) = x \quad \text{where } p \text{ is a path in } G \text{ with } p_0 = e\}.$$

Moreover, the states of G are unions of k-blocks and the edges of G are unions of

$(k+1)$-blocks. Each state of G has exactly d incoming edges, labeled distinctly from the set $\{0, 1, \ldots, d-1\}$.

Proof. The proof is an induction on k. When $k=0$, $G=G_0$. The edge e_i of G_0 is identified with the 1-block i of X_d. Thus the edges

$$\mathscr{E}_0=\{e_0, \ldots, e_{d-1}\}=\{0, \ldots, d-1\}$$

partition the space X_d into 1-blocks. The single state of G_0 is identified with the set X_d; in block notation, the single state of G_0 is the 0-block denoted by the empty word ε: no coordinates are specified.

Let $\mathscr{S}$ and $\mathscr{E}$ be the set of states and of edges of G. The state splitting rule says exactly that if G' is obtained from G by a round of splitting then

$$\mathscr{S} \leq \mathscr{S}' \leq \mathscr{E}$$

and

$$\mathscr{E}' = \mathscr{E}_0 \vee \sigma^{-1}\mathscr{S}'$$

where $\mathscr{S}'$ and $\mathscr{E}'$ are the states and edges of G' and

$$\sigma^{-1}\mathscr{S}'=\{\sigma^{-1}S'_1, \ldots, \sigma^{-1}S'_n\}$$

where

$$\mathscr{S}'=\{S'_1, \ldots, S'_n\}.$$

Thus if each $S \in \mathscr{S}$ is a union of k-blocks and each $e \in \mathscr{E}$ is a union of $(k+1)$-blocks, then each $S' \in \mathscr{S}'$, being a union of elements of $\mathscr{E}$, is a union of $(k+1)$-blocks and each $e' \in \mathscr{E}'$ is a union of $(k+2)$-blocks. Now for each edge $e' \in \mathscr{E}'$, we have $\sigma(e')=S'$, where S' is the terminal state of e' in graph G'. Thus, the incoming edges of state S' in $\mathscr{S}'$ are $0S', 1S', \ldots, (d-1)S'$. □

COROLLARY 2.2. *If graph G is as in Lemma* (2.1), *the map $L: \Sigma_G \to X_d$ is a conjugacy. Thus G presents X_d.*

Proof. If $x \in X_d$, then the unique path p in G labeled by x is given by $p_0p_1\ldots$, where $p_0 \supseteq x_0x_1\ldots x_k$, $p_1 \supseteq x_1x_2\ldots x_{k+1}$, etc. Thus $L^{-1}: X_d \to \Sigma_G$ is given by the $(k+1)$-*block map* $L^{-1}(x_ix_{i+1}\ldots x_{i+k})=e$ where e is that edge in graph G with $e \supseteq x_i \ldots x_{i+k}$. □

Marker automorphisms

Define, after Nasu [N], a *simple* automorphism of X_d to be an automorphism φ of the form

$$\varphi = L \circ \psi \circ L^{-1},$$

where $L: \Sigma_G \to X_d$ is the label conjugacy for some graph G obtained from G_0 by state splitting and ψ is an automorphism of Σ_G given by switching two fixed edges e_{i_0} and e_{i_1} in graph G, where e_{i_0} and e_{i_1} are *parallel* edges: they have a common initial state P and a common final state M.

In terms of X_d, φ is a *marker* automorphism: it acts on $x \in X_d$ only where a marker occurs in x as follows. If $L(e_{i_0})=a$ and $L(e_{i_1})=b$, then φ switches symbol

a with symbol b wherever a or b is followed by a k-block $c \subseteq M$ (recall that states of G are unions of k-blocks in X_d). To emphasize the marker M we denote φ by φ_M.

It follows from a more general result in [**BFK**] that

THEOREM 2.3. *The simple automorphisms generate* aut (X_d).

We concentrate on describing markers for automorphisms of X_d switching symbols 1 and 2 for definiteness. Marker automorphisms switching other symbols are conjugate to these.

Definition 2.4. A *k-block marker M* is a union of k-blocks that occurs as a union of states, each with parallel incoming edges labeled 1 and 2, occurring in a graph G obtained from G_0 by state splitting. We say that graph G *presents* marker M.

Observation 2.5. If a graph G simultaneously presents markers $M_1, \ldots, M_r$, then the automorphisms $\varphi_{M_1}, \ldots, \varphi_{M_r}$ pair-wise commute. If in addition the M_i are disjoint sets, then the product of $\varphi_{M_1}, \ldots, \varphi_{M_r}$ is given by the marker automorphism with marker $\bigcup_i M_i$.

Proof. The automorphism φ_{M_i} is given by $L \circ \psi_i \circ L^{-1}$, where ψ_i is the automorphism of Σ_G given by switching certain pairs of edges of graph G. The ψ_i pair-wise commute because each leaves any pair of edges switched by another set-wise fixed. If the M_i are disjoint, no two of the ψ_i switch the same pair of edges. Therefore, the set of pairs of edges switched by the product of all ψ_i, $1 \le i \le r$, is the union over $1 \le i \le r$ of the set of pairs switched by ψ_i. □

In fact, a converse to (2.5) is true, but we do not use it.

We now show that any k-block marker M is presented by a graph G that is obtained from G_0 *by k rounds* of state splitting. First we must characterize those partitions $\mathcal{S}$ of X_d obtained by state splitting the graph G_0.

If $a \subseteq X_d$ and $\mathcal{S}$ is a partition of X_d, we denote by $\mathcal{S}|a$ the induced partition of the set a.

LEMMA 2.6. *Let $\mathcal{S}$ be a partition of X_d coarser than the partition of X_d into all k-blocks. Then $\mathcal{S}$ is given by state splitting G_0 iff*

$$\mathcal{S} \ge \sigma(\mathcal{S}|a) \quad \textit{for all } 1\textit{-blocks } a \subseteq X_d.$$

Moreover, if the condition is satisfied, $\mathcal{S}$ is obtained from G_0 by k rounds of splitting.

Proof. We prove ($\Leftarrow$). The other direction is an easy consequence of Lemma 2.1. If a is a k-block, denote $|a| = k$. Now

$$\mathcal{S} \ge \bigvee_{|a|=1} \sigma(\mathcal{S}|a)$$

so

$$\begin{aligned}\bigvee_{|b|=l} \sigma^l(\mathcal{S}|b) &\ge \bigvee_{|b|=l} \sigma^l\left(\left[\bigvee_{|a|=1} \sigma(\mathcal{S}|a)\right] \middle| b\right)\\ &= \bigvee_{|b|=l} \bigvee_{|a|=1} \sigma^l([\sigma(\mathcal{S}|a)]|b)\\ &= \bigvee_{|b|=l} \bigvee_{|a|=1} \sigma^{l+1}(\mathcal{S}|ab)\\ &= \bigvee_{|c|=l+1} \sigma^{l+1}(\mathcal{S}|c), \quad l \ge 1.\end{aligned}$$

If we denote

$$\mathcal{S}_l = \bigvee_{|a|=k-l} \sigma^{k-l}(\mathcal{S}|a) \quad \text{and} \quad \mathcal{S}_k = \mathcal{S},$$

we have

$$\mathcal{S} = \mathcal{S}_k \geq \mathcal{S}_{k-1} \geq \cdots \geq \mathcal{S}_1 \geq \mathcal{S}_0 = \{X_d\}.$$

Let

$$\mathcal{E}_l = \mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_l, \quad 0 \leq l \leq k,$$

where $\mathcal{E}_0 = \{0, 1, \ldots, d-1\}$. We claim that $\mathcal{S}_{l+1}$ and $\mathcal{E}_{l+1}$ are the states and edges of a graph G_{l+1} obtained by one round of state splitting from a graph G_l, $0 \leq l \leq k-1$. It only remains to show

$$\mathcal{S}_l \leq \mathcal{E}_{l-1} = \mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_{l-1} = \mathcal{E}_0 \vee \sigma^{-1} \bigvee_{|b|=k-l+1} \sigma^{k-l+1}(\mathcal{S}|b).$$

If e and e' are in the same atom of the right-hand-side, then $e = ad$ and $e' = ad'$, where $|a| = 1$, and where for all $(k-l+1)$-blocks b, bd and bd' are in the same atom of $\mathcal{S}$. In particular, for all $(k-l)$-blocks c, cad and cad' are in the same atom of $\mathcal{S}$. Hence $e = ad$ and $e' = ad'$ are in the same atom of S_l. □

We denote the complement of a subset M of X_d by M^c.

LEMMA 2.7. *Let M be a finite union of k-blocks. The partition*

$$\mathcal{S} = \bigvee_{\substack{\{b:\, b \text{ is a} \\ \text{block, and} \\ 0 \leq |b| \leq k\}}} \{\sigma^{|b|}(M \cap b), \sigma^{|b|}(M^c \cap b)\}$$

is the unique coarsest partition having M as a union of atoms among all partitions of X_d obtained by rounds of state splitting from graph G_0. Moreover, the partition $\mathcal{S}$ can be obtained by k rounds of splitting from graph G_0.

Proof. We first show that $\mathcal{S}$ is a partition obtained by state splitting G_0. If a is a 1-block then

$$\begin{aligned}\sigma(\mathcal{S}|a) &= \bigvee_{\{b:\, 0 \leq |b| \leq k\}} \{\sigma^{|ba|}(M \cap ba), \sigma^{|ba|}(M^c \cap ba)\} \\ &= \{\phi, X_d\} \vee \bigvee_{\{b:\, 0 \leq |b| \leq k-1\}} \{\sigma^{|ba|}(M \cap ba), \sigma^{|ba|}(M^c \cap ba)\} \\ &\leq \mathcal{S}.\end{aligned}$$

Now the elements of $\mathcal{S}$ are unions of k-blocks, so $\mathcal{S}$ is obtained by k rounds of state splitting from G_0 by Lemma 2.6. We now show $\mathcal{P} \geq \mathcal{S}$ for any partition $\mathcal{P}$ of X_d obtained by splitting G_0 having M as a union of atoms. For each $P \in \mathcal{P}$, either $P \subseteq M$ or $P \subseteq M^c$. Hence for each block b, either $\sigma^{|b|}(P \cap b) \subseteq \sigma^{|b|}(M \cap b)$ or $\sigma^{|b|}(P \cap b) \subseteq \sigma^{|b|}(M^c \cap b)$. Hence

$$\sigma^{|b|}(\mathcal{P}|b) \geq \{\sigma^{|b|}(M \cap b), \sigma^{|b|}(M^c \cap b)\}.$$

Because $\mathcal{P}$ is obtained by splitting, we have by Lemma 2.6 and an induction on the length of b that

$$\mathcal{P} \geq \sigma^{|b|}(\mathcal{P} \mid b).$$

Thus

$$\mathcal{P} \geq \bigvee_{|b|\geq 0} \{\sigma^{|b|}(M \cap b), \sigma^{|b|}(M^c \cap b)\} = \mathcal{S}. \qquad \square$$

We can now show

THEOREM 2.8. *Any k-block marker M is presented by a graph G obtained from G_0 by k rounds of splitting.*

Proof. Let G, by Lemma 2.7, be the graph obtained from G_0 by k rounds of splitting whose states $\mathcal{S}$ give the unique coarsest partition of X_d among all graphs obtained by splitting G_0 and having M as a union of states. We must show G presents M as a marker. Let G' be any graph with states $\mathcal{S}'$ that presents M as a marker. The states of G' are invariant under the automorphism φ_M, and since $\mathcal{S} \leq \mathcal{S}'$, the states $\mathcal{S}$ of G are invariant as well, In particular, if $S \in \mathcal{S}$ is such that $S \subseteq M$, then $1S$ and $2S$ are contained in the same state $P \in \mathcal{S}$. Thus there are parallel edges labeled 1 and 2 from state P to state S in G. Thus graph G presents the marker M. $\square$

3. *Minimal markers*

In this section we show that any k-block marker M can be partitioned into a union of k-block markers that are minimal with respect to inclusion among all k-block markers. These markers are defined by a particular kind of state splitting.

Notation. Denote the union of 1-blocks $1 \cup 2$ by $\bar{0}$.

Definition 3.1. Let M be a marker presented by a graph G and let $U \subseteq M$ be any subset of M. The *U-complete* round of state splitting of G is defined as follows: each state P of G is partitioned into states

$$\{a_1Q_1, \ldots, a_kQ_k, \bar{0}M_1, \ldots, \bar{0}M_l\},$$

where

(i) M_i is a marker state with $\bar{0}M_i \subseteq P$ and $M_i \cap U \neq \varnothing$,

(ii) Q_j is a state with 1-block $a_j \notin \{1, 2\}$ or $\bar{0}Q_j \nsubseteq P$ or $Q_j \cap U = \varnothing$.

The U-complete round of splitting gives the finest possible partition of the states of G subject to the constraint that the set U remain contained in a marker in G'.

Definition 3.2. Let M be a marker presented by a graph G and let $U \subseteq M$ be any subset of M. A round of splitting on G is *U-preserving* if for each state P of G, the partition of P given by the splitting is coarser than the partition of P given by the U-complete splitting.

A U-preserving splitting preserves U as a subset of a marker in G'.

Definition 3.3. Let a be a k-block in X_d. Define the marker m_a to be the state containing a in the graph G_k obtained from graph G_0 from k rounds of a-complete splitting.

LEMMA 3.4. *The set of k-blocks in X_d is partitioned by*

$$\{m_a\colon a \text{ is a } k\text{-block in } X_d\}.$$

Proof. Suppose b is a k-block with $b \subseteq m_a$. We show $m_b = m_a$ giving that if $m_a \cap m_b \neq \emptyset$, then $m_a = m_b$. Let G_j be the graph resulting from j rounds of a-complete splitting applied to G_0. The k-blocks a and b are contained in the same single state of G_j, $0 \le j \le k$, since this is true of G_k. Thus the k rounds of splitting leading to G_k are also b-complete. Thus $m_a = m_b$. □

LEMMA 3.5. *Let G be a graph presenting marker M and let $U \subseteq M$. If G_1 is the graph obtained from G by n rounds of U-complete splitting and G_2 is any graph obtained from G by n rounds of U-preserving state splitting, then the partition of X_d given by the states of G_1 refines the partition of X_d given by the states of G_2.*

Proof. An easy induction on the number of rounds of splitting. □

We can now prove the main theorem of this section.

THEOREM 3.6. *Any k-block marker M is partitioned by*

$$\{m_a\colon a \text{ is a } k\text{-block contained in } M\}.$$

Proof. By Theorem 2.8, M is presented by a graph G obtained from G_0 by k rounds of state splitting. For any k-block $a \subseteq M$, each of the k rounds of splitting is a-preserving (because it is M-preserving). By Lemma 3.5, the partition of X_d given by the graph G' obtained from G_0 by k rounds of a-complete splitting refines the partition of X_d given by the states of G. Thus the state m_a of G' is contained in that state S of G with $a \subseteq S$. Now $a \subseteq M$, so $S \subseteq M$, so $m_a \subseteq M$. Now apply Lemma 3.4 to conclude that M is partitioned by $\{m_a\colon a$ is a k-block contained in $M\}$. □

We may introduce a tree $\mathcal{T}$ of markers defined as follows:

(i) the root of $\mathcal{T}$ is the 0-block marker ε.

(ii) the children of a marker m of length k are the markers

$$\{ma\colon a \text{ is a } (k+1)\text{-block contained in } m\}$$

in the partition of m into $(k+1)$-block markers.

COROLLARY 3.7. *Given a k-block a, there is unique marker minimal with respect to inclusion among all k-block markers that contain a: namely, m_a.*

Proof. By Theorem 3.6 any k-block marker M containing the k-block a also contains m_a. □

4. *Factoring a marker automorphism*

We now show that aut (X_d, σ) is generated by

$$\{\varphi_m\colon m \text{ is a minimal marker}\}.$$

We do this by showing that any marker automorphism φ_M factors into minimal marker automorphisms and then apply Theorem 2.3.

THEOREM 4.1. *Let M be an n-block marker for the automorphism φ_M of X_d switching the symbols 1 and 2 in $x \in X_d$ when followed in x by any n-block in M.*

(1) *The automorphism φ_M can be factored into the automorphisms $\varphi_{\bar{m}_1}, \ldots, \varphi_{\bar{m}_t}$, where $\{\bar{m}_1, \ldots, \bar{m}_t\}$ is the partition of M into minimal markers of length n.*

(2) *The automorphism φ_M can be iteratively factored as follows:*

$$\varphi_{M\cap m} = \varphi_{M\cap m^{(1)}} \circ \varphi_{M\cap m^{(2)}} \circ \cdots \circ \varphi_{M\cap m^{(r)}}$$

where m is a minimal marker of length $k \geq 0$ and $\{m^{(1)}, \ldots, m^{(r)}\}$ is the partition of m into minimal markers of length $k+1$. Moreover, the factors

$$\varphi_{M\cap m^{(1)}}, \ldots, \varphi_{M\cap m^{(r)}}$$

pair-wise commute.

Proof. Statement (1) follows from statement (2) and an induction on the length k of m: observe that $M = M \cap \varepsilon$ and that statement (2) enables us to work our way down the tree $\mathcal{T}$ of minimal markers to factor $M \cap \varepsilon$ as claimed.

We prove statement (2). Let $\bar{m}_1$ be one of the n-block minimal markers $\{\bar{m}_1, \ldots, \bar{m}_t\}$ that partition M. Let

$$\varepsilon = m_0^{(1)} \supseteq m_1^{(1)} \supseteq \cdots \supseteq m_n^{(1)} = \bar{m}_1$$

be the sequence of minimal markers leading from the root of the tree $\mathcal{T}$ to the n-block marker $\bar{m}_1$. For $1 \leq k \leq n-1$, let

$$\{m_k^{(1)}, m_k^{(2)}, \ldots, m_k^{(r_k)}\}$$

be the partition of $m_{k-1}^{(1)}$ into minimal markers of length k.

For $0 \leq k \leq n$, let G_k be the graph obtained from G_0 by k rounds of $m_n^{(1)}$-complete splitting. Notice that $m_k^{(1)}, m_k^{(2)}, \ldots, m_k^{(r_k)}$ all occur as marker states in graph G_k. Thus G_{k+1} is obtained from G_k by one round of $m_k^{(1)}$-complete splitting.

Let G_0' be any graph obtained from G_0 by state splitting that presents marker M. For $0 \leq k \leq n-1$, inductively define G_{k+1}' as the graph obtained from G_k' by one round of $M \cap m_k^{(1)}$-complete splitting.

We have two sequences of graphs:

$$G_0 \xrightarrow{\varepsilon} G_1 \xrightarrow{m_1^{(1)}} G_2 \xrightarrow{m_2^{(1)}} \cdots \xrightarrow{m_{n-1}^{(1)}} G_n$$

$$G_0' \xrightarrow{M\cap\varepsilon} G_1' \xrightarrow{M\cap m_1^{(1)}} G_2' \xrightarrow{M\cap m_2^{(1)}} \cdots \xrightarrow{M\cap m_{n-1}^{(1)}} G_n'.$$

We will show by induction on k that the partition $\mathcal{S}_k'$ of X_d given by the states of G_k' refines the partition $\mathcal{S}_k$ of X_d given by the states of G_k. This is clear for $k=0$ because $\mathcal{S}_0 = \{\varepsilon\} = \{X_d\}$.

Now suppose $\mathcal{S}_k \leq \mathcal{S}_k'$. We have

$$\mathcal{S}_{k+1} \leq \mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_k$$

and

$$\mathcal{S}_{k+1}' \leq \mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_k'$$

by the proof of Lemma 2.1. Now $\mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_k \leq \mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_k'$ by the inductive hypothesis. We need only show that if two atoms e and f of $\mathcal{E}_0 \vee \sigma^{-1}\mathcal{S}_k'$ are contained in the same atom of $\mathcal{S}_{k+1}'$, then e and f are contained in the same atom of $\mathcal{S}_{k+1}$.

Assume e and f are such atoms of $\mathscr{E}_0 \vee \sigma^{-1}\mathscr{S}'_k$. By the definition of $m_k^{(1)} \cap M$-complete state splitting, we have that $e = 1m'$ and $f = 2m'$, where:
(i) m' is an atom of $\mathscr{S}'_k$
(ii) $\bar{0}m'$ is contained in an atom p' of $\mathscr{S}'_k$
(iii) $m' \cap m_k^{(1)} \cap M \neq \varnothing$.
Now since $\mathscr{S}_k \leq \mathscr{S}'_k$, there are atoms $m, p \in \mathscr{S}_k$ with $m' \subseteq m$ and $p' \subseteq p$. Now $1m \cap p \supseteq 1m' \cap p' = 1m' \neq \varnothing$. But $\mathscr{S}_k \leq \mathscr{E}_0 \vee \sigma^{-1}\mathscr{S}_k$, so $1m \subseteq p$. Similarly, $2m \subseteq p$. Also, $m \cap m_k^{(1)} \supseteq m' \cap m_k^{(1)} \cap M \neq \varnothing$. Thus, by the definition of $m_k^{(1)}$-complete splitting, $\bar{0}m$ is an atom of $\mathscr{S}_{k+1}$. But $e \cup f = \bar{0}m' \subseteq \bar{0}m$; in particular e and f are in the same atom of $\mathscr{S}_{k+1}$. Thus $\mathscr{S}_{k+1} \leq \mathscr{S}'_{k+1}$, completing the induction.

We now show by induction that the graph G'_k presents each of

$$M \cap m_k^{(1)}, M \cap m_k^{(2)}, \ldots, M \cap m_k^{(r_k)}$$

as a marker, for $0 \leq k \leq n$. This is true for $k = 0$, since $M = M \cap \varepsilon = M \cap m_0^{(1)}$. Now suppose that the hypothesis is true for k. As G'_{k+1} is obtained from G'_k by a round of $M \cap m_k^{(1)}$-complete splitting, the graph G'_{k+1} also presents $M \cap m_k^{(1)}$ as a marker (perhaps spread over more states). As $\mathscr{S}_{k+1} \leq \mathscr{S}'_{k+1}$ and as $m_{k+1}^{(i)}$, $1 \leq i \leq r_{k+1}$, occur as states of the graph G_{k+1}, the sets $m_{k+1}^{(i)}$, $1 \leq i \leq r_{k+1}$, occur as unions of states in the graph G'_{k+1}. Hence the set $(M \cap m_k^{(1)}) \cap m_{k+1}^{(i)} = M \cap m_{k+1}^{(i)}$ occurs as a union of states in the graph G'_{k+1}, for $1 \leq i \leq r_{k+1}$. But all states of G'_{k+1} contained in $M \cap m_k^{(1)}$ are marker states. Thus $M \cap m_{k+1}^{(i)}$ is presented as a marker in graph G'_{k+1}, for $1 \leq i \leq r_{k+1}$. This completes the induction.

That

$$\varphi^{M \cap m_k^{(1)}} = \varphi^{M \cap m_{k+1}^{(1)}} \circ \varphi^{M \cap m_{k+1}^{(2)}} \circ \cdots \circ \varphi^{M \cap m_{k+1}^{(r_{k+1})}}$$

and that the factors commute follows from Observation (2.5). This completes the proof of statement (2).

Example 4.2. Minimal marker automorphisms do not always commute even if they have the same length. For example,

$$\varphi_{210}\varphi_{10\bar{0}} = \varphi_{10\bar{0}}\varphi_{110\bar{0}}\varphi_{2100}$$

and $\varphi_{210} \neq \varphi_{110\bar{0}}\varphi_{2100}$. The automorphism $\varphi_{210}\varphi_{10\bar{0}}$ has order 4. The automorphism $\varphi_{01\bar{0}}\varphi_{210}$ has infinite order, as can be seen by observing the orbit of the point $(2220)^n 1(2)^\infty$. In particular, the size of the orbit is at least $n/2$. Thus a union of minimal markers need not be a marker.

5. *A minimal marker algorithm*

We have already stated an algorithm that computes the minimal marker m_a containing a given k-block a: namely, perform k rounds of a-complete splitting on the graph G_0 and observe the contents of the state containing the k-block a in the resulting graph.

The algorithm we present below keeps track of only a few states of the graph after each round of a-complete splitting. In the algorithm, the elements of the set M_i are the states in the graph G_i that partition the marker state m_{i-1} containing the k-block a in graph G_{i-1}, for $i \geq 1$. The elements of the set P_i are the states in the graph G_i that partition the state p_{i-1} containing $\bar{0}m_{i-1}$ in graph G_{i-1}, $i \geq 1$. In understanding the algorithm it might be helpful to keep in mind that all of the states in graph G_i are of the form $b_1 b_2 \ldots b_i$ with $b_j \in \{\bar{0}, 0, 1, 2, \ldots, d-1\}$.

We present the algorithm for the case $d = 3$. The only change needed when $d > 3$ is in the initialization step 0, where the set P_1 should be set to $P_1 := \{\bar{0}, 0, 3, 4, \ldots, d-1\}$.

ALGORITHM 5.1. *Given a k-block a, construct the minimal marker m_a containing a.*

0. *Initialize*:
   $i := 1$,
   $M_0 := \{\varepsilon\}$,
   $m_0 := \varepsilon$,
   $P_0 := \{\varepsilon\}$,
   $P_1 := \{\bar{0}, 0\}$.
1. *Loop*:
   If $\bar{0}$ occurs in m_{i-1} then:
   $$u := \textit{prefix of } m_{i-1} \textit{ preceding the first occurrence of } \bar{0}.$$
   If $\bar{0}$ does not occur in m_{i-1} then:
   $$u := m_{i-1}$$
   $j := |u|$.
2. $M_i := \{ux \colon x \in P_{i-j}\}$.
3. $m_i :=$ *that marker in M_i such that $m_i \cap a \neq \emptyset$.*
4. *If $i = |a|$ then*:
   $$m_a := m_i \quad \textit{and stop,}$$
   otherwise
   $$P_{i+1} := \{\bar{0}m_i\} \cup \{1x \colon x \in M_i \textit{ and } x \neq m_i\} \cup \{2x \colon x \in M_i \textit{ and } x \neq m_i\}.$$
5. $i := i + 1$ *and go to* 1.

THEOREM 5.2. *Algorithm* 5.1 *correctly computes the minimal marker m_a containing the n-block a.*

Proof. Let G_i be the graph obtained from G_{i-1} by one round of a-complete state splitting, for $1 \leq i \leq n$. We make two inductive hypotheses:

(i) The sets in M_i are those states in G_i that partition the unique state m_{i-1} in G_{i-1} with $m_{i-1} \cap a \neq \emptyset$,

(ii) The sets in P_i are those states in G_i that partition the unique state p_{i-1} in G_{i-1} with $\bar{0}m_{i-1} \subseteq p_{i-1}$.

These statements are clear for the case $i = 1$. We assume (i) and (ii) are true for $1 \leq i \leq k-1$, and show that they are true for $i = k$, where $k \geq 2$.

We show (ii). By the application of step 4 when $i = k-2$ (or by step 0 if $k = 2$) we have $p_{k-1} = \bar{0}m_{k-2}$ in graph G_{k-1}. By hypothesis (i), state m_{k-2} in graph G_{k-2} is partitioned into the states in M_{k-1} in graph G_{k-1}. Thus, in graph G_{k-1}, state p_{k-1}

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has outgoing edges

$$\{1x: x\in M_{k-1}\}\cup\{2x: x\in M_{k-1}\}.$$

Now $m_{k-1}\in M_{k-1}$ is that unique state in graph G_{k-1} with $m_{k-1}\cap a\neq\varnothing$ (by step 3 when $i=k-1$). Thus, a round of a-complete splitting applied to G_{k-1} partitions state p_{k-1} into the elements of

$$P_k=\{\bar{0}m_{k-1}\}\cup\{1x: x\in M_{k-1} \text{ and } x\neq m_{k-1}\}\cup\{2x: x\in M_{k-1} \text{ and } x\neq m_{k-1}\}.$$

This proves (ii). □

In showing (i), it is helpful to establish the following

Claim. If p is a state of G_i and $u_ju_{j-1}\dots u_1$ is a string over $\{0,1,\dots,d-1\}$ where $i+j\le k$, then the following are equivalent:

(a) For $1\le l\le j$, either $u_l\notin\{1,2\}$ or $u'p\cap a=\varnothing$, where u' is the suffix of u of length $l-1$,

(b) up occurs as a state of G_{i+j}.

Proof of claim. If (a), then $u_lu_{l-1}\dots u_1p$ occurs in graph G_{i+l}, $1\le l\le j$, by the definition of a-complete splitting and an induction on l.

If not (a), let l be the least integer such that $u_l\in\{1,2\}$ and $u'p\cap a\neq\varnothing$. Since $u'p$ occurs as a state in graph G_{i+l-1} (by (a)⇒(b)) and since G_{i+l-1} was obtained from G_0 by rounds of a-complete splitting, $u'p$ is a *marker* state. Thus $\bar{0}u'p$ occurs as a state of G_{i+l}. Thus any state in G_{i+j} which contains up also contains $u_j\dots u_{l+1}\bar{0}u'p$. Therefore (b) is false. This proves the claim.

We show (i). If $u:=\varepsilon$ in step 1 when $i=k$, then $m_{k-1}=p_{k-1}$ and $M_k=P_k$ (by step 3). But we have already shown that P_{k-1} is partitioned by a round of a-complete splitting into the states in P_k.

If u is not set equal to ε in step 1, then $u=u_ju_{j-1}\dots u_1$ for some $j\ge 1$, and $m_{k-1}=u_ju_{j-1}\dots u_1p$ where p is, by step 2, some element of P_{k-1-j}. In fact $p=\varepsilon$ if $k-1-j=0$ (by step 0), or $p=\bar{0}m_{k-2-j}$ if $k-1-j>0$ (by step 4). In either case $p=p_{k-1-j}$, the unique state of G_{k-1-j} with $\bar{0}m_{k-1-j}\subseteq p_{k-1-j}$ (by inductive hypothesis (ii)).

By inductive hypothesis (ii), the state p_{k-1-j} in graph G_{k-1-j} is partitioned into states $P_{k-j}=\{q^{(1)},\dots,q^{(r)}\}$ in graph G_{k-j}. Since $m_{k-1}=u_ju_{j-1}\dots u_1p_{k-1-j}$, we have by the claim ((b)⇒(a)) that for $1\ge l\le j$,

$$\text{either}\quad u_l\notin\{1,2\}\quad \text{or } u'p_{k-1-j}\cap a=\varnothing,$$

where u' is the suffix of u of length $l-1$. Hence we also have

$$\text{either}\quad u_l\notin\{1,2\}\quad \text{or } u'q^{(i)}\cap a=\varnothing\quad \text{where } q^{(i)}\in P_{k-j}.$$

Thus, by the claim ((a)⇒(b)), $uq^{(i)}$ occurs as a state of G_k. Since $\{uq^{(1)},\dots,uq^{(r)}\}$ is a partition of $up_{k-1-j}=m_{k-1}$, we have that state m_{k-1} in graph G_{k-1} is partitioned into

$$\{ux: x\in P_{k-j}\}$$

in graph G_k. This proves hypothesis (i) and the theorem. □

Example 5.3. We apply the algorithm to $a = 020$.

i	M_i	m_i	P_i
0	$\{\varepsilon\}$	ε	$\{\varepsilon\}$
1	$\{\bar{0}, 0\}$	0	$\{\bar{0}, 0\}$
2	$\{0\bar{0}, 00\}$	$0\bar{0}$	$\{\bar{0}0, 1\bar{0}, 2\bar{0}\}$
3	$\{0\bar{0}0, 01\bar{0}, 02\bar{0}\}$	$0\bar{0}0$	$\{\bar{0}0\bar{0}, 100, 200\}$.

6. *An application to the dynamics of cubic polynomials*

In this section we describe a construction of Blanchard et al. [**BDK**] that motivated this paper. We apologize to those authors for the shortcomings of this description.

If $R: \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a rational map then the *Fatou set* F_R is defined by

$$F_R = \{z \in \bar{\mathbb{C}}: \exists \text{ a neighborhood } U \text{ of } z \text{ so that the iterates of } R, \text{ when restricted to } U, \text{ form a normal family}\}.$$

The *Julia set* J_R is defined to be the complement of F_R.

If p is a degree d polynomial over $\mathbb{C}$ all of whose critical points escape to infinity under iteration of p, then

$$J_p = \{z \in \mathbb{C}: \{p^n(z)\} \text{ is a bounded sequence}\}$$

and J_p is a Cantor set. As a dynamical system, (J_p, p) is conjugate to the one-sided d-shift [**B**].

Blanchard, Devaney and Keen have constructed automorphisms of (J_p, p) for cubic polynomials p [**BKK**]. In their construction, an automorphism of (J_p, p) is obtained by traversing a loop starting and ending at the polynomial p in the space $\mathcal{P}_3$ of cubic polynomials both of whose critical points escape to infinity. We are not qualified to delve into the parameterization or description of this space [**BH**].

For $p \in \mathcal{P}_3$ one can define (we do not) the rate-of-escape function $h_p: \mathbb{C} \to \mathbb{R}^+$ [**BH**]. The function h_p has the properties that

(i) $h_p(p(z)) = 3h_p(z), \quad z \in \mathbb{C}$

(ii) $J_p = \{z \in \mathbb{C}: h_p(z) = 0\}$

(iii) h_p is continuous and h_p is harmonic outside of J_p.

A polynomial $p \in \mathcal{P}_3$ is chosen by Blanchard et al. [**BDK**] so that the two critical points $c^{(1)}$ and $c^{(2)}$ of p are such that

$$h_p(p(c^{(1)})) < \rho < h_p(p(c^{(2)}))$$

and $\{z: h_p(z) = \rho\}$ is a Jordan curve enclosing J_p. In figure 1, we have labeled the curve $\{z: h_p(z) = \rho\}$ as Γ_ε. Let

$$D_\varepsilon = \{z: h_p(z) \le \rho\}.$$

The set $p^{-1}(D_\varepsilon) = \{z: h_p(z) \le \frac{1}{3}\rho\}$ has two connected components: A disk D_0 which maps by p in a degree 1 manner onto D_ε, and a disk $D_{\bar{0}}$ containing $c^{(1)}$ which maps in a degree 2 manner onto D_ε.

If $p(c^{(1)})$ is connected to Γ_ε by an arc γ along which $h_p(z)$ is increasing to $h_p(z) = \rho$, then the preimage of this arc divides the interior of $D_{\bar{0}}$ into two regions: U_1 and U_2. If we denote the interior of D_0 by U_0, we can coordinatize the Julia

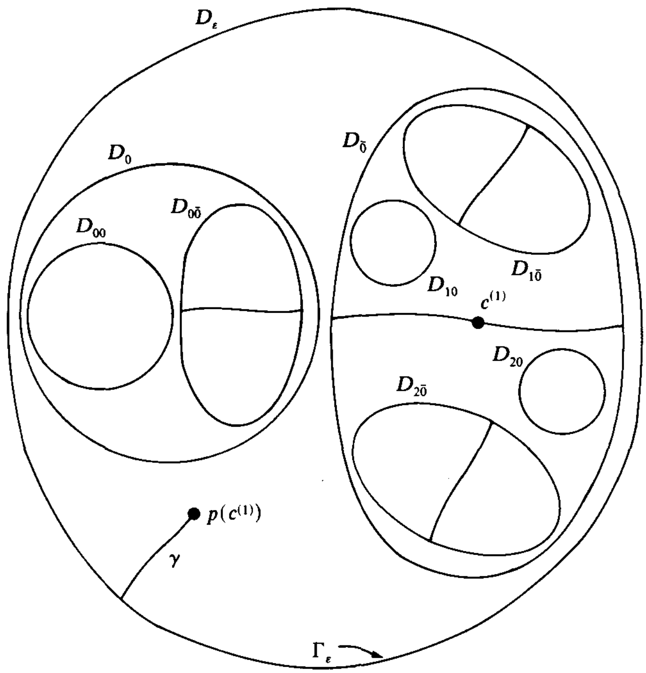

FIGURE 1

set J_p by $\psi: J_p \to X_3$ defined by

$$\psi(z) = a_0 a_1 a_2 \ldots,$$

where $p^n(z) \in U_{a_n}$. The map ψ is a conjugacy from (J_p, p) to (X_3, σ).

For each $k \geq 0$, we denote each connected component of the set $\{z: h_p(z) = \rho/3^k\}$ by D_u, where u is the set $\psi(J_p \cap D_u)$ in X_3. The set $u \subseteq X_3$ is actually a union of k-blocks since $p^k D_u = D_\varepsilon$. For example, in figure 2, $D_{0\bar{0}}$ is the connected component of $\{z: h_p(z) \leq (1/3^2)\rho\}$ that contains $\psi^{-1}(0\bar{0})$.

According to Blanchard et al. [**BDK**], the polynomial $p \in \mathcal{P}_3$ may be chosen so that the critical value $p(c^{(1)})$ is in the same connected component of $\{z: h_p(z) \leq (1/3^{k-1})\rho\}$ as $\psi^{-1}(a) \subseteq J_p$ where a is any $(k-1)$-block in X_3, for any $k > 0$. We address the essentially combinatorial question: What is the configuration of the level curves $h_p(z) = (1/3^k)\rho$ as a function of the location of the critical value $p(c^{(1)})$? This question was pointed out to me by Linda Keen and is of interest because Blanchard, Devaney and Keen have constructed a loop in the space of polynomials $\mathcal{P}_3$, parameterized by $0 \leq t \leq 1$, such that:

(i) $p_0 = p_1 = p$
(ii) $h_{p_t}(p_t(c_t^{(1)})) = \rho/3^k$, where $c_t^{(1)}$ is a critical point of p_t
(iii) $h_{p_t}(p_t(c_t^{(2)})) > \rho$
(iv) $p_t(c_t^{(1)})$ winds once around exactly one of the connected components, say D_u, of $\{z: h_{p_t}(z) < \rho/3^k\}$ and winds zero times around all other such components.

In fact, p_t is given by $p_t = \psi_t \circ \varphi_t \circ p \circ \psi_t^{-1}$ where ψ_t and φ_t are quasi-conformal homeomorphisms of $\mathbb{C}$ and φ_t is the identity on J_p, for $0 \leq t \leq 1$. Hence $p_t\psi_t = \psi_t p$

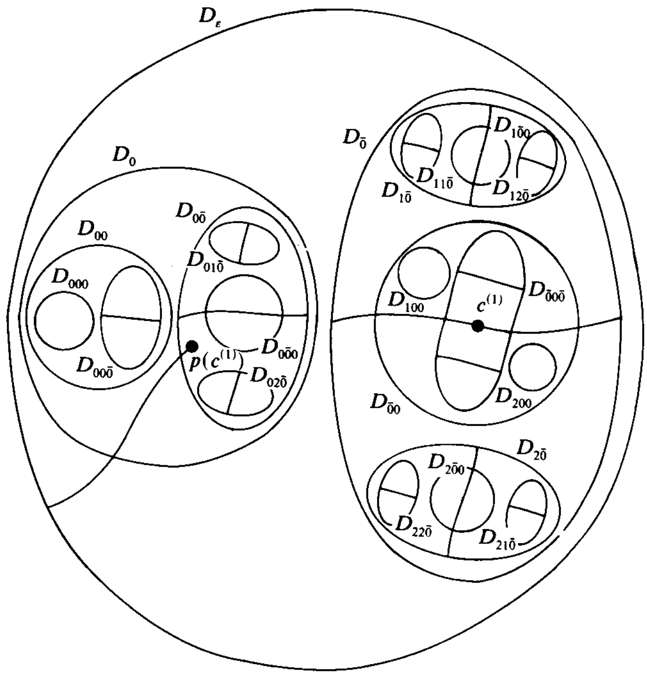

FIGURE 2

on J_p. Thus $\psi_t: J_p \to J_{p_t}$ is an isomorphism for $0 \le t \le 1$. In particular, $\psi_1: J_p \to J_p$ is an automorphism which, it turns out [**BDK**], in terms of the coordinates given by $\psi: J_p \to X_3$ is the marker automorphism switching symbols 1 and 2 when followed by the marker $\psi(J_p \cap D_u) = u$.

In Theorem 6.1, we will show that the markers constructed by Blanchard, Devaney and Keen, those of the form $\psi(J_p \cap D_u)$, where D_u is a single connected component of $\{z: h_p(z) \le \rho/3^k\}$ nested within the component of $\{z: h_p(z) \le \rho/3^{k-1}\}$ containing $p(c^{(1)})$, are exactly the minimal markers.

Now $\mathcal{P}_3$ is connected and by a separate argument there is a loop in $\mathcal{P}_3$ that cyclically permutes the symbols 0, 1, and 2 in J_p. Therefore by Theorems 2.3 and 4.1, any automorphism of the 3-shift X_3 may be realized by traversing a loop in $\mathcal{P}_3$.

THEOREM 6.1. *If $p \in \mathcal{P}_3$ is a polynomial with critical points $c^{(1)}$ and $c^{(2)}$, and $p > 0$, $k > 0$ are such that*

(i) $\rho < h_p(p(c^{(2)}))$,

(ii) $\Gamma_\varepsilon = \{z: h_p(z) = \rho\}$ *is a Jordan curve,*

(iii) $p(c^{(1)})$ *is in the same connected component of* $\{z: h_p(z) \le \rho/3^{k-1}\}$ *as* $\psi^{-1}(a) \subseteq J_p$ *where a is a $(k-1)$-block in X_3 and $\psi: J_p \to X_3$ is the conjugacy defined above,*

then for $0 \le j \le k$, the connected components of the set

$$\{z: h_p(z) \le \rho/3^j\}$$

are exactly

$\{D_S$: S is a state in the graph G_j obtained from the graph G_0 by j rounds of a-complete splitting$\}$.

In particular, the connected component of $\{z\colon h_p(z)\le \rho/3^{k-1}\}$ *containing the point* $p(c^{(1)})$ *is* D_{m_a}, *and the connected components of* $D_{m_a}\cap\{z\colon h_p(z)\le\rho/3^k\}$ *are* $D_{m_1},\ldots,D_{m_r}$ *where* $m_1,\ldots,m_r$ *are the minimal markers of length k that partition the marker* m_a.

Remark. The proof of the theorem does not really depend on the degree $d\ge 3$ of p. We state it for $d=3$ only for definiteness and simplicity.

Proof. We induct on k. The case $k=1$ is true by the definition of D_ε, D_0, and $D_{\bar{0}}$ given above. Supposing the theorem is true for k, we prove it for $k+1$. Let a be a k-block such that $p(c^{(1)})$ is in the same connected component of $\{z\colon h_p(z)\le\rho/3^k\}$ as $\psi^{-1}(a)$. Let S be any state in G_k. Now

$$p(D_S)\cap J_p=\psi^{-1}(\sigma(S)).$$

But

$$\sigma(S)=\bigcup\{S'\colon \text{state } S' \text{ follows state } S \text{ in } G_k\},$$

so the inductive hypothesis gives that the connected components of

$$p(D_S)\cap\{z\colon h(z)\le\rho/3^k\}$$

are

$$\{D_{S'}\colon \text{state } S' \text{ follows state } S \text{ in } G_k\}.$$

The remainder of the proof divides into two cases.

Case 1. $c^{(1)}\notin D_S$. We have $D_S\subseteq D_0\cup D_{\bar{0}}$ and $c^{(2)}\notin D_0\cup D_{\bar{0}}$, so $p|D_S$ is 1-to-1. So the connected components of $D_S\cap\{z\colon h_p(z)\le\rho/3^{k+1}\}$ are

$$\{p^{-1}(D_{S'})\cap D_S\colon \text{state } S' \text{ follows state } S \text{ in } G_k\}.$$

But $\psi(J_p\cap p^{-1}(D_{S'})\cap D_S)=S\cap\sigma^{-1}S'$, so this set is

$$\{D_{S\cap\sigma^{-1}S'}\colon \text{state } S' \text{ follows state } S \text{ in } G_k\}.$$

Since $p|D_S$ is 1-to-1, $\sigma|S$ is 1-to-1 also, so no parallel edges begin at state S in G_k. Thus state S in G_k is completely split into its following edges:

$$\{S\cap\sigma^{-1}S'\colon S' \text{ follows } S \text{ in } G_k\}.$$

This completes Case 1.

Case 2. $c^{(1)}\in D_S$. Again $D_S\subseteq D_0\cup D_{\bar{0}}$, so $c^{(2)}\notin D_S$. Thus $p|D_S$ is 2-to-1 except at $c^{(1)}$.

By the inductive hypothesis $\psi^{-1}(a)\subseteq D_{m_a}$ because m_a is the state in graph G_k with $a\subseteq m_a$. By assumption $p(c^{(1)})\in D_{m_a}$. Hence D_{m_a} is a connected component of

$$p(D_S)\cap\{z\colon h(z)\le\rho/3^k\}.$$

As $c^{(1)}\in D_S$, we have $D_S\subseteq D_{\bar{0}}$, so $S\subseteq\bar{0}$. Thus $S=p_k$, the unique state in G_k with $\bar{0}m_a\subseteq p_k$. Now $p^{-1}(D_{m_a})\cap D_S$ has a single connected component D mapping 2-to-1 onto D_{m_a} (except at $c^{(1)}$) because $p(c^{(1)})\in D_{m_a}$ and $c^{(1)}\in D_S$. Now

$$\psi(J_p\cap D)=\psi(J_p\cap p^{-1}(D_{m_a})\cap D_S)=\sigma^{-1}m_a\cap S=\bar{0}m_a.$$

Thus $D=D_{\bar{0}m_a}$. Any other connected component $D_{S'}$ of

$$p(D_S)\cap\{z\colon h_p(z)\le\rho/3^k\}$$

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is such that $p^{-1}(D_{S'}) \cap D_S$ has two connected components, $D^{(1)}$ and $D^{(2)}$, each mapping 1-to-1 onto $D_{S'}$. As $p \mid D^{(i)}$ is 1-to-1 onto $D_{S'}$, $\sigma \mid \psi(J_p \cap D^{(i)})$ is 1-to-1 onto S'. Now $D^{(i)} \subseteq D_{\bar{0}}$, so $\psi(J_p \cap D^{(i)}) \subseteq \bar{0}$. Thus

$$\{\psi(J_p \cap D^{(1)}), \psi(J_p \cap D^{(2)})\} = \{1S', 2S'\},$$

so the connected components of

$$D_S \cap \{z: h_p(z) \leq \rho/3^{k+1}\}$$

are

$$\{D_u: \text{state } u \text{ in graph } G_{k+1} \text{ is partitioned from state } S \text{ in graph } G_k\}.$$

This completes Case 2. □

Example 6.2. Figure 2 gives the nesting of the components of $\{z: h_p(z) \leq \rho/3^k\}$ for $k = 0, 1, 2, 3$ when $p(c^{(1)})$ is the same connected component of $\{z: h_p(z) \leq \rho/3^2\}$ as $\psi^{-1}(02)$. We list below the corresponding states of G_k for $k = 0, 1, 2, 3$.

k	States of G_k
0	$\{\varepsilon\}$
1	$\{\bar{0}, 0\}$
2	$\{\bar{0}0, 1\bar{0}, 2\bar{0}, 0\bar{0}, 00\}$
3	$\{\bar{0}0\bar{0}, 100, 200, 1\bar{0}0, 11\bar{0}, 12\bar{0}, 2\bar{0}0, 21\bar{0}, 22\bar{0}, 0\bar{0}0, 01\bar{0}, 02\bar{0}, 000, 00\bar{0}\}$

Compare to example (5.3).

Acknowledgement

Many thanks are due to Linda Keen and Bruce Kitchens for some very helpful discussions. Each deserves especial thanks: to Linda for the complex dynamics and to Bruce for the symbolic dynamics. Thanks are also due to the referee for his helpful corrections and suggestions.

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