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FUCHSIAN SUBGROUPS OF THE PICARD GROUP

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1. Introduction. The Picard group $\Gamma = PSL_2\ (Z(i))$ is the group of linear transformations

$$Z' = \frac{az+b}{cz+d} \quad ad - bc = \pm 1 \quad \text{with } a, b, c, d \text{ Gaussian integers.}$$

Γ is of interest both as an abstract group and in automorphic function theory [**10**]. In [**10**] Waldinger constructed a subgroup H of finite index which is a generalized free product, while in [**1**] Fine showed that Γ is a semidirect product with the subgroup H, contained as a subgroup of finite index in the normal factor. Tretkoff [**2**] used these to show that Γ is SQ-universal, while Mennicke [**8**] and Maskit [**7**] used Γ to generate faithful representations of fundamental groups of Riemann surfaces. Most recently Karrass and Solitar [**4**] characterized abelian subgroups of Γ. In this paper we will show that Γ is given directly as a generalized free product, and use this to characterize Fuchsian subgroups. In order to do this we find the conjugacy classes of elliptic and parabolic maps.

2. Decomposition of Γ. Γ contains the modular group $PSL_2(Z)$ as a subgroup. $PSL_2(Z)$ is known to be a free product [**5**]. Here we show that group theoretically is similar to the modular group.

THEOREM 1. *Γ is given directly as a free product of two groups G_1, G_2 with amalgamated subgroup H; G_1 is the free product of a symmetric group Σ_3 and the alternating group A_4 with a 3-cycle amalgamated, while G_2 is $\Sigma_3 * D_2$ with a 2-cycle amalgamated.*

The amalgamated subgroup $H \cong PSL_2(Z)$ {modular group}.

Proof. The result follows from a presentation of the Picard group Γ given by G. Sansone [**9**].

$$\Gamma = \{A, B, C, D; A^3 = B^2 = C^2 = D^2 = (AC)^2 = (AB)^2 = (CD)^2 = (BD)^2 = 1\}$$

where A is the transformation $z' = 1/(z+1)$, B is $z' = 1/z$, C is $z' = 1/(z+i)$, and D is $z' = -1/z$.

(i) We let $G_1 = \{A, B, C; A^3 = B^2 = (AB)^2 = 1, A^3 = C^3 = (AC)^2 = 1\} \cong \Sigma_3 * A_4$ with a 3-cycle amalgamated, and let $G_2 = \{B, C, D; B^2 = D^2 = (BD)^2 = 1, D^2 = C^3 = (CD)^2 = 1\} \cong \Sigma_3 * D_2$ with a 2-cycle amalgamated. Then Γ is $G_1 * G_2$ with the identifications $B = B$, $C = C$.

(ii) In G_1, the subgroup generated by B, C is their free product, $z_2 * z_3$ while

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this is also true in G_2; these follow from combinatoral arguments. Therefore the identifications induce isomorphisms and Γ is a generalized free product with the subgroup $H \cong z_2 * z_3 \cong PSL_2(Z)$ amalgamated.

A decomposition of Γ as a free product with amalgamation was computed in a different manner by Karrass & Solitar [**4**]. This was used to investigate abelian and nilpotent subgroups of Γ.

3. Conjugacy classes. Γ is also presented by [**1**] $\{a, l, t, u; a^2 = l^2 = (al)^2 = (tl)^2 = (ul)^2 = (at)^3 = (ual)^3 = tut^{-1}u^{-1} = 1\}$ where a is $z' = -1/z$, $l : z' = -z$, $t : z' = z + 1$, $u : z' = z + i$. We will be using this presentation as well as the previous one. Now although Γ is similar as a group to $PSL_2(Z)$ they differ greatly in their action on the complex plane **C**. While $PSL_2(Z)$ is discontinuous in the upper half-plane and Fuchsian with the real line as a fixed circle [**5**], Γ is nowhere discontinuous in **C** and therefore has no Fuchsian subgroups of finite index [**5**]. If C is a circle we let $P(C)$ be the Fuchsian Stabilizer in C in Γ {the subgroup of Γ which maps both C and the interior of C on itself} and $P_N(C)$ the normal closure in Γ of $P(C)$. Waldinger [**10**], exhibited several classes of circles C_i for which $|\Gamma : P_N(C_i)|$ is finite. We extend these results. To do this we must find the conjugacy classes of elliptic and parabolic elements in Γ.

THEOREM 2. *There are only seven conjugacy classes of elliptic elements in* Γ*; five for those of order* 2, *and two for those of order* 3. *In particular, any elliptic map of order is conjugate to one of:* $l(z) = -z$, $al(z) = 1/z$, $ul(z) = -z + i$, $tl(z) = -z + 1$, $utl(z) = -z + (1 + i)$, *while any elliptic map of order* 3 *is conjugate to* $at(z) = -1/(z + 1)$, *or* $ual(z) = 1/(z + i)$.

Proof. From Theorem 1, $\Gamma \cong G_1 * G_2$ with H amalgamated.

$$G_1 \cong \{A, B, C; A^3 = B^2 = AB^2 = 1, A^3 = C^3 = (AC)^2 = 1\}$$
$$G_2 \cong \{B, C, D; B^2 = D^2 = (BD)^2 = 1, B^2 = C^3 = (CD)^2 = 1\}$$

with A, B, C, D as before.

Now in a generalized free product $* (G_1, G_2; H)$ any element of finite order is conjugate to an element of finite order in one of the factors [**6**]. Therefore to find the conjugacy classes of elliptic elements in Γ, we must find the conjugacy classes of elements of finite order in G_1 and in G_2.

Now

$$G_1 \cong \{A, B, C; A^3 = B^2 = (AB)^2 = 1, A^3 = C^3 = (AC)^2 = 1\}$$
$$\cong \{A, B; A^3 = B^2 = (AB)^2 = 1\} * \{A, C; A^3 = C^3 = (AC)^2 = 1\}$$

with the amalgamation $A = A$. Therefore by the same argument as above, to find the conjugacy classes in G_1, we must find the conjugacy classes in the factors of G_1.

Now $\{A, B; A^3 = B^2 = (AB)^2 = 1\} \cong \Sigma_3$ has two conjugacy classes of elements of order 2 with representatives B, AB, and one conjugacy class of elements of order 3 with representative A. While in $\{A, C; A^3 = C^3 = (AC)^2 =$

$1\} \cong A_4$, there is one conjugacy class of elements of order 2 with representative AC and two conjugacy classes in order 3 with representatives A, C. Hence in G_1, there are three conjugacy classes of elements of order 2 with representatives B, AB, AC, and two conjugacy classes in order 3 with representatives A, C.

In an identical manner we investigate $G_2 \cong \{B, C, D \cdot B^2 = D^2 = (BD)^2 = (CD)^2 = 1\}$ to find there are four classes of elements in order 2 with representatives B, D, BD, CD, and one class in order 3 with representative C. Therefore, in Γ as a whole we have the following representatives for elliptic maps; B, AB, AC, D, BD in order 2 and A, C in order 3.

B is $z' = 1/z = al(z)$; AB is $z' = -z/(z+1) = ala(z)$ which is conjugate (by a) to $tl(z) = -z + 1$; AC is $z' = -z/(l+i)(z+1) =$ aut $la(z)$ which is conjugate to utl; BD is $z' = -z = l(z)$ while D is $z' = -1/z = a(z)$. But from $(ual)^3 = (ul)^2 = (al)^2 = l^2 = 1$ it follows that $uau^{-1}aua = l$; hence (uau^{-1}) $a(uau^{-1}) = lu^{-1} = ul$ and so a is conjugate to $ul(z) = -z + i$.

In order 3, A is $z' = -1/(z+1) = at(z)$ while C is $z' = 1/(z+i) = ual(z)$.

We would like to note that there is also a direct proof of the above theorem which does not involve the decomposition of Γ as a generalized free product. This entails the minimizing of elements in matrices and while is lengthier does have the property that it will generalize to certain other matrix groups which do not necessarily decompose as generalized free products.

Using the above theorem we get

THEOREM 3. *If G is a normal subgroup of Γ and G contains an elliptic element then the index $|\Gamma : G|$ is finite.*

Proof. Say T is elliptic and $T \in G$. Since $G \lhd \Gamma$, then $G \supseteq N(T)$ {normal closure in Γ of T} implies $|\Gamma : G| \leqq |\Gamma : N(T)|$. We will show $|\Gamma : N(T)| < \infty$ for all elliptic maps T.

Now $|\Gamma : N(T)| = |\Gamma : N(T*)|$ for all conjugates $T*$ of T, so we must only show that the normal closures of our conjugacy class representatives have finite index; to do this we show the factor groups have finite order. To do this we will set our conjugacy class representatives equal to 1 in the presentation of Γ to obtain presentations for the factor groups.

Now $\Gamma = \{a, l, t, u; a^2 = l^2 = (al)^2 = (tl)^2 = (ul)^2 = (at)^3 = (ual)^3 = tut^{-1}u^{-1} = 1\}$. From Theorem 2, any elliptic map is conjugate to l, tl, al, ul, utl, at, or ual, so

(1) $\Gamma/N(l) \cong \{a, t, u; a^2 = t^2 = u^2 = (tu)^2 = (at)^3 = (au)^3 = 1\}$ which has order 24 [**10**].

(2) $\Gamma/N(al) \cong \{a, t, u; a^2 = (at)^3 = (at)^2 = (au)^2 = u^3 = tut^{-1}u^{-1} = 1\}$ $\cong \{a; a^2 = 1\} \cong Z_z$ so order 2.

(3) $\Gamma/N(ul) \cong \{a, t, l; a^2 = l^2 = (al)^2 = a^3 = (at)^3 = (tl)^2 = 1, tl = lt\}$ $\cong \{l; l^2 = 1\} \cong Z_z$ so order 2.

(4) $\Gamma/N(tl) \cong \{l; l^2 = 1\} \cong Z_z$ so order 2.

(5) $\Gamma/N(utl) \cong \{a, t, u; a^2 = t^2 = u^2 = (atu)^2 = (tu)^2 = (at)^3 = 1\}$ $\cong \{a, t, z; a^2 = t^2 = (at)^3 = 1, z^2 = 1, [a, z) = 1, [t, z] = 1\} \cong Z_2 \times \Sigma_3$ so order 12.

(6) $\Gamma/N(at) \cong \{a, l; a^2 = l^2 = (al)^2 = 1\} \cong Z_2 \times Z_2$ so order 4.
(7) $\Gamma/N(ual) \cong \{a, l; a^2 = l^2 = (al)^2 = 1\} \cong Z_2 \times Z_2$ so order 4.

We can restate this as

COROLLARY. *If $G \lhd \Gamma$ and G has an elliptic element then $|\Gamma : G|$ divides* 24 $\{$*divides* 2, 4, 12 *or* 24 *depending on elliptic map*$\}$.

Applying this to our Fuchsian groups $P(C)$ we get

COROLLARY. *If the circle C and the interior of C are both fixed by any elliptic map in Γ then the index $|\Gamma : P_N(C)| < \infty$. In fact $|\Gamma : P_N(C)|$ divides* 2, 4, 12 *or* 24.

If we do not require the interior of C to be mapped on itself we get a stronger statement. Letting $L(C)$ be the general stabilizer of the circle C in Γ, (the subgroup of Γ which maps C on itself) and $L_N(C)$ its normal closure, we get

THEOREM 4. *If the circle C is fixed by either an elliptic and/or parabolic map in Γ then $|\Gamma : L_N(C)| < \infty$ (in fact $|\Gamma : L_N(C)| \mid 24$).*

Proof. If the circle C is fixed by an elliptic map then the result follows from Theorem 3. If C is fixed by a parabolic map we need the following.

LEMMA. *A parabolic map in Γ is conjugate within Γ itself to a translation.*

Proof. If T is parabolic, $T \in \Gamma$, then $T = (az + b)/(cz + d)$ with $a + d = \pm 2$. Then the fixed point of T is $(a - d)/2c$ which is a Gaussian rational. If α/B is a Gaussian rational, with $(\alpha, B) = 1$, then

$$V_1(Z) = (az + \alpha)/(cz + B)$$

is in Γ (where a, c, B, α are Gaussian integers with $aB - c\alpha = 1$ and a, c, exist because $(\alpha, B) = 1$), and maps 0 to α/B. So V_1^{-1} maps α/B to 0 and then $a = \{z' = -1/z\}$ takes 0 to ∞. So $aV_1^{-1} : \alpha/B \to \infty$.

Therefore, conjugating T by aV_1^{-1} gives T^* which is also parabolic [**5**], and in Γ and has fixed point infinity. A parabolic map in Γ with fixed point ∞ is a translation [**5**], proving the lemma.

Say parabolic T fixes circle C and say $V^{-1}TV = T^*$ is a translation, say $z' = z + \alpha$. Then $V(C)$ is a fixed circle of T^* which must be the line L through α and the origin. But any line through the origin is fixed by $l(z) = -z$ (although not in a Fuchsian manner). Then $V^{-1}l\,V$ fixes C so $L(C)$ contains an elliptic map. Then from Theorem 3, $|\Gamma : L_N(C)| < \infty$; in fact it divides 24.

4. Fuchsian subgroups.

Using the above we can characterize normal Fuchsian subgroups.

THEOREM. *A finitely generated normal Fuchsian subgroup F of Γ is either a free group or provides a faithful representation of a fundamental group of a Riemann surface of genus $\geqq 2$.*

The theorem is also true if F is not normal but if the normal closure of F in Γ has infinite index.

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Proof. Say $F \lhd \Gamma$ and F is finitely generated Fuchsian. Then since F is discontinuous in C, $|\Gamma : F| = \infty$ [**5**]. Therefore from before F cannot have any elliptic maps. (If it did it would be of finite index.)

A finitely generated Fuchsian group F has a presentation;

$$F \cong \{a_i, b_i, c_j, d_k, i = 1 \dots n, j = 1 \dots s, k = 1 \dots t; cjlj = 1;$$
$$c_1^{-1} \dots C_s^{-1} d_1^{-1} \dots d_t^{-1}[a_1, b_1] \dots [a_n, b_n] = 1\}$$

Since F is elliptic map free, $j = 0$ and we get

$$F \cong \{a_i, b_i, d_k, i = 1 \dots n, k = 1 \dots t; d_1 \dots d_t[a_1, b_1] \dots [a_n, b_n] = 1\}$$

If $t \neq 0$ then F is a free product of torsion free cyclic groups [**3**] and is therefore a free group.

If $t = 0$ then

$$F \cong \{a_i, b_i, i = 1, \dots n; [a_1, b_1] \dots [a_n, b_n] = 1\}$$

which is a presentation for the fundamental group of Riemann surface of genus n. If $n = 1$, F is free abelian of rank 2 and cannot be Fuchsian [**5**] so $n > 1$, and F provides a faithful representation of a fundamental group of a Riemann surface of genus $n \geqq 2$.

If F is not normal but $|\Gamma : N(F)| = \infty$, then by the same argument F must be elliptic map free, and satisfy the same requirements as above.

It is interesting to note that free groups of all finite ranks as well as faithful representations of fundamental groups of Riemann surfaces of all finite genuses do appear in Γ [**7**; **8**], although not necessarily as normal Fuchsian subgroups.

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