BULL. AUSTRAL. MATH. SOC. 62E20, 62M99
VOL. 27 (1983), 329-337.

ON THE ASYMPTOTIC BEHAVIOUR OF THE JACKKNIFE FOR STOCHASTIC PROCESSES

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The limiting behaviour of the J_∞ jackknife estimator for parameters associated with stochastic processes is shown to depend on the nature of the underlying process through the asymptotic behaviour of the estimator being jackknifed. In particular, the jackknifed versions of certain estimators associated with renewal processes are shown to have an asymptotic normal distribution.

1. Introduction

Although the original motivation for considering the jackknifing technique was bias reduction, it is the asymptotic properties of the jack-knife that make it a useful general tool in data analysis. Gray, Watkins and Adams [3] extended the jackknife technique to estimation problems for certain parameters associated with stochastic processes. They developed the J_n estimator by partitioning the observed sample path into n segments and then applying the standard jackknifing procedure to the estimates based on the resulting segments. They proved that if the under-lying process was sufficiently well behaved then the bias reduction properties of the jackknife improved as the partition was made finer and this led them to propose the J_∞ jackknife estimator.

Gray *et al* [3] developed the asymptotic theory for the J_∞ estimator when the underlying stochastic process has stationary independent

Received 4 January 1983.

increments. The purpose of this paper is to show that the behaviour of the J_∞ estimator depends on the underlying stochastic process through the behaviour of the estimator being jackknifed. This result is then used to obtain some asymptotic results for the J_∞ estimator when the underlying process is a renewal process.

2. Notation

Let $\{X(t) : t \in [a, b]\}$ be a stochastic process and suppose that the probability law of $X(t)$ depends on the parameter θ for every $t \in [a, b]$. Then if $b \geq t_2 > t_1 \geq a$ define $\hat{\theta}(t_1, t_2)$ to be an estimator of θ of the form

$$\hat{\theta}(t_1, t_2) = [I(t_2)-I(t_1)]/(t_2-t_1) ,$$

where $\{I(t) : t \in [a, b]\}$ is a stochastic process determined by the process $\{X(t) : t \in [a, b]\}$, such that almost every realisation of $\{I(t)\}$ is piecewise continuous. The reader is referred to Gray and Schucany [2], Chapter 4, for examples of such estimators. For the purpose of discussing asymptotic results we will take $\{X(t)\}$ defined for all $t \geq a$.

If the process $\{X(t)\}$ is observed for $t \in [a, a+T]$ and we are interested in $f(\theta)$ for some real valued function f then we construct the J_n jackknifed estimator for $f(\theta)$ as follows. Let $t_i = a + iT/n$, $i = 0, 1, \ldots, n$ and set $\hat{\theta} = \hat{\theta}(a, a+T)$ and

$$\hat{\theta}_n^i = [n\hat{\theta}-\hat{\theta}(t_{i-1}, t_i)]/(n-1) , \quad i = 1, 2, \ldots, n .$$

Define the estimator $J_n(f(\hat{\theta}))$ by

$$J_n(f(\hat{\theta})) = nf(\hat{\theta}) - (n-1)n^{-1} \sum_{i=1}^{n} f(\hat{\theta}_n^i) .$$

By analogy with jackknife estimation results for independent random samples, an estimator for the variance of $J_n(f(\hat{\theta}))$ is

$$s_{n,T}^2 = \sum_{i=1}^{n} \left[\left[nf(\hat{\theta})-(n-1)f(\hat{\theta}_n^i)\right]-J_n(f(\hat{\theta}))\right]^2/[n(n-1)] .$$

https://doi.org/10.1017/S0004972700025843 Published online by Cambridge University Press

Another statistic we need to consider is

$$\hat{\sigma}^2_{n,T} = \frac{1}{n([nT]-1)} \sum_{j=1}^{[nT]} \left\{\hat{\theta}\left(a+(j-1)/n,\ a+j/n\right)-\hat{\theta}(a,\ a+[nT]/n)\right\}^2 ,$$

where $[nT]$ denotes the largest integer not exceeding nT. If $\operatorname{var} \hat{\theta}(a, a+1) = \sigma^2$ and the process $\{I(t)\}$ has stationary independent increments then it is easy to show that $\hat{\sigma}^2_{n,T}$ is a consistent estimator for σ^2, as $T \to \infty$.

Now let Γ be the set of possible non-zero jump sizes for the process $\{I(t)\}$. For each $\gamma \in \Gamma$ let N_γ denote the number of jumps of size γ occurring in $I(t)$ for $t \in [a, a+T]$. Set $N = \sum_{\gamma\in\Gamma} N_\gamma$. Following Gray *et al* [3] define the statistics

$$J_\infty\left(f(\hat{\theta})\right) = f(\hat{\theta}) - \sum_{\gamma\in\Gamma} N_\gamma\left[f\left(\hat{\theta}-(\gamma/T)\right)-f(\hat{\theta})+(\gamma/T)f'(\hat{\theta})\right] , \tag{1}$$

$$s_T^2 = \sum_{\gamma\in\Gamma} N_\gamma\left(f\left(\hat{\theta}-(\gamma/T)\right)-f(\hat{\theta})\right)^2$$

and

$$\hat{\sigma}_T^2 = 1/T \sum_{\gamma\in\Gamma} \gamma^2 N_\gamma ,$$

where $f'(x)$ denotes the first derivative of f at x. Note that if the observed sample path is continuous then $J_\infty\left(f(\hat{\theta})\right) = f(\hat{\theta})$ and so the J_∞ statistic coincides with the original estimator for $f(\theta)$. For a discussion of the bias properties of $J_n\left(f(\hat{\theta})\right)$, $J_\infty\left(f(\hat{\theta})\right)$ and the simple estimator $f(\hat{\theta})$ the reader is referred to Gray and Schucany [2].

The following result, proved by Watkins [4], provides sufficient conditions for the convergence of $J_n\left(f(\hat{\theta})\right)$, $s^2_{n,T}$, and $\hat{\sigma}^2_{n,T}$ as $n \to \infty$ and is included for completeness.

THEOREM 1. *Let f be a real valued differentiable function. Suppose $\{I(t) : t \in [a, a+T]\}$ defined above is such that almost every realisation is piecewise continuous and of bounded variation on $[a, a+T]$.*

Further, suppose that for each $t \in [a, a+T]$, $I(t)$ *is continuous at* t *with probability* 1 . *Then*

$$\lim_{n\to\infty} J_n\big(f(\hat{\theta})\big) = J_\infty\big(f(\hat{\theta})\big) \quad \textit{almost surely,}$$

$$\lim_{n\to\infty} s^2_{n,T} = s^2_T \quad \textit{almost surely}$$

and

$$\lim_{n\to\infty} \hat{\sigma}^2_{n,T} = \hat{\sigma}^2_T \quad \textit{almost surely.}$$

3. Asymptotic behaviour of the J_∞ estimator

We will now investigate the distribution of $J_\infty\big(f(\hat{\theta})\big)$ as the record length T becomes larger.

THEOREM 2. *Suppose that* $\hat{\theta} \xrightarrow{p} \theta$ *and there is a random variable* X *and a non-negative function* $g(T)$ *such that*

$$g(T)(\hat{\theta}-\theta) \xrightarrow{\mathcal{D}} X \quad \textit{as} \quad T \to \infty .$$

If f *is a real valued function with bounded second derivative in a neighbourhood of* θ , Γ *is a bounded set and* $\hat{\sigma}^2_T g(T)/T \xrightarrow{p} 0$ *as* $T \to \infty$, *then*

$$g(T)\big(J_\infty\big(f(\hat{\theta})\big)-f(\theta)\big) \xrightarrow{\mathcal{D}} f'(\theta)X \quad \textit{as} \quad T \to \infty .$$

Proof. Since f has a continuous derivative in a neighbourhood of θ it follows that

$$g(T)\big(f(\hat{\theta})-f(\theta)\big) \xrightarrow{\mathcal{D}} f'(\theta)X \quad \text{as} \quad T \to \infty .$$

So from equation (1) and Slutsky's Theorem it is sufficient to show that

$$g(T)\sum_{\gamma\in\Gamma} N_\gamma\big[f\big(\hat{\theta}-(\gamma/T)\big)-f(\hat{\theta})+(\gamma/T)f'(\hat{\theta})\big] \xrightarrow{p} 0 \quad \text{as} \quad T \to \infty .$$

Suppose that $|f''(t)| < M$ for all $t \in (\theta-2\delta, \theta+2\delta)$ where $\delta > 0$ and let $\gamma_0 = \sup\{|\gamma| : \gamma \in \Gamma\}$. Then given $\varepsilon > 0$, there exists a T_ε such that, for $T > T_\varepsilon$,

$$P\left(\hat{\theta} \in (\theta-\delta,\ \theta+\delta)\right) \geq 1 - \varepsilon \quad \text{and} \quad \gamma_0/T < \delta .$$

Thus, for $T > T_\varepsilon$, with probability at least $1 - \varepsilon$,

$$f\left(\hat{\theta}-(\gamma/T)\right) = f(\hat{\theta}) - (\gamma/T)f'(\hat{\theta}) + \left(\gamma^2/2T^2\right)f''\left(\theta_\gamma\right) ,$$

where $\left|f''\left(\theta_\gamma\right)\right| < M$. So, for $T > T_\varepsilon$,

$$\left|g(T) \sum_{\gamma\in\Gamma} N_\gamma\left[f\left(\hat{\theta}-(\gamma/T)\right)-f(\hat{\theta})+(\gamma/T)f'(\hat{\theta})\right]\right| = \left|\left(g(T)/2T^2\right) \sum_{\gamma\in\Gamma} \gamma^2 N_\gamma f''\left(\theta_\gamma\right)\right|$$
$$\leq M\hat{\sigma}_T^2 g(T)/T ,$$

with probability greater than $1 - \varepsilon$ and the result follows since $\hat{\sigma}_T^2 g(T)/T \xrightarrow{p} 0$.

Thus the asymptotic behaviour of $J_\infty\left(f(\hat{\theta})\right)$ depends on the underlying process through the limiting distribution of the estimator $\hat{\theta}$. It is now possible to obtain limit results for $J_\infty\left(f(\hat{\theta})\right)$ when the underlying process $\{I(t)\}$ does not have stationary independent increments. One immediate consequence of Theorem 2 is the following.

COROLLARY 1. *Let f be a real valued function with bounded second derivative in a neighbourhood of θ and suppose Γ is a bounded set. If $\hat{\theta} \xrightarrow{p} \theta$ and $\hat{\sigma}_T^2/T \xrightarrow{p} 0$ as $T \to \infty$ then*

$$J_\infty\left(f(\hat{\theta})\right) \xrightarrow{p} f(\theta) \quad \text{as} \quad T \to \infty .$$

That is, provided $\hat{\sigma}_T^2/T \xrightarrow{p} 0$, if $\hat{\theta}$ is a consistent estimator for θ then $J_\infty\left(f(\hat{\theta})\right)$ is consistent for $f(\theta)$.

The next result gives a consistent estimator for $f'(\theta)$ based on s_T^2.

THEOREM 3. *Suppose $\hat{\theta} \xrightarrow{p} \theta$ and $\hat{\sigma}_T^2 \xrightarrow{p} \eta$ as $T \to \infty$, for some constant η. If f is a real valued function with continuous first derivative at θ and Γ is a bounded set then*

$$Ts_T^2 \xrightarrow{p} \eta\left(f'(\theta)\right)^2 \quad \text{as} \quad T \to \infty .$$

Proof. Let $\gamma_0 = \sup\{|\gamma| : \gamma \in \Gamma\}$ and suppose $f'(t)$ is continuous for $t \in (\theta-2\delta_1, \theta+2\delta_1)$, where $\delta_1 > 0$. So given $\delta_2 > 0$ there is a δ_3, $0 < \delta_3 < \delta_1$, such that

$$|(f'(t))^2-(f'(\theta))^2| < \delta_2 \quad \text{if} \quad |t-\theta| < 2\delta_3 .$$

Since $\hat{\theta} \xrightarrow{P} \theta$, given $\varepsilon > 0$ there exists a T_ε such that, for $T > T_\varepsilon$,

$$P(|\hat{\theta}-\theta| < \delta_3) > 1 - \varepsilon \quad \text{and} \quad \gamma_0/T < \delta_3 .$$

Hence, for $T > T_\varepsilon$,

$$f(\hat{\theta}-(\gamma/T)) = f(\hat{\theta}) - (\gamma/T)f'(\theta_\gamma)$$

and

$$\left|(f'(\theta_\gamma))^2-(f'(\theta))^2\right| < \delta_2 \quad \text{for all} \quad \gamma \in \Gamma ,$$

with probability greater than $1 - \varepsilon$. So with probability at least $1 - \varepsilon$ we have

$$\begin{aligned}\left|Ts_T^2-(f'(\theta))^2\hat{\sigma}_T^2\right| &= \left|(1/T) \sum_{\gamma\in\Gamma} \gamma^2 N_\gamma\left[(f'(\theta_\gamma))^2-(f'(\theta))^2\right]\right| \\ &\le \hat{\sigma}_T^2\delta_2 .\end{aligned}$$

The result now follows since δ_2 is arbitrary and $\hat{\sigma}_T^2 \xrightarrow{P} \eta$ as $T \to \infty$.

We can now apply the above results to obtain Theorem 6.4 of Gray *et al* [3] for processes with stationary independent increments.

COROLLARY 2. *Suppose $\{I(t) : t \ge a\}$ has stationary independent increments, $E\hat{\theta}(a, t) = \theta$ for $t > a$ and $\operatorname{Var} \hat{\theta}(a, a+1) = \sigma^2$. If f is a real valued function with bounded second derivative in a neighbourhood of θ, Γ is a bounded set and $\hat{\sigma}_T^2 \xrightarrow{P} \sigma^2$ then*

$$(J_\infty(f(\hat{\theta}))-f(\theta))\Big/\left(\left[\sum_{\gamma\in\Gamma} N_\gamma(f(\hat{\theta}-(\gamma/T))-f(\hat{\theta}))^2\right]^{\frac{1}{2}}\right) \xrightarrow{D} N(0, 1) \quad \textit{as} \quad T \to \infty .$$

Proof. Since $\{I(t)\}$ has stationary independent increments,

$\operatorname{Var} \hat{\theta}(a, a+T) = \sigma^2/T$. Writing

$$\hat{\theta}(a, a+T) = T^{-1} \sum_{i=1}^{[T]} \hat{\theta}(a+i-1, a+i) + \hat{\theta}(a+[T], a+T)(1-[T]/T)$$

we have from the classical central limit theorem for independent random variables and Slutsky's Theorem that

$$T^{\frac{1}{2}}(\hat{\theta}-\theta) \xrightarrow{\mathcal{D}} N\left(0, \sigma^2\right) \quad \text{as} \quad T \to \infty .$$

Also $\hat{\sigma}_T^2/T^{\frac{1}{2}} \xrightarrow{p} 0$ and so the result follows from Theorems 2 and 3.

In particular, if $\{I(t)\}$ is a Poisson process with intensity θ then $\Gamma = \{1\}$ and $\hat{\sigma}_T^2 = N_1/T \xrightarrow{p} \theta = \operatorname{Var} \hat{\theta}(a, a+1)$ as $T \to \infty$. Therefore if f is a real valued function with bounded second derivative in a neighbourhood of θ then

$$\text{(2)} \qquad \frac{J_\infty\left(f(\hat{\theta})\right)-f(\theta)}{\left[T\hat{\theta}\left(f\left(\hat{\theta}-T^{-1}\right)-f(\hat{\theta})\right)^2\right]^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} N(0, 1) \quad \text{as} \quad T \to \infty .$$

4. Jackknifing renewal processes

Suppose we have a stochastic process with events occurring at times

$$X_1,\ X_1+X_2,\ X_1+X_2+X_3,\ \ldots ,$$

where $\{X_1, X_2, \ldots\}$ are independent, non-negative, continuous random variables. Further let $X_2, X_3, \ldots$ have distribution function $F(x)$ and assume that $EX_2 = \mu$ and $\operatorname{Var} X_2 = \sigma^2 < \infty$. The renewal process $\{I(t) : t \geq 0\}$ counts the number of events that have occurred up to time t and is defined by

$$\begin{aligned} I(t) &= 0 \quad \text{if} \quad X_1 > t \\ &= \sup\left\{n : \sum_{i=1}^{n} X_i \leq t\right\} , \text{ otherwise.} \end{aligned}$$

In general, $\{I(t) : t \geq 0\}$ will not have stationary independent increments. If X_1 has distribution function F then $\{I(t)\}$ is an ordinary renewal process; if X_1 has distribution function

$\mu^{-1}\int_0^x (1-F(t))dt$ then $\{I(t)\}$ is an equilibrium renewal process and has stationary increments; otherwise $\{I(t)\}$ is called a modified renewal process.

Consider $\hat{\theta} = \hat{\theta}(0, T) = I(T)/T$, the average number of events per unit time. Since the X_i have continuous distributions, the conditions of Theorem 1 are satisfied and so if f is a differentiable, real valued function, $J_\infty(f(\hat{\theta}))$ is the almost sure limit of $J_n(f(\hat{\theta}))$ as $n \to \infty$.

Moreover from Cox [1], Section 3.3, we have that

$$T^{\frac{1}{2}}(\hat{\theta}-\mu^{-1}) \xrightarrow{\mathcal{D}} N(0, \sigma^2/\mu^3) \quad \text{as} \quad T \to \infty$$

regardless of the distribution of X_1, and

$$\hat{\sigma}_T^2 = T^{-1} \sum_{\gamma\in\Gamma} \gamma^2 N_\gamma = \hat{\theta} \xrightarrow{P} \mu^{-1} \quad \text{as} \quad T \to \infty .$$

Combining these results with Theorems 2 and 3 we obtain the following theorem for estimates of functions of the mean waiting time for events.

THEOREM 4. *Given $\{I(t) : t \geq 0\}$ defined above, if f is a real valued function with bounded second derivative in a neighbourhood of μ^{-1} then*

$$\sqrt{T}\left(J_\infty(f(\hat{\theta}))-f(\mu^{-1})\right) \xrightarrow{\mathcal{D}} N\left(0, [f'(\mu^{-1})]^2\sigma^2/\mu^3\right)$$

and

$$Ts_T^2 = T^2\hat{\theta}\left(f(\hat{\theta}-T^{-1})-f(\hat{\theta})\right)^2 \xrightarrow{P} \mu^{-1}[f'(\mu^{-1})]^2 \quad \textit{as} \quad T \to \infty .$$

In particular if the coefficient of variation of the waiting time distribution $\sigma/\mu = k$, where k is known, then

$$\frac{J_\infty(f(\hat{\theta}))-f(\mu^{-1})}{\left[T\hat{\theta}\left(f(\hat{\theta}-T^{-1})-f(\hat{\theta})\right)^2\right]^{\frac{1}{2}}} \xrightarrow{\mathcal{D}} N(0, k^2) \quad \textit{as} \quad T \to \infty .$$

https://doi.org/10.1017/S0004972700025843 Published online by Cambridge University Press

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https://doi.org/10.1017/S0004972700025843 Published online by Cambridge University Press