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POLYNOMIALS WITH $\{0, +1, -1\}$ COEFFICIENTS AND A ROOT CLOSE TO A GIVEN POINT

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ABSTRACT. For a fixed algebraic number α we discuss how closely α can be approximated by a root of a $\{0, +1, -1\}$ polynomial of given degree. We show that the worst rate of approximation tends to occur for roots of unity, particularly those of small degree. For roots of unity these bounds depend on the order of vanishing, k, of the polynomial at α.

In particular we obtain the following. Let B_N denote the set of roots of all $\{0, +1, -1\}$ polynomials of degree at most N and $B_N(\alpha, k)$ the roots of those polynomials that have a root of order at most k at α. For a Pisot number α in $(1, 2]$ we show that

$$\min_{\beta \in B_N \setminus \{\alpha\}} |\alpha - \beta| \asymp \frac{1}{\alpha^N},$$

and for a root of unity α that

$$\min_{\beta \in B_N(\alpha,k) \setminus \{\alpha\}} |\alpha - \beta| \asymp \frac{1}{N^{(k+1)\lceil \frac{1}{2}\phi(d) \rceil + 1}}.$$

We study in detail the case of $\alpha = 1$, where, by far, the best approximations are real. We give fairly precise bounds on the closest real root to 1. When $k = 0$ or 1 we can describe the extremal polynomials explicitly.

1. **Introduction.** We are interested in studying how well an algebraic number α can be approximated by a root $\beta \neq \alpha$ of a $\{0, +1, -1\}$ polynomial of a given degree. In particular if we fix α (typically itself a root of a $\{0, +1, -1\}$ polynomial) and plot the roots of all $\{0, +1, -1\}$ polynomials of degree at most N how does the size of the zero-free region around α vary with N. For example, Figure 1 shows the roots of all $\{0, +1, -1\}$ polynomials of degree at most eight. We give a related picture (Figure 2) for roots of all $\{-1, +1\}$ polynomials of degree twelve, showing some of the fractal behaviour visible for higher degrees. Similar pictures have been produced for $\{0, 1\}$ polynomials by Odlyzko and Poonen [11], and for polynomials of low two-norm by Yamamoto [14]. Barnsley and Harrington [2] consider the limiting case (as the bound N on the degree tends to infinity) showing that every α in the annulus $1/\sqrt{2} < |\alpha| < 1$ is a root of some $\{0, +1, -1\}$ power series (see also [1, Section 8.2] for pictures of the boundary of the zero accessible region).

Let B_N denote the set of roots of all $\{0, +1, -1\}$ polynomials of degree at most N and $B_N(\alpha, k)$ the roots of those polynomials that have a root of order at most k at α. Around points away from the unit circle that are themselves roots of $\{0, +1, -1\}$ polynomials or

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power series, we show that the distance to the nearest root decreases exponentially with degree:

$$-\min_{\beta\in B_N\setminus\{\alpha\}} \log|\alpha-\beta| \asymp N.$$

For points on the unit circle which are not roots of unity but which have small Mahler measure we show a similar exponential decrease. For Pisot or Salem numbers in (1,2] we can make this fairly precise

$$-\min_{\beta\in B_N\setminus\{\alpha\}} \log|\alpha-\beta| \sim (\log\alpha)N.$$

For a d-th root of unity the growth rate is only subexponential,

$$-\min_{\beta\in B_N\setminus\{\alpha\}} \log|\alpha-\beta| \ll \sqrt{N}\log N.$$

For roots of unity the closeness of a root depends critically on the order of vanishing k of the corresponding polynomial at α (off the unit circle the order of vanishing is bounded and generally less significant). For fixed k we show that the decrease is merely polynomial and give the correct order of growth (the slowest growth occurring when $d = 1, 2, 3, 4$ or 6):

$$\min_{\beta\in B_N(\alpha,k)\setminus\{\alpha\}} |\alpha-\beta| \asymp \frac{1}{N^{(k+1)\lceil\frac{1}{2}\phi(d)\rceil+1}}.$$

The most interesting case seems to be $\alpha = 1$ where the best approximations are overwhelmingly real, as is immediately apparent on looking at a plot. For example, Figure 3 shows a detail of the plot of the roots of all $\{-1,+1\}$ polynomials of degree fifteen. This latter picture was generated by the CECM Roots of Polynomials Interface (URL: http://www.cecm.sfu.ca/organics/papers/odlyzko/support/polyform.html) developed by Loki Jorgensen. Although the region around 1 appears very similar in Figures 1 and 2 we show in Theorem 10 that the limited order of vanishing at 1 possible in the $\{-1,+1\}$ case actually leads to a significantly worse rate of approximation to 1.

In Section 3 we therefore concentrate on bounding the closest real root to 1 and on making the k dependence of the implied constants in

$$\min_{\beta\in B_N(1,k)\setminus\{1\}} |1-\beta| \asymp \frac{1}{N^{(k+2)}}$$

explicit. When the multiplicity k of the root at 1 is restricted to 0 or 1 we determine the growth precisely

$$\min_{\beta\in B_N(1,0)\setminus\{1\}} |1-\beta| \sim \frac{4}{N^2}, \qquad \min_{\beta\in B_N(1,1)\setminus\{1\}} |1-\beta| \sim \frac{32}{N^3},$$

and in Section 4 give the extremal polynomial. Such explicitness seems inaccessible for higher orders.

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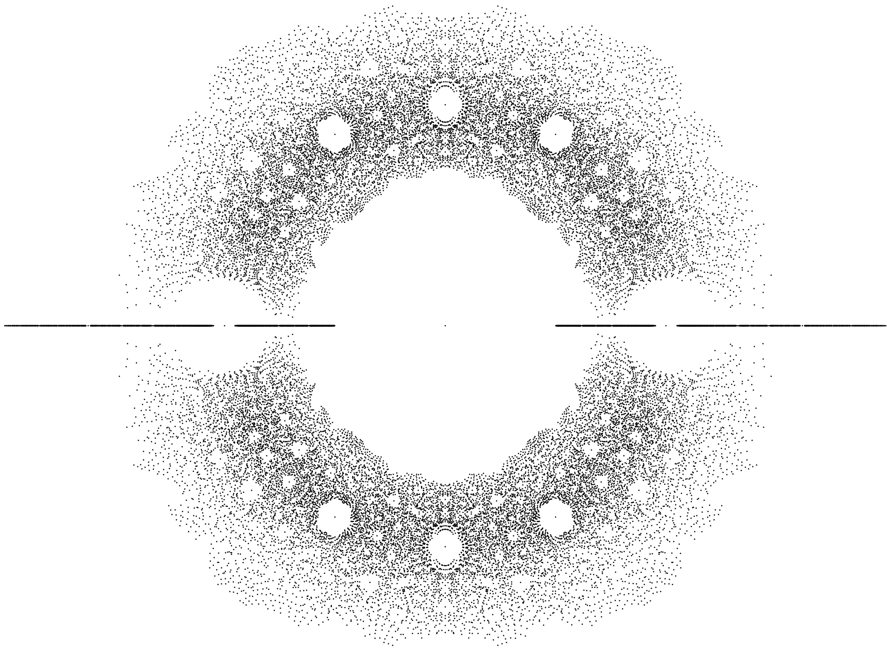

FIGURE 1: Zeros of all polynomials with $\{0,+1,-1\}$ coefficients and degree at most eight.

2. **Results for general points.** We recall the definition of the Mahler measure M of a polynomial:

$$M\Big(a_d\prod_{i=1}^{d}(x-\alpha_i)\Big) := |a_d|\prod_{i=1}^{d}\max\{1,|\alpha_i|\}.$$

For an algebraic α we shall use $M(\alpha)$ to denote the Mahler measure of the minimal polynomial of α. We shall write $\partial(F)$ for the degree of F and $F^j(x)$ for the j-th derivative of $F(x)$.

THEOREM 1. *Let α be a fixed algebraic number. Let F be a $\{0,+1,-1\}$ polynomial of degree N with a root of order $k\geq 0$ at α, and (not necessarily distinct) roots $\beta_1,\ldots,\beta_m$ not equal to α.*

Then, for fixed k and m,

$$|\alpha-\beta_1|\cdots|\alpha-\beta_m| \geq \frac{c_1(m,k,\alpha)}{M(\alpha)^{\delta N}(N+1)^{c_2+m\epsilon}},$$

with

$$\delta := \begin{cases} 1 & \text{if } \alpha \text{ is real,} \\ \frac{1}{2} & \text{if } \alpha \text{ is complex,} \end{cases} \qquad \epsilon := \begin{cases} 0 & \text{if } |\alpha|\neq 1, \\ 1 & \text{if } |\alpha|=1, \end{cases}$$

and

$$c_2 = c_2(k,\alpha) := \delta(k+1)d_1,$$

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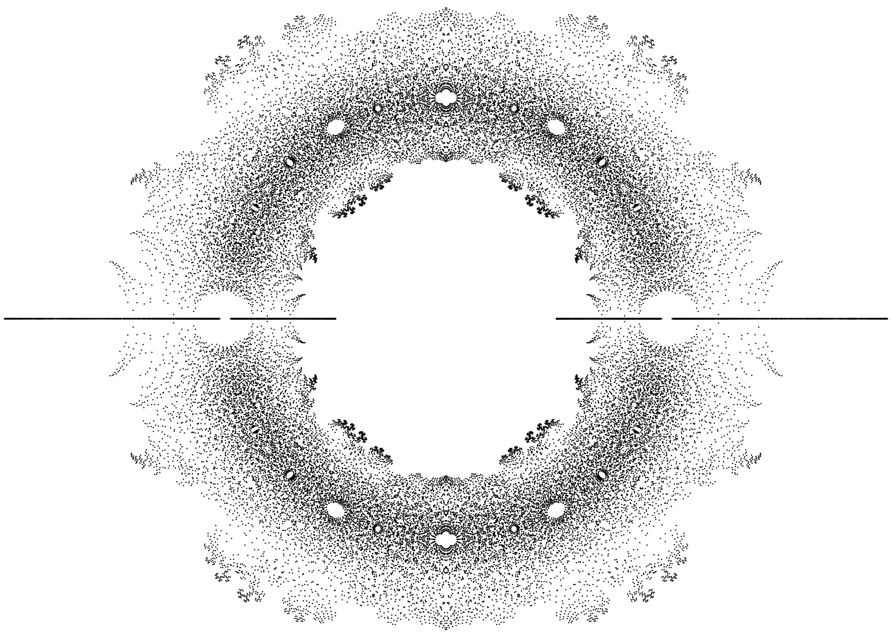

FIGURE 2: All zeros of all degree twelve polynomials with $\{+1, -1\}$ coefficients.

where d_1 denotes the number of conjugates of α (including α) that lie on the unit circle.

Explicit expressions for the constant $c_1(m, k, \alpha)$ can be found in the proof of Theorem 1. In particular when α is a d-th root of unity we obtain

$$c_1(m, k, \alpha) = (k!)^{\lceil \frac{1}{2}\phi(d) \rceil} e^{-m},$$

where $\phi(d)$ is the usual Euler phi-function.

For a fixed multiplicity k we see in Theorem 1 a clear difference between the roots of unity where the distance can decrease at most polynomially and non-roots where exponential growth is allowed. Notice also that exceptionally good approximations prevent the remaining roots of that polynomial from coming too close. Taking $m = 1$ in Theorem 1 gives a lower bound

$$|\alpha - \beta| \geq \frac{c(k, \alpha)}{M(\alpha)^{\delta N} N^{\delta(k+1)d_1+\epsilon}},$$

for the smallest root, β. Taking $m = 2$ it is clear that we can not hope to come close to achieving this unless the remaining roots β_i satisfy $|\alpha - \beta_i| \gg N^{-1}$ when $|\alpha| = 1$ or $\gg 1$ when $|\alpha| \neq 1$. This strongly suggests that the best approximations should occur as single roots and that for real α they should probably be real rather than a pair of conjugate roots

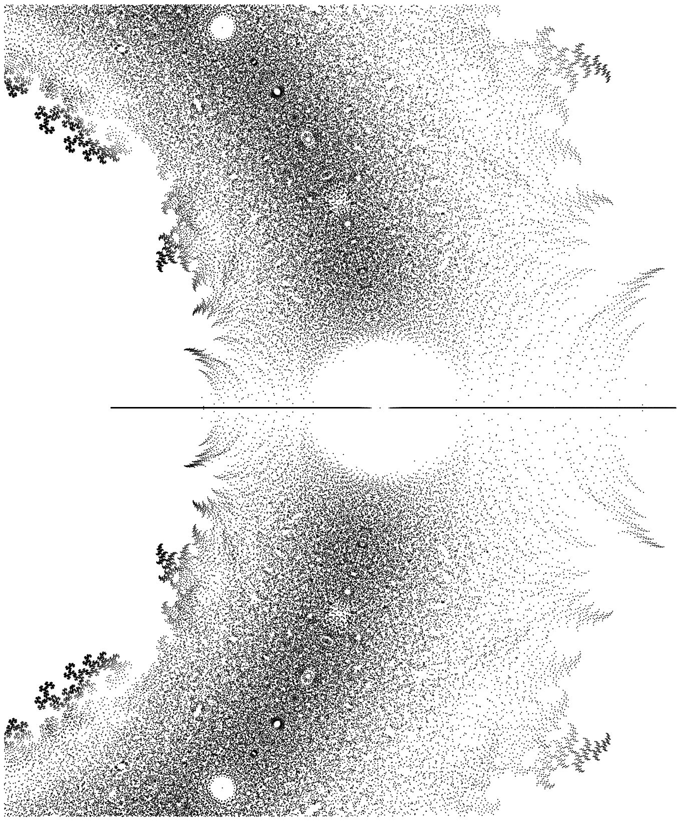

FIGURE 3: Detail around 1 showing zeros of all degree fifteen polynomials with $\{+1, -1\}$ coefficients.

(it must certainly be the case in Corollary 3 and 4 where we have sharpness in this lower bound). Note, when β is a double root or α is real and β complex applying Theorem 1 with $m = 2$ gives

$$|\alpha - \beta| \geq \frac{c(k, \alpha)}{M(\alpha)^{\frac{1}{2}\delta N} N^{\frac{1}{2}\delta(k+1)d_1+\epsilon}}.$$

Now if α is not a root of unity then the maximum multiplicity k of a root at α is bounded. To see this observe that for α to be a root of a $\{0, +1, -1\}$ polynomial it must be an algebraic integer and hence, by Kronecker's theorem, if not a root of unity it must have a conjugate α_i off the unit circle. It is straightforward to see that away from the unit circle the multiplicity is necessarily bounded. In [3] we gave explicit bounds on this multiplicity, Borwein-Erdélyi-Kós [5, Theorem 4.2] in fact show more precisely that

$$k < c \min_{|\alpha_i| \neq 1} \frac{1}{|1 - |\alpha_i||}$$

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for some absolute constant c. It is not known whether the multiplicity for non roots of unity is bounded by an absolute constant (independent of α). It is an interesting problem to decide which non roots of unity α are multiple roots of $\{0, +1, -1\}$ polynomials. Of course any root of a $\{0, +1, -1\}$ polynomial must certainly lie in the annulus $\frac{1}{2} \leq |\alpha| \leq 2$. Conversely we know from results of Bombieri-Vaaler [4] that if the minimal polynomial of α has measure less than $2^{1/k}$ then α must be a k-th order root of some $\{0, +1, -1\}$ polynomial (in particular as mentioned in [3] there are at least two Salem numbers which must be a fourth order root and infinitely many examples with a triple root). It is not known whether there exists a root of multiplicity five or more.

For α a d-th root of unity Theorem 1 gives

$$|\alpha - \beta| \geq e^{-1} \frac{(k!)^{\lceil \frac{1}{2}\phi(d) \rceil}}{(N+1)^{(k+1)\lceil \frac{1}{2}\phi(d) \rceil + 1}}.$$

Although for a fixed k this is only polynomial in N, it is easy to see that for appropriately large N the multiplicity k can be made arbitrarily large. However for a given N Borwein-Erdélyi-Kós [5, Theorem 2.4] have shown that

$$k < \left\lfloor \frac{16}{7} \sqrt{N} \right\rfloor + 1.$$

From the above comments we readily deduce lower bounds independent of the multiplicity k, which decrease exponentially with N when α is not a root of unity but only sub-exponentially when α is a root of unity:

COROLLARY 1. *For a fixed algebraic α, any root $\beta \neq \alpha$ of a $\{0, +1, -1\}$ polynomial of degree N satisfies*

$$|\alpha - \beta| > \exp\bigl(-c(\alpha)N + O(\log N)\bigr), \quad c(\alpha) := \delta \log M(\alpha),$$

if α is not a root of unity and

$$|\alpha - \beta| > \exp\bigl(-c(\alpha)\sqrt{N} \log N + O(\sqrt{N})\bigr), \quad c(\alpha) := \frac{8}{7} \left\lceil \frac{1}{2}\phi(d) \right\rceil,$$

if α is a d-th root of unity.

For an α off the unit circle that is a root of a $\{0, +1, -1\}$ polynomial or a $\{0, +1, -1\}$ power series it is easily seen that we can construct roots exponentially close to α. We shall assume that $|\alpha| > 1$, otherwise we work with α^{-1} and the reciprocals of the polynomials, $x^{\partial(f)} f(x^{-1})$.

THEOREM 2. *Suppose that α is fixed with $|\alpha| > 1$.*

(i) If there exist $\{0, +1, -1\}$ polynomials F, G with a root of order exactly $k \geq 0$ and $s > k$ at α respectively, then, for fixed F and G,

$$H_N(x) := x^{N - \partial(G)} G(x) - F(x), \quad N > \partial(FG),$$

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is a $\{0,+1,-1\}$ *polynomial of degree N with a root of order k at* α *and* $m := (s-k)$ *roots* $\beta_j \neq \alpha$ *with*

$$\beta_j - \alpha = \frac{c_j(\alpha)}{\alpha^{N/m}}\left(1 + O\left(\frac{N^{s+2}}{\alpha^{N/m}}\right)\right),$$

where

$$c_j(\alpha) := \alpha^{\partial(G)}\frac{F^k(\alpha)/k!}{G^s(\alpha)/s!}e^{2\pi j i/m}, \quad j = 1,\ldots,m.$$

If α *is real and* $m = 1$ *then the root is also real.*

(ii) If there exists a power series

$$F(z) = \sum_{i=0}^{\infty} c_i x^i, \quad c_i \in \{0,+1,-1\}$$

with a root of order exactly $k \geq 1$ *at* α^{-1}*, then, for fixed F, the polynomial reciprocal of the truncations*

$$H_N(x) := \sum_{i=0}^{N} c_{N-i}x^i,$$

are $\{0,+1,-1\}$ *polynomials of degree N with k roots* β_i *(counted with multiplicity and not necessarily distinct from* α*) such that*

$$|\alpha - \beta_i| \leq \frac{c(\alpha)}{|\alpha|^{N/k}}.$$

If α *is real and* $k = 1$ *then the root* β *is also real.*

For real α in (1,2) truncations of the beta-expansion of 1 thus yield exponentially good approximations:

COROLLARY 2. *If* α *is a fixed real in* $(1,2]$*, then there exists a* $\{0,+1,-1\}$ *polynomial of degree N with a real root* $\beta \neq \alpha$ *such that*

$$|\alpha - \beta| \leq \frac{c(\alpha)}{\alpha^N},$$

for some constant $c(\alpha)$*.*

If α is a Pisot number (that is a real algebraic integer $\alpha > 1$ with all its conjugates strictly inside the unit circle) in (1,2] we thus obtain the correct order of growth for the minimal distance. We let B_N denote the set of roots of all $\{0,+1,-1\}$ polynomials of degree at most N.

COROLLARY 3. *If* α *is a fixed Pisot number in (1,2], then*

$$\min_{\beta\in B_N\setminus\{\alpha\}} |\alpha - \beta| \asymp \frac{1}{\alpha^N},$$

where the implied constants are allowed to depend on α*.*

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Notice that from Theorem 1 any complex root β must have $|\alpha-\beta| \geq c(\alpha)\alpha^{-N/2}$, so that the approximations β in Corollary 3 will certainly be real (for large N). For a Salem number (that is a real algebraic integer $\alpha > 1$ with one conjugate α^{-1} inside the unit circle and the remaining conjugates on the unit circle) in (1,2) the dominant term is again α^{-N} although a polynomial function remains undetermined. Similarly if α is a complex Pisot number (that is α is a complex algebraic integer with $|\alpha| > 1$, all of whose conjugates other than $\bar{\alpha}$ lie strictly inside the unit circle) which is a single root of some $\{0, +1, -1\}$ polynomial (or α^{-1} a single root of a $\{0, +1, -1\}$ power series) then the correct order of approximation is again precisely $|\alpha|^{-N}$.

When α is an algebraic number (on or off the unit circle), that is not a root of unity but whose Mahler measure is small, we show that there are roots exponentially close to α:

THEOREM 3. *Suppose that α is a fixed algebraic with*

$$1 < M(\alpha) < 2.$$

Then there exists a $\{0, +1, -1\}$ polynomial of degree at most N with a root $\beta \neq \alpha$ such that

$$|\alpha-\beta| < c(\alpha) N^{1+(2\delta d/L)} \left(\frac{M(\alpha)}{2}\right)^{2\delta N/L(L+1)},$$

where $L = L(\alpha)$ is the highest order of a root at α possible for a $\{0, +1, -1\}$ polynomial, d is the degree of α, and

$$\delta := \begin{cases} 1 & \text{if } \alpha \text{ is real,} \\ \frac{1}{2} & \text{if } \alpha \text{ is complex.} \end{cases}$$

For roots on the unit circle with $M(\alpha) \geq 2$ the situation is less clear. From Dirichlet's Theorem we can at least say that for any fixed $\alpha = e^{2\pi i\theta}$ on the unit circle that is not a root of unity there must certainly be infinitely many N such that $|\theta - p/N| < N^{-2}$ for some integer p, and hence have

$$|\alpha-\beta| < \frac{c}{N^2}$$

for some root β of $(x^N - 1)$. Notice that if α is a d-th root of unity we can only obtain $|\alpha-\beta| < c/dN$ from such polynomials.

There remains the case when α is a root of unity. For fixed k we show the following:

THEOREM 4. *Let α be a fixed d-th root of unity and $k \geq 0$ a fixed positive integer. For N sufficiently large there exists a $\{0, +1, -1\}$ polynomial of degree at most N with a root of order k at α and a root $\beta \neq \alpha$ with*

$$|\alpha-\beta| \leq \frac{c(\alpha, k)}{N^{(k+1)\lceil \frac{1}{2}\phi(d) \rceil + 1}}.$$

From the lower bound of Theorem 1 this is the optimal order of growth for fixed k. We let $B_N(\alpha, k)$ denote the set of roots of all $\{0, +1, -1\}$ polynomials of degree at most N with a root of order at most k at α.

COROLLARY 4. *For a fixed root of unity α, and fixed integer $k \geq 0$,*

$$\min_{\beta \in B_N(\alpha,k)\setminus\{\alpha\}} |\alpha-\beta| \asymp \frac{1}{N^{(k+1)\lceil \frac{1}{2}\phi(d)\rceil+1}},$$

where the implied constants are allowed to depend on α and k.

The bounds of Theorems 2 and 4, together with a variant of the construction in Theorem 2 allowing the multiplicity of the root at 1 to grow as a function of N, give us an upper bound analogue of Corollary 1:

THEOREM 5. *If α is a fixed algebraic integer with $M(\alpha) < 2$, then there is a constant $c(\alpha) > 0$ such that, for sufficiently large N, there is a $\{0,+1,-1\}$ polynomial of degree at most N with a root $\beta \neq \alpha$ satisfying*

$$|\alpha-\beta| < \exp\bigl(-c(\alpha)N\bigr)$$

if α is not a root of unity, and

$$|\alpha-\beta| < \exp\bigl(-c(\alpha)(N\log N)^{1/3}\bigr)$$

if α is a root of unity.

We have here considered the rate of approximating a fixed α by roots of $\{0,+1,-1\}$ polynomials of degree at most N. A somewhat similar question would be to ask for the minimum separation of two distinct roots α, β, of a $\{0,+1,-1\}$ polynomial F of degree at most N. We observe that bounds of Mignotte [10] using the discriminant Δ of the polynomial give

$$|\alpha-\beta| \geq \frac{|\Delta|^{1/2}}{N^{(N+2)/2}M(F)^{N-1}} \geq \frac{1}{(N+1)^{(N+1)}}$$

on observing that

$$M(F) = \exp\Bigl(\int_0^1 \log|F(e^{2\pi i t})|\,dt\Bigr) \leq ||F||_2 \leq \sqrt{N+1}.$$

It is an old problem of Mahler [9] to determine whether this inequality for $M(F)$ can be significantly sharpened (Littlewood [8] asks a number of related questions for the sup norm).

3. **Roots close to 1.** We now concentrate on roots close to 1. From Theorem 1 we know that a complex root, β, of a $\{0,+1,-1\}$ polynomial of degree N with a root of order k at 1 satisfies

$$|\beta-1| \geq c\frac{\sqrt{k!}}{N^{3/2+k/2}}.$$

Hence we restrict ourselves to real roots β where the rate of approximation is, as we saw in Corollary 4, substantially better.

Let $P(N,k)$ denote the set of polynomials of degree at most N, with $\{0,+1,-1\}$ coefficients, and a root of order exactly k at 1.

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We define $\theta(N,k)$ to be the largest real number θ in $[0,1)$ such that $f(\theta)=0$ for some f in $P(N,k)$. Reversing the order of the coefficients we could plainly equivalently define $\theta(N,k)^{-1}$ to be the smallest real root $\theta>1$.

Corollary 4 tells us that for fixed k the growth in terms of N is precisely

$$1-\theta(N,k)\asymp\frac{1}{N^{k+2}}.$$

For $k\geq 2$ the optimal constants in these bounds are not clear.

We give the following upper bound on $\theta(N,k)$:

THEOREM 6. *For a fixed integer $k\geq 0$ we have*

$$\theta(N,k)\leq 1-\frac{4^{k+1}(k+1)!}{N^{k+2}}+O\Big(\frac{c_3}{N^{k+3}}\Big),$$

where $c_3=c_3(k)$ is independent of N.

We also give a similar lower bound:

THEOREM 7. *For a fixed k and polynomial $g(x)$ in $\mathbf{Z}[x]$, with $g(1)\neq 0$, such that*

$$G(x)=(x-1)^{k+1}g(x)$$

has $\{0,+1,-1\}$ coefficients,

$$\theta(N,k)\geq 1-\frac{c_2}{N^{k+2}}+O\Big(\frac{c_4}{N^{2k+3}}\Big),$$

where

$$c_2=c_2(g,k):=\frac{\big(\partial(G)+1\big)^{k+2}}{|g(1)|},$$

and $c_4=c_4(g,k)$ is independent of N.

SOME NOTES. The polynomials $G(x)=\prod_{i=1}^{k+1}(x^{2^{i-1}}-1)$ give us

$$c_2=2^{\frac{1}{2}(k+1)(k+4)}$$

which, as we shall see, is sharp for $k=0,1$ (but not for higher k). For example when $k=2$ the polynomial $G(x):=(x-1)(x^2-1)(x^3-1)$ gives $c_2=7^4/6$.

In general one expects there to be suitable $G(x)$ of degree $O(k^2)$ (this would be optimal). It can be shown (see for example [5, Theorem 2.7]) there is a $\{0,+1,-1\}$ polynomial of degree $O(k^2\log k)$ with a root of at least, though not necessarily exactly, multiplicity k at 1. Hence, for infinitely many k, we can take

$$c_2\leq\exp\big(2k\log k+O(k\log\log k)\big).$$

This compares favorably with the constant

$$(k+1)!\,4^{k+1}=\exp\big(k\log k+O(k)\big)$$

in the lower bound.

For $k = 0$ and $k = 1$ we can determine the growth precisely:

$$\theta(N, 0) = 1 - \frac{4}{N^2} + O\Big(\frac{1}{N^3}\Big),$$
$$\theta(N, 1) = 1 - \frac{32}{N^3} + O\Big(\frac{1}{N^4}\Big),$$

For $k = 2$ our bounds give

$$\frac{384}{N^4} \leq \theta(N, 2)\big(1 + o(1)\big) \leq \frac{400\frac{1}{6}}{N^4}.$$

In the next section we describe the optimal polynomials explicitly in the cases $k = 0$ or 1.

4. **Precise results for** $k = 0$ **or** 1. Let $F(x; N, k)$ denote a polynomial of degree N with $\{0, +1, -1\}$ coefficients and a root at $\theta(N, k)$.

THEOREM 8. *For $k = 0$ and $N \geq 2$ the extremal polynomials $F(x; N, 0)$ take the form*

$$\pm\frac{(x^{2m+1} - 2x^m + 1)}{(1-x)}, \qquad \text{if } N = 2m,$$

$$\pm\frac{(x^{2m+2} - x^{m+1} - x^m + 1)}{(1-x)}, \qquad \text{if } N = 2m + 1.$$

For $k = 1$ and $N \geq 4$ the extremal polynomials $F(x; N, 1)$ take the form

$$\pm\frac{(x^{4m+1} - 2x^{3m+1} + x^{2m+2} - x^{2m+1} + 2x^m - 1)}{(x-1)}, \qquad \text{if } N = 4m,$$

$$\pm\frac{(x^{4m+2} - x^{3m+2} - x^{3m+1} + x^{m+2} - x^{m+1} + 2x^m - 1)}{(x-1)}, \qquad \text{if } N = 4m + 1,$$

$$\pm\frac{(x^{4m+3} - x^{3m+3} - x^{3m+2} + x^{2m+3} - x^{2m+2} + x^{m+1} + x^m - 1)}{(x-1)}, \qquad \text{if } N = 4m + 2,$$

$$\pm\frac{(x^{4m+4} - 2x^{3m+3} + x^{m+2} + x^m - 1)}{(x-1)}, \qquad \text{if } N = 4m + 3.$$

It is perhaps more enlightening to instead write out the pattern of coefficients $a_0 a_1 \ldots a_N$ of $F(x, N, k) = \sum_{i=0}^N a_i x^i$ (we assume without loss of generality that $a_0 = 1$):

For $k = 0$

$$\underbrace{1 \ldots 1}_{m}\ \underbrace{-1 \ldots -1}_{m+1}, \qquad \text{if } N = 2m,$$

$$\underbrace{1 \ldots 1}_{m}\ 0\ \underbrace{-1 \ldots -1}_{m+1}, \qquad \text{if } N = 2m + 1.$$

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For $k = 1$

$$\underbrace{1\dots 1}_{m}\ \underbrace{-1\dots -1}_{m+1}\ 0\ \underbrace{-1\dots -1}_{m-1}\ \underbrace{1\dots 1}_{m}, \qquad \text{if } N = 4m,$$

$$\underbrace{1\dots 1}_{m}\ -1\ 0\ \underbrace{-1\dots -1}_{2m-1}\ 0\ \underbrace{1\dots 1}_{m}, \qquad \text{if } N = 4m+1,$$

$$\underbrace{1\dots 1}_{m}\ 0\ \underbrace{-1\dots -1}_{m+1}\ 0\ \underbrace{-1\dots -1}_{m-1}\ 0\ \underbrace{1\dots 1}_{m}, \qquad \text{if } N = 4m+2,$$

$$\underbrace{1\dots 1}_{m}\ 0\ 0\ \underbrace{-1\dots -1}_{2m+1}\ \underbrace{1\dots 1}_{m+1}, \qquad \text{if } N = 4m+3.$$

5. **Some special subclasses.** If we restrict ourselves to $\{-1, 1\}$ or $\{0, 1\}$ coefficients then much of the behaviour observed at ± 1 still occurs at ± 1 in the first case and at -1 in the latter. For example if we let $\theta^*(N, k)$ denote the largest real root $\theta < 1$ of any $\{-1, 1\}$ polynomial of degree at most N with a root of order k at 1, and $\theta^\dagger(N, k)$ the smallest real root $\theta > -1$ of any $\{0, 1\}$ polynomial of degree at most N with a root of order k at -1, then for fixed multiplicity k we still have:

THEOREM 9. *For a fixed integer $k \geq 0$,*

$$|\theta^*(N, k) - 1| \asymp \frac{1}{N^{k+2}},$$
$$\left|\theta^\dagger(N, k) + 1\right| \asymp \frac{1}{N^{k+2}}.$$

However the maximum order of vanishing at ± 1 is significantly less for a $\{-1, 1\}$ or $\{0, 1\}$ than for a $\{0, +1, -1\}$ polynomial. Consequently if B_N, B_N^* and $B_N^\dagger$, denote respectively the zeros of all $\{0, +1, -1\}$, $\{-1, +1\}$ and $\{0, 1\}$ polynomials of degree at most N, then the Corollary 1 and Theorem 5 bounds

$$\exp(-c_1 N^{1/2} \log N) \leq \min_{\beta \in B_N \setminus \{1\}} |\beta - 1| \leq \exp\bigl(-c_2 (N \log N)^{1/3}\bigr),$$

must be drastically reduced in these special cases:

THEOREM 10.

$$\exp\left(-c_1 \frac{(\log N)^3}{\log\log N}\right) \leq \min_{\beta \in B_N^* \setminus \{1\}} |\beta - 1| \leq \exp\left(-c_2 (\log N)^2\right),$$

$$\exp\left(-c_1 (\log N)^2\right) \leq \min_{\beta \in B_N^\dagger \setminus \{-1\}} |\beta + 1| \leq \exp\left(-c_2 (\log N)^2\right),$$

for some positive constants c_1, c_2.

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In the $\{0, 1\}$ case we actually obtain the explicit constants

$$c_2 = \big(1 + o(1)\big)(4 \log 3)^{-1}, \quad c_1 = \big(1 + o(1)\big)(\log 2)^{-1}.$$

Although the rates of approximation at +1 or −1 are thus very different in these special cases, since the polynomials with high multiplicity roots form such a small proportion of the polynomials, it is not suprising that the pictures remain similar in appearance (particularly for small degree).

5.1. *Proof of Theorem 1.* Suppose that $F(x) = \sum_{i=0}^{N} a_i x^i$ is a $\{0, +1, -1\}$ polynomial with a k-th order root at α and roots $\beta_1, \dots, \beta_m$. We set

$$G(x) := \frac{F(x)}{(x - \beta_1) \dots (x - \beta_m)},$$

so that

$$|\alpha - \beta_1| \cdots |\alpha - \beta_m| = \left| \frac{F^k(\alpha)/k!}{G^k(\alpha)/k!} \right|.$$

Suppose that $a_d \prod_{i=1}^{d}(x - \alpha_i)$ is the minimal polynomial of α. Then, by integrality,

$$|a_d|^{N-k} \prod_{i=1}^{d} \left| \frac{F^k(\alpha_i)}{k!} \right| \geq 1,$$

where if α is complex with $\alpha = \alpha_1 = \overline{\alpha_2}$

$$\left| \frac{F^k(\alpha)}{k!} \right| = \left(\left| \frac{F^k(\alpha_1)}{k!} \right| \left| \frac{F^k(\alpha_2)}{k!} \right| \right)^{1/2}.$$

Hence if

$$\delta := \begin{cases} 1 & \text{if } \alpha \text{ is real,} \\ 1/2 & \text{if } \alpha \text{ is complex,} \end{cases} \qquad \mu := \begin{cases} 2 & \text{if } \alpha \text{ is real,} \\ 3 & \text{if } \alpha \text{ is complex,} \end{cases}$$

we have

$$|\alpha - \beta_1| \cdots |\alpha - \beta_m| \geq \left(|a_d|^{\delta(N-k)} \left| \frac{G^k(\alpha)}{k!} \right| \prod_{i=\mu}^{d} \left| \frac{F^k(\alpha_i)}{k!} \right|^{\delta} \right)^{-1}.$$

For $|\alpha_i| \leq 1$ we use the trivial bounds

$$\left| \frac{F^k(\alpha_i)}{k!} \right| \leq \begin{cases} (N+1)^{k+1}/k! & \text{if } |\alpha_i| = 1, \\ (1 - |\alpha_i|)^{-(k+1)} & \text{if } |\alpha_i| < 1. \end{cases}$$

For $|\alpha_i| > 1$ we make use of the vanishing of F at α_i. Let

$$H(x) := \frac{F(x)}{\big(1 - (x/\alpha_i)\big)^k}$$

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so that

$$|\alpha_i|^k \left|\frac{F^k(\alpha_i)}{k!}\right| = |H(\alpha_i)|.$$

Now the coefficients of $H(x) = \sum_{j=0}^{N-k} h_j x^j$ plainly satisfy

$$|h_l| = \left|\sum_{j=0}^{l} \binom{j+k-1}{k-1} \alpha_i^{-j} a_{l-j}\right| \leq (1-|\alpha_i|^{-1})^{-k},$$

and

$$\left|\frac{F^k(\alpha_i)}{k!}\right| \leq \frac{|\alpha_i|^{-k}}{(1-|\alpha_i|^{-1})^k} \sum_{j=0}^{N-k} |\alpha_i|^j \leq \frac{|\alpha_i|^{N-k+1}}{(|\alpha_i|-1)^{k+1}}.$$

It remains to estimate $G^k(\alpha)/k!$. Set

$$K(x) := \frac{G(x)}{(x-\alpha)^k} = \frac{F(x)}{(x-\alpha)^k(x-\beta_1)\cdots(x-\beta_m)},$$

so that $G^k(\alpha)/k! = K(\alpha)$.

Notice that if

$$\frac{\sum r_i x^i}{\big(1-(x/u)\big)} = \sum s_i x^i$$

then

$$|s_i| = \left|\sum_{j=0}^{i} r_{i-j} u^{-j}\right| \leq \max_{0\leq j\leq i} |r_j| \begin{cases} (i+1)\max\{1,|u|^{-1}\}^i & \text{for any } u,\\ (1-|u|^{-1})^{-1} & \text{if } |u|>1. \end{cases}$$

Now if $|\alpha| > 1$ we can assume that all the $|\beta_i| > 1$ (otherwise $|\alpha-\beta_i|$ is greater than a constant and we can omit those β_i and adjust the constant accordingly). Hence the coefficients of $K(x) = \sum_{j=0}^{N-m-k} k_j x^j$ clearly satisfy

$$k_j \leq |\alpha|^{-k}(1-|\alpha|^{-1})^{-k} \prod_{i=1}^{m} |\beta_i|^{-1}(1-|\beta_i|^{-1})^{-1},$$

and

$$\left|\frac{G^k(\alpha)}{k!}\right| \leq (|\alpha|-1)^{-k} \prod_{i=1}^{m}(|\beta_i|-1)^{-1} \sum_{j=0}^{N-m-k} |\alpha|^j \leq |\alpha|^{N-m-k+1}(|\alpha|-1)^{-(k+1)} \prod_{i=1}^{m}(|\beta_i|-1)^{-1}.$$

Hence when $|\alpha| > 1$ and all the $|\beta_i| > 1$ we obtain

$$|\alpha-\beta_1|\cdots|\alpha-\beta_m| \geq \frac{C_1(\alpha,m,k,\vec{\beta})}{M(\alpha)^{\delta N}(N+1)^{\delta(k+1)d_1}},$$

where

$$C_1(\alpha,m,k,\vec{\beta}) := B_1(\alpha,k)|\alpha|^m \prod_{i=1}^{m} \big|\,|\beta_i|-1\big|,$$

with

$$B_1(\alpha,k) := |a_d|^{\delta} M(\alpha)^{\delta(k-1)} (k!)^{\delta d_1} \prod_{|\alpha_i|\neq 1} \big|\,|\alpha_i|-1\big|^{\delta(k+1)}.$$

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The result follows since we can clearly assume $|\beta_i|-1>\frac{1}{2}(|\alpha|-1)$ (or else we can omit that term from the product). The result for $|\alpha|<1$ follows by working with α^{-1} and β_i^{-1}.

If $|\alpha|=1$ we similarly see that the coefficients of $G(x)=\sum_{j=0}^{N-m} g_j x^j$ satisfy

$$|g_j| \leq (j+1)^m \prod_{i=1}^{m} \max\{1, |\beta_i|^{-1}\}^j,$$

and hence

$$\left|\frac{G^k(\alpha)}{k!}\right| \leq \prod_{i=1}^{m} \max\{1,|\beta_i|^{-1}\}^{N-m} \frac{(N-m)^k}{k!} \sum_{j=0}^{N-m} (j+1)^m$$
$$\leq \prod_{i=1}^{m} \max\{1,|\beta_i|^{-1}\}^{N-m} \frac{N^{k+m+1}}{k!}.$$

Thus in this case

$$|\alpha-\beta_1|\cdots|\alpha-\beta_m| \geq \frac{C_1(\alpha,m,k,\vec{\beta})}{M(\alpha)^{\delta N}(N+1)^{\delta(k+1)d_1+m}},$$

where

$$C_1(\alpha,m,k,\vec{\beta}) := B_1(\alpha,k)\prod_{i=1}^{m}\min\{1,|\beta_i|\}^{N-m},$$

with $B_1(\alpha,k)$ as above.

The result follows since we can assume that $|\beta_i|>1-(N+1)^{-1}$ (otherwise $|\alpha-\beta_i|>1/(N+1)$ and the result follows by simply omitting the term $|\alpha-\beta_i|$ from the product). ■

5.2. *Proof of Theorem 2.* (i) Observing that the derivatives of H_N satisfy

$$|H_N^j(\alpha)| = O(N^{j+1}|\alpha|^{N-j}),$$

expanding H_N around α gives

$$H_N(x) = (x-\alpha)^k\left(\alpha^{N-\partial(G)}\frac{G^s(\alpha)}{s!}(x-\alpha)^m - \frac{F^k(\alpha)}{k!} + E(x)\right)$$

where, for $N|x-\alpha|<1/2$,

$$E(x) = O(|x-\alpha|) + O(N^{s+2}|\alpha|^N|x-\alpha|^{m+1}).$$

The result follows at once from Rouché's Theorem.

(ii) We write

$$F_N(x) = \sum_{i=0}^{N} c_i x^i$$

for the N-th truncation of F.

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Observing that

$$\left|\frac{F_N^j(\alpha^{-1})}{j!}\right| \le (1-|\alpha|^{-1})^{-(j+1)},$$

and

$$F_N^j(\alpha^{-1}) = F^j(\alpha^{-1}) + O\Big(N^j\alpha^{-N}(1-|\alpha|^{-1})^{-(j+1)}\Big),$$

expanding $F_N(x)$ around α^{-1} gives

$$F_N(x) = (x-\alpha^{-1})^k\frac{F^k(\alpha^{-1})}{k!} + E(x)$$

where

$$E(x) = O\left(\alpha^{-N}\right) + O\left(|x-\alpha^{-1}|^{k+1}\right),$$

for $|x-\alpha^{-1}| \le \frac{1}{2}\min\left\{N^{-1},\ (1-|\alpha|^{-1})\right\}$. The result is plain (with the β_i denoting the reciprocals of roots of F_N). ∎

5.3. *Proof of Corollary 2.* We first recall the definition [13] of the beta-expansion $\{c_n\}$ (of 1) for α;

$$c_n := \lfloor \alpha\gamma_{n-1}\rfloor, \quad \gamma_n := \alpha\gamma_{n-1} - c_n, \quad \gamma_0 := 1.$$

Notice that for α in (1,2) all the $c_i = 0$ or 1.

For $\alpha \neq 2$ we write

$$F(x) := 1 - \sum_{i=1}^{\infty} c_i x^i,$$

so that $F(\alpha^{-1}) = 0$ (the beta-expansion of 1 for α). Moreover by Descartes' Rule of Signs α^{-1} is a simple root (the only real root in (0,1)).

If the sequence $\{c_i\}$ terminates in zeros (that is α is a simple beta number) then α^{-1} is a simple root of the $\{0,+1,-1\}$ polynomial F and the result follows from Theorem 2(i). If the sequence $\{c_i\}$ is infinite then by Theorem 2(ii) the polynomial reciprocal of the N-th truncation of F has a real root $\beta \neq \alpha$ suitably close to α.

We should remark that Parry's proof [12, Theorem 5] of the denseness of the simple beta-numbers in $(1,\infty)$ shows that the β converge to α.

For $\alpha = 2$ we similarly take $F = 1 - \sum_{i=1}^{\infty} x^i$.

Corollary 3 follows at once from the upper bound of Corollary 2 and the lower bound of Theorem 1 on observing that for a Pisot number $M(\alpha) = \alpha$. ∎

5.4. *Proof of Theorem 3.* We assume that $|\alpha| \le 1$. Suppose that L is the maximum multiplicity at α possible for a root of a $\{0,+1,-1\}$ polynomial. We are assuming that $M(\alpha) < 2$ so that $L \ge 1$ but that α is not a root of unity so that $L = L(\alpha)$ is finite. We first use the box principle to show the existence of a $\{0,+1,-1\}$ polynomial F with $F^L(\alpha) = 0$ and $F^j(\alpha)$ small for all $j < L$. The vanishing of the L-th derivative at α is to ensure that at least one of the earlier derivatives is non-vanishing. Suppose that $\alpha_1, \ldots, \alpha_r$ are the real conjugates and $\alpha_{r+1}, \overline{\alpha_{r+1}} \ldots, \alpha_{r+s}, \overline{\alpha_{r+s}}$ the complex conjugates of α. We write $d = r + 2s$ for the degree of α.

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For a polynomial of the form $f=\sum_{i=0}^{N-1}a_ix^i$, with coefficients a_i in $\{0,1\}$, we consider a vector $\overline{u}(f)$ in $\mathbf{R}^{2L+d}$ with components consisting of

$$u(f)_j:=\begin{cases}\operatorname{Re} f^i(\alpha) & \text{if } j=2i+1,\ i=0,\ldots,L-1,\\ \operatorname{Im} f^i(\alpha) & \text{if } j=2i+2,\ i=0,\ldots,L-1,\\ f^L(\alpha_i) & \text{if } j=2L+i,\ i=1,\ldots,r,\\ \operatorname{Re} f^L(\alpha_{r+i+1}) & \text{if } j=2L+r+2i+1,\ i=0,\ldots,s-1,\\ \operatorname{Im} f^L(\alpha_{r+i+1}) & \text{if } j=2L+r+2i+2,\ \ i=0,\ldots,s-1.\end{cases}$$

When α is real we ignore the $\operatorname{Im} f^i(\alpha)$ entries. We set

$$A:=N^{1+(2\delta d/L)}\left(\frac{2}{M(\alpha)}\right)^{-2\delta N/L(L+1)},\qquad c:=(4\sqrt{2})^{1+(\delta d/L)},$$

and assume that $cA\le 1$ (if $cA>1$ then Theorem 3 is immediate). Observing that each of the 2^N polynomials have

$$|f^j(\alpha)|\le N^{j+1},\qquad |f^L(\alpha_i)|\le N^{L+1}\max\{1,|\alpha_i|\}^N,$$

the box principle shows that we must have two $\overline{u}(f_1)$, $\overline{u}(f_2)$, with

$$|u(f_1)_j-u(f_2)_j|<\frac{1}{\sqrt{2}},\qquad j>2L,$$

and

$$|u(f_1)_{2i+1}-u(f_2)_{2i+1}|,\ |u(f_1)_{2i+2}-u(f_2)_{2i+2}|\le\frac{c}{\sqrt{2}}A^{L-i},$$

for $i=0,\ldots,L-1$ (these restrictions requiring the product of

$$\prod_{i=1}^{d}\left(\left\lfloor\sqrt{2}\,2N^{L+1}\max\{1,|\alpha_i|\}^N\right\rfloor+1\right)\prod_{j=0}^{L-1}\left(\left\lfloor\sqrt{2}c^{-1}A^{j-L}2N^{j+1}\right\rfloor+1\right)^{1/\delta}<2^N$$

boxes). Hence $F=f_1-f_2$ will be a $\{0,+1,-1\}$ polynomial of degree at most $(N-1)$ with

$$|F^j(\alpha)|\le cA^{L-j},\quad j=0,\ldots,L-1,\qquad |F^L(\alpha_i)|<1,\quad i=1,\ldots,d.$$

Since $\prod_{i=1}^d|F^L(\alpha_i)|$ is an integer we must certainly have $F^L(\alpha)=0$. Moreover, since F cannot have a root of order $(L+1)$ at α, we must have $F^J(\alpha)\ne 0$ for some $0\le J\le L-1$.

Suppose that $G(x)$ is a fixed polynomial with a root of order L at α and consider

$$H(x)=x^{\partial(G)}F(x)+G(x).$$

Then H is a $\{0,+1,-1\}$ polynomial with

$$H(x)=(x-\alpha)^J\left(\alpha^{\partial(G)}\frac{F^J(\alpha)}{J!}+\frac{G^L(\alpha)}{L!}(x-\alpha)^{L-J}+E(x)\right)$$

where

$$E(x)=O(N^{L+1}|x-\alpha|^{L-J+1})+\sum_{j=J+1}^{L-1}O\big(A^{L-J}(A^{-1}|x-\alpha|)^{j-J}\big),$$

for $N|x-\alpha|<1/2$.

Hence by Rouché's Theorem H has $(L-J)$ roots in the disc $|x-\alpha|\le CA$ for a sufficiently large constant $C=C(L)$. ∎

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5.5. *Proof of Theorem 4.* Suppose that α is a d-th root of unity. We first construct $\{0,+1,-1\}$ polynomials of degree N, with specified vanishing at α, whose first non-vanishing derivative is large:

For a constant c and fixed k we set

$$g_k(x) = x^{D+1}\frac{(x^D-1)}{(x-1)}\prod_{i=0}^{k-1}(x^{2^iD}-1), \quad D := \lfloor N/dc(2^k+1)\rfloor$$

and take

$$G_k(x) = x^m g_k(x^{dc}), \quad m := N-(2^k+1)dcD.$$

Hence $G_k(x)$ is a $\{0,+1,-1\}$ polynomial of degree N with a root of order k at α and

$$\frac{G^k(\alpha)}{k!} = 2^{\frac{1}{2}k(k-1)}D^{k+1}(dc)^k\alpha^{m-k}.$$

We next show the existence of $\{0,+1,-1\}$ polynomials with a prescribed order of vanishing at α whose first non zero derivative at α is small.

We first suppose that $\phi(d)\neq 1,2$. Let $\mathcal{B}$ be a set of positive integers with $b\leq B$ for each of the b in $\mathcal{B}$, then, by the box principle, there are certainly integers

$$a_i = b_i - b_i', \quad b_i, b_i' \in \mathcal{B},$$

not all zero, such that

$$0 < \left|\sum_{j=0}^{\phi(d)-1} a_j\alpha^j\right| \leq \frac{2\sqrt{2}\phi(d)B}{|\mathcal{B}|^{\frac{1}{2}\phi(d)}}.$$

The non-vanishing is immediate since the a_i are integers (not all zero) and the degree of α over $\mathbf{Q}$ is $\phi(d)$.

We set $M = \lfloor D/2^{k+1}(k+1)\rfloor$ and take

$$\mathcal{B} := \{b \ : 0\leq b < M^{k+1}\}, \quad B := M^{k+1}.$$

Now for any $0\leq b< M^{k+1}$ we can write

$$b = \sum_{l=0}^{k} b_l M^l, \quad 0\leq b_l < M,$$

and hence construct a $\{0,+1,-1\}$ polynomial

$$F(x;b) := \sum_{l=0}^{k}\Big(\prod_{j=0}^{k-l-1} x^{2^j}-1\Big)\Big(\sum_{t=0}^{b_l-1} x^{t2^{k-l}}\Big)\Big(\prod_{j=k-l}^{k-1} x^{M2^j}-1\Big)x^{Ml2^k},$$

of degree

$$\partial\big(F(x;b)\big) < 2^k(k+1)M =: L,$$

with a k-th order root at 1 and

$$\frac{F^k(1;b)}{k!} = 2^{\frac{1}{2}k(k-1)}b.$$

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With the integers b_i, b_i' from the box principle we set

$$F_k(x) = \sum_{j=0}^{\phi(d)-1} \big(F(x^d; b_j) - x^{Ld} F(x^d; b_j')\big) x^j.$$

Then $F_k(x)$ is a $\{0, +1, -1\}$ polynomial of degree

$$\partial(F_k) \leq 2Ld \leq Dd,$$

with a k-th order root at α and

$$\begin{aligned}\left|\frac{F^k(\alpha)}{k!}\right| &= \left|\alpha^{k(d-1)} d^k \sum_{j=0}^{\phi(d)-1} (b_j - b_j')\alpha^j\right| \\ &\leq \frac{2\sqrt{2}\phi(d)d^k}{M^{(k+1)(\frac{1}{2}\phi(d)-1)}}.\end{aligned}$$

For $\phi(d) = 1$ or 2 we simply set $F_k(x) = \prod_{i=0}^{k-1}(x^{2^i} - 1)$.

Hence in each case $F_k(x)$ is a $\{0, +1, -1\}$ polynomial of degree at most Dd with a k-th order root at α and

$$\left|F_k^k(\alpha)\right| \leq C(k, \alpha) \frac{D^{k+1}}{D^{(k+1)\lceil \frac{1}{2}\phi(d)\rceil}}.$$

We set

$$H_k(x) = F_k(x) + G_{k+1}(x)$$

and observe that

$$H_k(x) = (x - \alpha)^k \left(\frac{F^k(\alpha)}{k!} + \frac{\big(F_k^{k+1}(\alpha) + G_{k+1}^{k+1}(\alpha)\big)}{(k+1)!}(x - \alpha) + E(x) \right)$$

with

$$E(x) = O\big((Dd)^{k+3}|x - \alpha|^2\big), \qquad \text{for } |x - \alpha| \leq \frac{1}{2N}.$$

Since $|F_k^{k+1}(\alpha)| \leq (Dd)^{k+2}$ it is clear by Rouché's Theorem that for a suitably large constant $c = c(d)$ the polynomials $H_k(x)$ have a root β_k with

$$\begin{aligned}\beta_k - \alpha &= -\frac{(k+1)F_k^k(\alpha)}{\big(G_{k+1}^{k+1}(\alpha) + F_k^{k+1}(\alpha)\big)} \left(1 + O\left(\frac{1}{D^{(k+1)\lceil \frac{1}{2}\phi(d)\rceil}}\right)\right) \\ &= O\left(\frac{1}{D^{(k+1)\lceil \frac{1}{2}\phi(d)\rceil+1}}\right).\end{aligned}$$

∎

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5.6. *Proof of Theorem 5.* When α is not a root of unity the result follows from Theorems 2 and 3.

It is plainly enough to show the existence of a $\{0, +1, -1\}$ polynomial G of degree at most N with a root $\beta \neq 1$ satisfying

$$|\beta - 1| \leq \exp\bigl(-c(N \log N)^{1/3}\bigr),$$

the result for a general d-th root of unity then following from considering the polynomial $G(x^d)$.

Suppose that we have a $\{0, +1, -1\}$ polynomial F of degree $(M-1)$ with a root of order exactly L at 1. Then

$$G(x) := x^M(x^{MD} - 1)F(x^D)\left(\frac{x^D - 1}{x - 1}\right) - F(x)$$

is a $\{0, +1, -1\}$ polynomial with a root of order L at 1 and degree $N < 3MD$.

Expanding G around 1 and using the trivial bounds

$$|G^j(1)| \leq (3MD)^{j+1},$$

it is easy to see that for $(3MD)|x - 1| < 1/2$ we have

$$G(x) = (x-1)^L \frac{F^L(1)}{L!}\bigl(-1 + MD^{L+2}(x-1) + E(x)\bigr)$$

where, since $|F^L(1)|/L! \geq 1$,

$$E(x) = O\left(\frac{M^{L+1}}{(L+1)!}|x-1|\right) + O\left(\frac{(3MD)^{L+3}}{(L+2)!}|x-1|^2\right).$$

Observing that the choice

$$D = \left\lfloor \frac{3eM}{L} \right\rfloor + 1$$

gives

$$\frac{1}{(L+1)!}\left(\frac{3M}{D}\right)^{L+1} = O(1)$$

and hence

$$E(x) = O\left(\frac{1}{(MD)}(MD^{L+2}|x-1|)\right) + O\left(\frac{1}{L}(MD^{L+2}|x-1|)^2\right),$$

it is easily seen that G has a root $\beta \neq 1$ with

$$\beta - 1 = \frac{1}{MD^{L+2}}\left(1 + 0\left(\frac{1}{L}\right)\right).$$

Now we can assume (see for example [5, Theorem 2.7]) that

$$L > c_1\sqrt{\frac{M}{\log M}}$$

for some absolute constant c_1, so that

$$N \le c_2\frac{M^2}{L} \le c_3 M^{3/2}(\log M)^{1/2}$$

and

$$|\beta - 1| \le \exp\bigl(-c_4 L\log(M/L)\bigr) \le \exp\left(-c_5\sqrt{M\log M}\right) \le \exp\bigl(-c_6(N\log N)^{1/3}\bigr),$$

as required. ∎

5.7. *Proof of Theorem 6.* We need a preliminary lemma:

LEMMA 1. *Suppose that the polynomial F has bounded coefficients $|a_i| \le A$. Then for a fixed positive integer k we have*

$$|F^k(1)| \le \frac{A}{4^k}N^{k+1}\left(1 + O_{p,k}\left(\frac{1}{N}\right)\right) + \sum_{j=0}^{k-1} O_{p,k}\bigl(N^{k-j}|F^j(1)|\bigr).$$

Here $O_{p,k}$ denotes that the implied constant in the order result is permitted to depend on p and k.

PROOF. Setting

$$Q_j(x) := \prod_{i=0}^{j-1}(x - i), \quad Q_0(x) := 1$$

it is readily seen that the $Q_j(x)$ can be written

$$Q_j(x) = x^j + \sum_{i=0}^{j-1}\gamma_{ij}Q_i(x)$$

for appropriate constants γ_{ij}.

Hence in particular, if $F(x) = \sum_{s=0}^{N} a_s x^s$,

$$\begin{aligned} F^j(1) &= \sum_{s=0}^{N} a_s Q_j(s) \\ &= \sum_{s=0}^{N} a_s s^j + \sum_{i=0}^{j-1}\gamma_{ij}F^i(1). \end{aligned}$$

Thus for a fixed polynomial $p(x) = x^k + \sum_{j=0}^{k-1} b_j x^j$ we have

$$S := N^k\sum_{s=0}^{N} a_s p(s/N)$$

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$$= \sum_{j=0}^{k} b_j N^{k-j}\Big(F^j(1) - \sum_{i=0}^{j-1} \gamma_{ij} F^i(1)\Big)$$

$$= \sum_{j=0}^{k} F^j(1)\Big(b_j N^{k-j} - \sum_{i>j} \gamma_{ji} b_i N^{k-i}\Big)$$

$$= F^k(1) + \sum_{j=0}^{k-1} O\Big(C(j,k,p) N^{k-j} |F^j(1)|\Big).$$

It is easily seen that

$$S \le AN^{k+1} \sum_{s=0}^{N} \left| p\left(\frac{s}{N}\right) \right| \left(\frac{1}{N}\right)$$

$$= AN^{k+1} \left(\int_0^1 |p(x)|\, dx\right)\left(1 + O_p\left(\frac{1}{N}\right)\right).$$

The result follows on observing that, by a classical result of Korkin and Zolotarev [7],

$$\inf_p \int_0^1 |p(x)|\, dx = 4^{-k},$$

achieved for

$$p(x) = \frac{1}{4^k} U_k(2x-1)$$

where U_n denotes the n-th Chebyshev polynomial of the second kind

$$U_n(x) := \frac{\sin(n \arccos x)}{\sin(\arccos x)}.$$

∎

PROOF OF THEOREM 6. Taking the Taylor expansion of F around 1 we have

$$-\frac{F^k(1)}{k!} = \sum_{j=k+1}^{N} \frac{F^j(1)}{j!}(\theta - 1)^{j-k} \qquad (1)$$

We may clearly assume that

$$|\theta - 1| < \frac{c(k)}{N^{k+2}} < \frac{1}{2N},$$

for some suitably large $c(k)$ else there is nothing to show. Hence, from the trivial bound

$$|F^j(1)| \le N^{j+1}$$

we have

$$\Big|\sum_{j=k+2}^{N} \frac{F^j(1)}{j!}(\theta-1)^{j-k}\Big| \le \frac{2}{(k+2)!}|\theta-1|^2 N^{k+3} = O(|\theta-1| N^{k+1}),$$

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where the implied constant in the O is allowed to depend on k. From the above lemma we have

$$\left|\frac{F^{k+1}(1)}{(k+1)!}\right| \leq \frac{N^{k+2}}{(k+1)!\,4^{k+1}}\left(1+O\left(\frac{1}{N}\right)\right) + O\left(N\frac{|F^k(1)|}{k!}\right).$$

So (1) becomes

$$\left|\frac{F^k(1)}{k!}\right|\left(1+O\left(\frac{1}{N^{k+1}}\right)\right) \leq \frac{N^{k+2}}{(k+1)!\,4^{k+1}}\left(1+O\left(\frac{1}{N}\right)\right)|\theta-1|,$$

and the result follows on observing that $\left|F^k(1)\right|/k!$ is an integer and hence at least 1. ∎

5.8. *Proof of Theorem 7.* The lower bound is completely constructive:

LEMMA 2. *Suppose that $f(x)$ and $g(x)$ are integer polynomials, with $f(1), g(1) \neq 0$, such that the polynomials*

$$F(x) := (x-1)^k f(x), \quad G(x) := (x-1)^{k+1} g(x)$$

have $\{0, +1, -1\}$ coefficients.

Then for

$$N \geq \left(\partial(F) + \partial(G) + 1\right)$$

the polynomial

$$H(x) := x^e G(x^d)\left(\frac{x^d-1}{x-1}\right) + F(x),$$

where

$$d := \left\lfloor \frac{N-\partial(F)}{\partial(G)+1} \right\rfloor, \quad e := N - d\partial(G) - (d-1),$$

is a $\{0, +1, -1\}$ polynomial of degree N with a k-th order root at 1 and a root of size

$$1 - \frac{c_2}{N^{k+2}} + O\left(\frac{c_4}{N^{2k+3}}\right),$$

where

$$c_2 = c_2(f, g, k) := \frac{f(1)}{g(1)}\left(\partial(G)+1\right)^{k+2},$$

and $c_2 = c_2(f, g, k)$ is independent of N.

PROOF. From the Taylor expansions about 1 we have;

$$H(x) = (x-1)^k\left\{f(1) + (x-1)d^{k+2}g(1) + E(x)\right\},$$

where, for $|x-1| < \frac{1}{d}$,

$$E(x) = O(|x-1|) + O(d^{k+3}|x-1|^2).$$

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From examining sign changes, H must have a root at

$$(x-1)=\frac{-f(1)}{d^{k+2}g(1)}\big(1+O(d^{-(k+1)})\big),$$

as claimed. ∎

To show Theorem 7 there remains only to justify that for any k we can always find a suitable polynomial f with $f(1)=1$. This is immediate from the following simple construction:

Let $n_1,\ldots,n_k$ and $m_1,\ldots,m_k$ be two sets of integers satisfying

$$(n_1\cdots n_k, m_1\cdots m_k)=1$$

and

$$n_t > n_1+\cdots+n_{t-1}, \quad m_t > m_1+\cdots+m_{t-1}$$

for all $t=2,..,,k$.

Let A and B be two positive integers such that

$$An_1\cdots n_k - Bm_1\cdots m_k = 1$$

then, writing

$$u := 1+n_1+\cdots+n_k, \quad v := 1+m_1+\cdots+m_k,$$

the polynomials

$$F(x) := \left(\frac{x^{Au}-1}{x^u-1}\right)\prod_{i=1}^{k}(x^{n_i}-1) =: (x-1)^k f(x),$$
$$G(x) := \left(\frac{x^{Bv}-1}{x^v-1}\right)\prod_{i=1}^{k}(x^{m_i}-1) =: (x-1)^k g(x),$$

have $\{0,+1,-1\}$ coefficients and

$$U(x) := x^{\partial(G)+1}F(x)-G(x) =: (x-1)^k u(x),$$

is a $\{0,+1,-1\}$ polynomial with $u(1)=1$. ∎

5.9. *Proof of Theorem 8.* Suppose that

$$F(x)=\sum_{i=0}^{N}a_i x^i$$

is a $\{0,+1,-1\}$ polynomial with the extremal root $\theta(N,k)$ in (0,1). We shall use simple perturbation ideas to show that the coefficients must have the stated patterns.

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THE CASE $k=0$. We assume (taking $\pm F(x)$ as necessary) that $F(1)<0$.

Suppose that $n \geq 0$ is such that

$$a_n = -1 \text{ or } 0, \quad a_m = 1 \text{ for all } m < n,$$

then

$$a_{n+t} = -1, \text{ for all } t \geq 1,$$

since otherwise

$$\tilde{F}(x) = F(x) + x^n(1-x^t)$$

would be a $\{0,+1,-1\}$ polynomial with

$$\tilde{F}(1) = F(1) < 0, \quad \tilde{F}(\theta) = \theta^n(1-\theta^t) > 0,$$

and hence would have a root in $(\theta,1)$ contradicting the maximality of θ.

Clearly to have any positive real roots $F(x)$ must have at least one sign change, and thus the coefficients must take the form;

$$1\ldots1\ \{\ 0 \text{ or } 1\ \}\ -1\ldots-1.$$

Now $F(1)=-1$ otherwise, taking an n with $a_n = 0$ or -1,

$$\tilde{F}(x) = F(x) + x^n$$

would have $\{0,+1,-1\}$ coefficients,

$$\tilde{F}(1) = F(1)+1 < 0, \quad \tilde{F}(\theta) = \theta^n > 0,$$

and a larger root in (0,1). The form given follows immediately.

THE CASE $k=1$. We suppose (taking $\pm F$ as needed) that

$$F(x) = (x-1)f(x), \quad f(1) > 0.$$

Now if $n \geq 1$ is such that

$$a_n = 1, \quad a_{n-1} = 0 \text{ or } -1,$$

then

$$a_{n+t} = 1, \quad 0 \leq t \leq N-n.$$

To see this suppose that for some $r \geq 1$

$$a_{n+r} = 0 \text{ or } -1, \quad a_{n+j} = 1,\ 0 \leq j < r,$$

then

$$\begin{aligned} \tilde{F}(x) &:= F(x) + x^{n+r} - x^{n+r-1} - x^n + x^{n-1} \\ &= (x-1)\big(f(x) + x^{n-1}(x^r-1)\big) =: (x-1)\tilde{f}(x), \end{aligned}$$

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has $\{0, +1, -1\}$,

$$\tilde{f}(1) = f(1) > 0, \quad \tilde{f}(\theta) = -\theta^{n-1}(1-\theta^r) < 0,$$

contradicting the maximality of θ.

In the same way if

$$a_m = 1, \quad a_{m+1} = 0 \text{ or } -1,$$

then

$$a_{m-t} = 1, \quad 0 \leq t \leq m,$$

using the perturbed polynomials

$$\tilde{F}(x) = F(x) + x^{m-r}(x^r - 1)(x-1).$$

By Descartes' Rule of Signs the coefficients of F must have at least two sign changes, and therefore must take the form

$$1\ldots 1\ \{0 \text{ or } -1\}\ldots\{0 \text{ or } -1\}\ 1\ldots 1.$$

Further we cannot have the configurations

$$a_{r-1} = 0 \text{ or } -1, \quad a_r = 0, \quad a_{r+s} = 0, \quad a_{r+s+1} = 0 \text{ or } -1,$$

else the polynomial

$$\tilde{F}(x) = F(x) + x^{r-1}(x-1)(x^{s+1}-1)$$

would have a larger root than θ. Hence the coefficients of F take the form

$$1\ldots 1\ \{0 \text{ or } -1\} - 1\ldots - 1\ \{0 \text{ or } -1\} - 1\ldots - 1\ \{0 \text{ or } -1\}\ 1\ldots 1.$$

Finally we must also have

$$F'(1) = f(1) = 1,$$

since if $f(1) \geq 2$ then, taking an n such that $a_n = 1$, $a_{n-1} = 0$ or -1, we could perturb

$$\tilde{F}(x) = F(x) + x^{n-1}(1-x) = (x-1)\big(f(x) - x^{n-1}\big),$$

to obtain a larger root.

There remains only to show algebraically the exact form of F:

We know from the above that F must take the form

$$F(x) = 1 + \cdots + x^{m-1} - x^m - \cdots - x^{m+k-1} + x^{m+k} + \cdots + x^{m+k+l-1} + \lambda_1 x^{m+j} + \lambda_2 x^m + \lambda_3 x^{m+k-1}$$

for some $\lambda_1, \lambda_2, \lambda_3 = 0$ or 1, and positive integers $m, j, k, l \geq 1$ such that

$$F(1) = 0, \quad F'(1) = 1.$$

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Thus, writing

$$\lambda := \lambda_1 + \lambda_2 + \lambda_3,$$

we have

$$l = k - m - \lambda$$

giving

$$k \geq 2 + \lambda$$

and, after some rewriting,

$$F'(1) = \left(k - \frac{1}{2}\lambda\right)\left(k - 2m - \frac{3}{2}\lambda + \lambda_3\right) + \frac{1}{4}(2-\lambda)(\lambda - 2\lambda_3) + \lambda_1 j = 1.$$

If $\lambda_1 = 0$ then

$$\begin{aligned}
(\lambda_2, \lambda_3) = (1,0) &\Rightarrow (2k-1)(2k-4m-3) = 3, k \geq 3,\\
(\lambda_2, \lambda_3) = (0,1) &\Rightarrow (2k-1)(2k-4m-1) = 5, k \geq 3,\\
(\lambda_2, \lambda_3) = (1,1) &\Rightarrow (k-1)(k-2m-2) = 1, k \geq 4,\\
(\lambda_2, \lambda_3) = (0,0) &\Rightarrow k(k-2m) = 1, k \geq 2,
\end{aligned}$$

and plainly the only solution is given by $(\lambda_2, \lambda_3) = (0,1)$, $k = 3$, $m = 1$, $l = 1$, corresponding to $N = 4$.

Hence we can assume that $\lambda_1 = 1$. Now from the bounds

$$1 \leq \lambda_1 j \leq k - 2, \quad -\frac{1}{4} \leq \frac{1}{4}(2-\lambda)(\lambda - 2\lambda_3) \leq \frac{1}{4},$$

it is not hard to see that if $\left(k - 2m - \frac{3}{2}\lambda + \lambda_3\right) \geq \frac{1}{2}$ then

$$F'(1) \geq \frac{1}{2}\left(k - \frac{1}{2}\lambda\right) - \frac{1}{4} + 1 \geq \frac{1}{2}\left(2 + \frac{1}{2}\lambda\right) + \frac{3}{4} \geq 2$$

while if $(k - 2m - \frac{3}{2}\lambda + \lambda_3) \leq -1$ then

$$F'(1) \leq -\left(k - \frac{1}{2}\lambda\right) + \frac{1}{4} + (k-2) \leq -\frac{1}{4}.$$

So $k = \left\lfloor 2m + \frac{3}{2}\lambda - \lambda_3 \right\rfloor$ and

$$\begin{aligned}
(\lambda_2, \lambda_3) = (1,0) &\Rightarrow k = 2m+3,\ j = 1,\ l = m+1,\ (N = 4m+3),\\
(\lambda_2, \lambda_3) = (0,1) &\Rightarrow k = 2m+2,\ j = 1,\ l = m,\ (N = 4m+1),\\
(\lambda_2, \lambda_3) = (1,1) &\Rightarrow k = 2m+3,\ j = m+2,\ l = m,\ (N = 4m+2),\\
(\lambda_2, \lambda_3) = (0,0) &\Rightarrow k = 2m+1,\ j = m+1,\ l = m,\ (N = 4m),
\end{aligned}$$

giving the polynomials of the stated forms. ∎

5.10. *Proof of Theorem 9.* The lower bound is immediate from Theorem 6. For the upper bound we use the $\{-1, +1\}$ and $\{0, 1\}$ polynomials

$$F_k^*(x) := \prod_{i=0}^{k-1}(x^{2^i} - 1), \quad F_k^\dagger(x) := \prod_{i=0}^{k-1}(x^{3^i} + 1),$$

to form the $\{-1, +1\}$ and $\{0, 1\}$ polynomials

$$H^*(x) := x^{2^k} F_{k+1}^*(x^d)\left(\frac{x^d - 1}{x - 1}\right) + F_k^*(x)$$

and

$$H^\dagger(x) := x^{3^k} F_{k+1}^\dagger(x^d)\left(\frac{x^{2\lfloor d/2\rfloor} - 1}{x^2 - 1}\right) + F_k^\dagger(x), \quad d \text{ odd},$$

and proceed just as in Lemma 2.

As regards the implied constants in these bounds notice that the lower bound constant $4^{k+1}(k+1)!$ still holds in these cases, while in place of Theorem 7 the above polynomials readily yield

$$c_2^* \le 2^{k^2+2k+2}, \quad c_2^\dagger \le \frac{(3^{k+1} + 1)^{k+2}}{2^{k+1}3^k},$$

(although we have made no attempt to obtain optimal constants here). ∎

5.11. *Proof of Theorem 10.* The lower bounds follow from Theorem 1, a result of Boyd [6] showing that the order of vanishing of a $\{-1, +1\}$ polynomial at 1 satisfies

$$k \ll \frac{(\log N)^2}{\log\log N},$$

and a simple observation of Borwein-Erdélyi-Kós [5] that for a $\{0, 1\}$ polynomial the order of vanishing at -1 satisfies

$$k \le \frac{\log(N + 1)}{\log 2}.$$

For the upper bounds we follow the Proof of Theorem 5.

In the $\{-1, 1\}$ case we take F to be the $\{-1, 1\}$ polynomial

$$F := \prod_{i=0}^{L-1}(x^{2^i} - 1), \quad M = 2^L,$$

and

$$L := \left\lfloor \frac{\log(N/18e)}{\log 4} \right\rfloor, \quad D := \left\lfloor \frac{3eM}{L} \right\rfloor + 1,$$

so that

$$G(x) := x^M(x^{MD} - 1)F(x^D)\left(\frac{x^D - 1}{x - 1}\right) - F(x),$$

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is a $\{-1,1\}$ polynomial of degree at most N.

In the $\{0,1\}$ case we take

$$F := \prod_{i=0}^{L-1}(x^{3^i}-1), \quad M = \binom{3^L+1}{2},$$

choose

$$L := 2\left\lfloor \frac{\log(N/27e)}{2\log 9} \right\rfloor, \quad D := 2\left\lfloor \frac{3eM}{2L} \right\rfloor + 1,$$

and set

$$G(x) := x^M(x^{MD}-1)F(x^D)\left(\frac{x^{2\lfloor D/2\rfloor}-1}{x^2-1}\right) - F(x).$$

Hence M and D are odd and $-G(-x)$ is a $\{0,1\}$ polynomial of degree at most N.

It is readily checked (in the manner of the proof of Theorem 5) that in both cases $G(x)$ has a root $\beta \neq 1$ with

$$|\beta - 1| = O\Big(\exp\Big(-\big(1+o(1)\big)L\log D\Big)\Big) = O\Big(\exp\big(-c_2(\log N)^2\big)\Big),$$

for some constant $c_2 > 0$. ∎

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