A CHARACTERISTIC SUBGROUP OF A p-STABLE GROUP

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1. Introduction. Let p be a prime, and let S be a Sylow p-subgroup of a finite group G. J. Thompson (**13**; **14**) has introduced a characteristic subgroup $J_R(S)$ and has proved the following results:

(1.1) *Suppose that p is odd. Then G has a normal p-complement if and only if $C(Z(S))$ and $N(J_R(S))$ have normal p-complements.*

(1.2) *Suppose that G is p-solvable and contains a normal p-subgroup P such that $C(P) \subseteq P$. Assume that* $\mathrm{SL}(2, p)$ *is not involved in G. Then*

$$G = C(Z(S))N(J_R(S)).$$

Recently, Thompson introduced (**15**) a characteristic subgroup $J_O(S)$ that is quite similar to $J_R(S)$ but also satisfies a "replacement theorem" (Theorem 3.1).[1] He then proved (Corollary 3.5) that for p odd, (1.2) holds without assuming p-solvability if we substitute $J_O(S)$ for $J_R(S)$.

In this paper we prove the following results.

(1.3) *Suppose that p is odd. Then G has a normal p-complement if and only if $N(Z(J_O(S)))$ has a normal p-complement.*

(1.4) *Suppose that p is odd and that G contains a normal p-subgroup P such that $C(P) \subseteq P$. Assume* $\mathrm{SL}(2, p)$ *is not involved in G. Then $Z(J_O(S))$ is a characteristic subgroup of G.*

(1.5) *Suppose that p is odd and that* $\mathrm{SL}(2, p)$ *is not involved in G. Then two elements of S are conjugate in G if and only if they are conjugate in $N(Z(J_O(S)))$.*

In §2 we state some stronger forms of these results, and in §9 we obtain some analogues for a characteristic subgroup $ZJ^*(S)$ having the property that $C_S(ZJ^*(S)) \subseteq ZJ^*(S)$.

These results were obtained in several stages. The proofs of (1.1) and (1.2) are also valid, with slight changes, for $J_O(S)$ and a number of similarly defined subgroups. We first proved (1.3), (1.4), and (1.5) for all of these subgroups under the additional assumptions that G was p-solvable in (1.4) and that all proper subgroups of G were p-solvable in (1.5).

Both (1.3) and (1.5) are ultimately derived from (1.4). The exclusion of $\mathrm{SL}(2, p)$ in (1.4) guarantees (Lemma 6.3) that G is *p-stable* by a definition similar to that introduced by Gorenstein and Walter in (**7**). This means that certain p-subgroups P and elements x cannot satisfy the commutator condition

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[1]We define $J_R(S)$ and $J_O(S)$ in §2.

$[[P, x], x] = 1$. An easy example (Example 10.1) shows that this condition or some related hypothesis is necessary to obtain the conclusion of (1.4).

Originally, we had assumed p-solvability in (1.4) in order to reduce it, by induction, to the case where G had a normal p-subgroup P for which the Sylow p-subgroup of G/P was cyclic. Thompson used his Replacement Theorem to show that this type of argument was unnecessary for $J_O(S)$ in (1.2). Thus, for p odd and $J_O(S)$ in place of $J_R(S)$, he obtained (1.2) and an important case of (1.4), without assuming p-solvability (Corollary 3.5 and Theorem 3.2). Using Thompson's results and our previous methods, we showed that a counter-example to (1.4) must contain a subgroup P and an element x such that $[[[P, x], x], x] = 1$. By generalizing the Replacement Theorem for odd primes, we obtained $[[P, x], x] = 1$ and thus derived a contradiction that completed the proof. Unfortunately, the condition $[[P, x], x] = 1$ is unexceptional when $p = 2$. In fact, (1.4) fails rather spectacularly for $p = 2$, although some characteristic subgroup other than $Z(J_O(S))$ might satisfy (1.4). We give some examples of this failure, and some counter-examples to other generalizations, in §10. In §3, we include the statements and proofs of several results of Thompson. Thus, this paper is self-contained except for the first two lemmas of §6 and some standard results and techniques. In some other, unpublished work, Thompson has proved that we may replace $Z(J_O(S))$ by $J_O(S)$ in (1.3) if $p \geqq 5$, and in (1.4) if $p \geqq 7$ and G is p-solvable.

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2. Definitions and statement of main results. All groups considered in this paper will be finite. Let G be a finite group. For every finite set S, denote the number of elements of S by $|S|$. We write $H \subseteq G$ $(H \subset G)$ to indicate that H is a subgroup (a proper subgroup) of G. If $H \subseteq G$ and K is a subset of G, denote the normalizer and centralizer of K in H by $N_H(K)$ and $C_H(K)$. If K contains a single element x, let $C_H(x) = C_H(K)$. When there is no danger of confusion, we write $N(K)$ and $C(K)$ for $N_G(K)$ and $C_G(K)$. For every $x \in G$ and every element or subset y of G, let $y^x = x^{-1}yx$.

For every pair of elements x and y of G, let $[x, y]$ be the commutator $x^{-1}y^{-1}xy$. For $x \in G$ and $H \subseteq G$, let $[H, x]$ be the subgroup of G generated by all the commutators of the form $[h, x]$ for $h \in H$. Define $[x, H]$ similarly, and also $[H, K]$ for $H, K \subseteq G$. For elements or subgroups $x_1, x_2, \dots, x_n$ of G we define iterated commutators inductively by

$$[x_1, x_2, \dots, x_{n-1}, x_n] = [[x_1, x_2, \dots, x_{n-1}], x_n] \qquad (n \geqq 3)$$

and

$$[x_1, x_2; 0] = x_1, \quad [x_1, x_2; n] = [[x_1, x_2; n - 1], x_2] \qquad (n \geqq 1).$$

A *subquotient* of G is a factor group of the form H/K, where $H, K \subseteq G$ and

K is a normal subgroup of H. Let 1 denote both the identity element and the identity subgroup of G. We say that a finite group G_0 is *involved* in G if G_0 is isomorphic to a subquotient of G.

G is called an *elementary Abelian* group if G is a direct product of groups of the same prime order p. In this case, we shall occasionally consider G to be an additive group and therefore a vector space over the field $\mathrm{GF}(p)$ of p elements. Thus, the automorphisms of G correspond to the linear transformations of G over $\mathrm{GF}(p)$. We will denote the zero element of a vector space by 0 or 1.

An *operator group* A on G is a group A in which every $a \in A$ corresponds to an automorphism $x \rightarrow x^a$ of G with the property that $x^{ab} = (x^a)^b$ for all $x \in G$ and $a, b \in A$. (Different elements of A may correspond to the same automorphism of G.) In particular, a group of linear transformations on a vector space V may be considered as an operator group on V. Let

$$C_A(G) = \{a \in A\colon x^a = x \text{ for all } x \in G\}$$

and

$$C_G(A) = \{x \in G\colon x^a = x \text{ for all } a \in A\}.$$

(We use braces to denote sets, rather than the groups they generate.) A is said to be *faithful* if $C_A(G) = 1$. A is said to be *irreducible* on G if G is an elementary Abelian group and if 1 and G are the only subgroups H of G such that $H^a = H$ for all $a \in A$.

If M and N are normal subgroups of G and $N \subseteq M$, we consider G to be an operator group on M/N by the definition

$$(xN)^g = (g^{-1}xg)N \qquad (x \in M,\ g \in G).$$

We define $C_G(M/N)$ and $C_{M/N}(G)$ and irreducibility of G on M/N according to the definitions for operator groups.

Let p be a prime. An element or subgroup of G is called a *p-element* or *p-subgroup* if its order is a power of p. It is called a *p'-element* or *p'-subgroup* if its order is not divisible by p. We adhere to the convention that $p^0 = 1$, thus the identity element is considered to be both a p-element and a p'-element, and the identity subgroup is considered similarly. We say that G is *p-solvable* if each of its composition factors is either a p-group or a p'-group. It is easy to show that the product of normal p-subgroups is a normal p-subgroup, and that, therefore, G contains a unique maximal normal p-subgroup, denoted by $O_p(G)$. Similarly, G contains a unique maximal normal p'-subgroup, denoted by $O_{p'}(G)$. Let $O_{p',p}(G)$ be the subgroup of G that contains $O_{p'}(G)$ and satisfies

$$O_{p',p}(G)/O_{p'}(G) = O_p(G/O_{p'}(G)).$$

We say that G has a *normal p-complement* (namely, $O_{p'}(G)$) if $G = O_{p',p}(G)$. Let $|G|_p$ be the order of a Sylow p-subgroup of G.

For each Abelian subgroup A of G, let $m(A)$ be the *rank*, or minimal number of generators of A; and let $d_R(G)$ be the maximum of the numbers $m(A)$.

Similarly, let $d_O(G)$ be the maximum of the orders of the Abelian subgroups of G. Define

$$\mathscr{A}_R(G) = \{A\colon A \subseteq G,\ A \text{ Abelian, and } m(A) = d(G)\}$$

and

$$\mathscr{A}_O(G) = \{A\colon A \subseteq G,\ A \text{ Abelian, and } |A| = d_O(G)\}.$$

Then $J_R(G)$ and $J_O(G)$ are the subgroups of G generated by the elements of $\mathscr{A}_R(G)$ and $\mathscr{A}_O(G)$, respectively. These definitions will be used only when G is a p-group. Since we will not use $d_R(G)$, $\mathscr{A}_R(G)$, and $J_R(G)$, we let

$$d(G) = d_O(G), \quad \mathscr{A}(G) = \mathscr{A}_O(G), \quad \text{and} \quad J(G) = J_O(G)$$

throughout this paper.

We require concepts of p-stability and p-constraint similar to those of Gorenstein and Walter (**7**) and Gorenstein (**6**).

DEFINITION 2.1. *Let p be an odd prime. Let G be a finite group such that $O_p(G) \neq 1$. Then G is p-stable if it satisfies the following condition:*

Let P be an arbitrary p-subgroup of G such that $O_{p'}(G)P$ is a normal subgroup of G. Suppose that $x \in N(P)$ and $\bar{x}$ is the coset of $C(P)$ containing x. If $[P, x, x] = 1$, then $\bar{x} \in O_n(N(P)/C(P))$.

DEFINITION 2.2. *Let p be an odd prime. Let G be a finite group such that $O_p(G) \neq 1$, and let P be a Sylow p-subgroup of $O_{p',p}(G)$. Then G is p-constrained if $C_G(P) \subseteq O_{p',p}(G)$.*

For an arbitrary prime p, let $\mathscr{M}_p(G)$ be the set of all subgroups M of G that are maximal with respect to the property that $O_p(M) \neq 1$.

DEFINITION 2.3. *Let p be an odd prime and let G be a finite group. We say that G is p-stable (p-constrained) if every element of $\mathscr{M}_p(G)$ is p-stable (p-constrained) according to Definition* 2.1 (*Definition* 2.2).

Note that if $O_p(G) \neq 1$, then $\mathscr{M}_p(G) = \{G\}$, therefore the above definitions are consistent. If p does not divide $|G|$, then G is trivially p-stable and p-constrained since $\mathscr{M}_p(G)$ is the empty set. By Lemma 1.2.3 and Theorem B of Hall and Higman (**10**), a p-solvable group H is p-constrained regardless of p, and is p-stable except possibly when $p = 3$ and H has a non-abelian Sylow 2-subgroup.

More generally, for every prime p, let V_2 be a two-dimensional vector space over $\mathrm{GF}(p)$ and let $\mathrm{SL}(V_2)$ be the *special linear group* on V_2, i.e., the group of all linear transformations of determinant one on V_2. We define the *quadratic* group $\mathrm{Qd}(p)$ for the prime p to be the semi-direct product of V_2 by $\mathrm{SL}(V_2)$.

We say that a finite group H of linear transformations over a finite field of characteristic p is called a *p-stable linear group* if no element of H has the quadratic polynomial $(x - 1)^2$ as its minimal polynomial. Suppose that $O_p(H) = 1$ and H is irreducible on V. Turning from multiplicative to additive notation, we obtain that

$$[v, x, x] = (v^{-1} + v^x)^{-1} + (v^{-1} + v^x)^x = -v^{x-1} + v^{(x-1)x} = v^{(x-1)^2}$$

for $v \in V$ and $x \in H$. Thus the semi-direct product HV is a p-stable group if and only if H is a p-stable linear group. Thus $\mathrm{Qd}(p)$ is not a p-stable group. Conversely, in Lemma 6.3 we show that a group G is p-stable if $\mathrm{Qd}(p)$ is not involved in G.

The main results of this paper are the following.

THEOREM A. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Assume that G is p-stable and that $C(O_p(G)) \subseteq O_p(G)$. Then $Z(J(S))$ is a characteristic subgroup of G.*

THEOREM B. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Assume that $\mathrm{Qd}(p)$ is not involved in G. Suppose that W is a non-empty subset of S, $g \in G$, and that W^g is contained in S. Then there exist $c \in C(W)$* and *$n \in N(Z(J(S)))$ such that $g = cn$.*

THEOREM C. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Assume that G is p-stable and p-constrained. Suppose that W is a non-empty subset of S, $g \in G$, and W^g is contained in S. Then there exist $c \in O_{p'}(C(W))$ and $n \in N(Z(J(S)))$ such that $g = cn$.*

THEOREM D. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Then G has a normal p-complement if and only if $N(Z(J(S)))$ has a normal p-complement.*

These results yield several corollaries.

COROLLARY 2.1. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Assume either that $\mathrm{Qd}(p)$ is not involved in G or that G is p-stable and p-constrained. Then*:

(a) *Two elements or subsets of S are conjugate in G if and only if they are conjugate in $N(Z(J(S)))$.*

(b) *For every $x \in G$, $Z(J(S))^x \cap S \subseteq Z(J(S))$.*

Corollary 2.1 (b) states that $Z(J(S))$ is *strongly closed*, and therefore *weakly closed*, in S with respect to G, as defined by Wielandt (**19**, p. 205). For every subgroup H of G, let $O^p(H)$ be the subgroup of G generated by all the p'-elements of H. By Corollary 2.1 (b) and a theorem of P. Hall and Wielandt (**9**, p. 212), we obtain the following corollary.

COROLLARY 2.2. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Let $N = N(Z(J(S)))$. Assume either that $\mathrm{Qd}(p)$ is not involved in G or that G is p-stable and p-constrained. Then $G/O^p(G)$ is isomorphic to $N/O^p(N)$.*

Note. In order to apply the Hall-Wielandt theorem, we use the facts that $Z(J(S))$ is Abelian and p is odd. An alternate proof of Corollary 2.2 that does not require these facts may be obtained from Corollary 2.1 (a) by using (**11**, pp. 485–488; in particular, Theorem 3.6 and Remark 3).

3. Two theorems of Thompson. In this section we require several results of Thompson, including his generalization of (1.2). All the theorems and corollaries in this section were proved by Thompson, but only Theorem 3.1 and Corollary 3.1 have been previously published.

The following two elementary lemmas will sometimes be used without explicit reference.

LEMMA 3.1. *Let G be a group. Suppose that $x \in G$ and that $H \subseteq G$. Then $x \in N(H)$ if and only if $[x, H] \subseteq H$.*

Proof. For $h \in H$, $h^x = h[h, x]$.

LEMMA 3.2. *Let G be a group and let $x, y, z \in G$. Then:*

(a) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$ *and* $[yz, x] = [y, x]^z[z, x] = [y, x][y, x, z][z, x]$.

(b) *Suppose that $A \subseteq G$ and that A is Abelian. If $[x, A]$ is Abelian and $a, b \in A$, then $[x, a, b] = [x, b, a]$.*

(c) *Suppose that $H \subseteq G$. Then H normalizes $[x, H]$, and $[x, H; n] \subseteq [x, H; m]$ for all non-negative integers m and n such that $n \geqq m$.*

Proof. One may obtain (a) by straightforward calculation. Then (b) follows from the first equation in (a), and (c) follows by induction from the identity

$$[x, y]^z = [x, z]^{-1}[x, yz].$$

LEMMA 3.3. *Suppose that G is a finite p-group and that H is a non-identity normal subgroup of G. Then $H \cap Z(G) \neq 1$.*

Proof. The proof is found in (**16**, Theorem 14, p. 144).

THEOREM 3.1. (REPLACEMENT THEOREM (**15**)). *Let S be a finite p-group, and let $A \in \mathscr{A}(S)$.*

(a) *Suppose that $x \in S$ and that $[x, A]$ is Abelian. Let $M = [x, A]$. Then $MC_A(M) \in \mathscr{A}(S)$.*

(b) *Suppose that B is an Abelian subgroup of S that is normalized by A. If $[B, A, A] \neq 1$, then there exists $A^* \in \mathscr{A}(S)$ such that $A \cap B \subset A^* \cap B$ and $[A^*, A, A] = 1$.*

Proof. (a) Let $C = C_A(M)$. Clearly, MC is an Abelian group, thus it suffices to show that $|MC| \geqq |A|$. Since M is Abelian and $C_S(A) = A$,

$$|MC| = |M||C|/|C \cap M| = |M||C|/|A \cap M| = |M||C_A(M)|/|C_M(A)|.$$

Hence we wish to show that $|M/C_M(A)| \geqq |A/C_A(M)|$.

Suppose that $a, b \in A$ and that $[x, a] \equiv [x, b]$ modulo $C_M(A)$. Since

$$[x, a]^{-1}[x, b] = (x^{-1}x^a)^{-1}(x^{-1}x^b) = (x^a)^{-1}x^b,$$

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then $(x^a)^{-1}x^b \in C_M(A)$. But A centralizes $C_M(A)$, therefore

$$(x^a)^{-1}x^b = ((x^a)^{-1}x^b)^{a^{-1}} = x^{-1}x^{ba^{-1}} = [x, ba^{-1}].$$

Thus $[x, ba^{-1}] \in C_M(A)$. For every $c \in A$, $1 = [x, ba^{-1}, c] = [x, c, ba^{-1}]$, by Lemma 3.2 (b). Thus $ba^{-1} \in C_A(M)$.

Now let $a_1, \ldots, a_n$ be a set of coset representatives for $C_A(M)$ in A. By the above paragraph, no two of the elements $[x, a_1], \ldots, [x, a_n]$ are congruent, modulo $C_M(A)$. Hence $|M/C_M(A)| \geqq |A/C_A(M)|$, as desired.

(b) Assume that $[B, A, A] \neq 1$. Now, AB is a group in which B and $N_B(A)$ are normal subgroups. Since $[B, A, A] \neq 1$, we have that $[B, A] \not\subseteq A$. By Lemma 3.1, B does not normalize A. Therefore, $B/N_B(A)$ is a non-trivial normal subgroup of $AB/N_B(A)$. By Lemma 3.3 there exists $b \in B$ such that the coset $bN_B(A)$ lies in $Z(AB/N_B(A))$. Let $M = [b, A]$ and $A^* = MC_A(M)$. Since $bN_B(A) \in Z(AB/N_B(A))$, we have that

$$M = [b, A] \subseteq N_B(A)$$

and therefore $A^* \subseteq N_S(A)$. Thus

$$[A^*, A, A] \subseteq [[N_S(A), A], A] \subseteq [A, A] = 1.$$

Moreover, as $b \notin N_B(A)$, we have that $M = [b, A] \not\subseteq A$. Since

$$A \cap B \subseteq C_A(M) \subseteq A^*,$$

then

$$A^* \cap B \supseteq M(A \cap B) \supset A \cap B.$$

COROLLARY 3.1 (**14**). *Suppose that S is a finite p-group and that B is an Abelian normal subgroup of S. Then there exists $A \in \mathscr{A}(S)$ such that B normalizes A and $[B, A, A] = 1$.*

Proof. Choose $A \in \mathscr{A}(S)$ such that $A \cap B$ is maximal. By the Replacement Theorem, $[B, A, A] = 1$. Hence

$$[B, A] \subseteq C_S(A) = A;$$

thus $B \subseteq N(A)$, by Lemma 3.1.

COROLLARY 3.2. *Suppose that p is an odd prime and that S is a Sylow p-subgroup of a finite p-stable group G. Let $P = O_p(G)$, and assume that $C(P) = P$. Then P is the only element of $\mathscr{A}(S)$.*

Remark 3.1. In this result and in the following results in this section that use p-stability (except Corollary 3.3), we require the definition of p-stability only for *Abelian* p-subgroups P and elements x such that $[P, x, x] = 1$.

Proof. If $P = 1$, then $G = 1$. Thus we may assume that $O_p(G) = P \neq 1$. Since $P \subseteq C(P)$, P is Abelian. Since $[O_{p'}(G), P] \subseteq O_{p'}(G) \cap P = 1$ and $C(P) \subseteq P$, we have that $O_{p'}(G) = 1$. Suppose that $\mathscr{A}(S)$ contains an element A different from P. Choose A such that $A \cap P$ is maximal. Assume that

$[P, A, A] \neq 1$, and take A^* to satisfy the Replacement Theorem. Since $A \cap P \subset A^* \cap P$, $A^* \subseteq P$. Hence $A^* = P$. But then

$$[P, A, A] = [A^*, A, A] = 1. \tag{3.1}$$

Since $O_{p'}(G) = 1$ and $C(P) = P$, then

$$O_p(G/C(P)) = O_p(G/P) = 1.$$

By (3.1) and the definition of p-stability, $A \subseteq P$. This contradiction completes the proof.

LEMMA 3.4. *Let p be a prime and let G be a group of linear transformations on a finite-dimensional vector space V over a finite field of characteristic p. If P is a non-identity p-subgroup of G, then $1 \subset C_V(P) \subset V$.*

Proof. Since $P \neq 1$, we have that $C_V(P) \subset V$. In the semi-direct product PV of V by P, V is a non-trivial normal subgroup. By Lemma 3.3,

$$C_V(P) = V \cap Z(PV) \neq 1.$$

COROLLARY 3.3. *Let p be an odd prime and let G be a p-stable linear group of transformations of a finite-dimensional vector space V over a finite field F of characteristic p. Suppose that A is a non-identity Abelian p-subgroup of G. Then $|A| < |V/C_V(A)|$. Therefore,*

$$|A| \leqq |V|/|F|p.$$

Proof. Let H be the semi-direct product of V by G. Then $V = C(V) \subseteq O_p(H)$. Let S be a Sylow p-subgroup of H that contains AV. Now, $v^{x-1} = [v, x]$ for $v \in V$ and $x \in G$. Since G is a p-stable linear group, no non-identity element x of G satisfies $[V, x, x] = 1$. We may merely repeat the proof of Corollary 3.2 to show that $\mathscr{A}(S) = \{V\}$. Thus

$$|A||C_V(A)| = |AC_V(A)| < |V|.$$

By Lemma 3.4, $C_V(A) \neq 1$. Hence

$$|A| \leqq |V|/p|C_V(A)| \leqq |V|/p|F|.$$

LEMMA 3.5. *Let S be a Sylow p-subgroup of a finite group G. Then:*

(a) *If $A \in \mathscr{A}(S)$ and $A \subseteq P \subseteq S$, then $Z(J(S)) \subseteq A$ and $Z(J(S)) \subseteq Z(J(P))$;*

(b) *If P is a p-subgroup of G that contains $J(S)$, then $J(S) = J(P)$.*

Proof. (a) Since $Z(J(S))A$ is Abelian and $|A| = d(S)$, we have that $Z(J(S)) \subseteq A$. If $A \subseteq P \subseteq S$, then

$$d(P) = |A| = d(S) \quad \text{and} \quad Z(J(S)) \subseteq A \subseteq J(P) \subseteq J(S).$$

Hence $Z(J(S)) \subseteq Z(J(P))$.

(b) As in (a), $d(P) = d(S)$. Hence $J(S) \subseteq J(P)$. Also $|J(P)| \leqq |J(S)|$ since P is conjugate to a subgroup of S. Therefore, $J(S) = J(P)$.

LEMMA 3.6 (Frattini argument). *Let H be a normal subgroup of a finite group G. Let S be a Sylow p-subgroup of H. Then $G = HN(S)$.*

Proof. Let $g \in G$. Then S^g is a Sylow p-subgroup of H. Therefore, there exists $h \in H$ such that $S^g = S^h$. Then $gh^{-1} \in N(S)$, thus $g = (gh^{-1})h \in N(S)H$. Therefore $G = N(S)H = HN(S)$.

THEOREM 3.2. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Suppose that B is an Abelian normal p-subgroup of G. If G is p-stable, then $Z(J(S)) \cap B$ is a normal subgroup of G.*

Proof. The theorem is trivial if $B = 1$. Assume that $B \neq 1$; thus $O_p(G) \neq 1$. Let $C = Z(J(S)) \cap B$, and let L be the largest normal subgroup of G contained in $N(C)$. Now, $L \cap S$ is a Sylow p-subgroup of L. By the Frattini argument,

$$G = LN(L \cap S) = LN(J(L \cap S)). \tag{3.2}$$

If $J(S) \subseteq L \cap S$, then $J(S) = J(L \cap S)$ by Lemma 3.5. But in this case

$$G = LN(J(L \cap S)) = LN(J(S)) \subseteq LN(C) = N(C).$$

Thus we may assume that

$$J(S) \nsubseteq L \cap S. \tag{3.3}$$

Let L_1 be any normal subgroup of G contained in L. Let M be the subgroup of G that contains L_1 and satisfies $M/L_1 = O_p(G/L_1)$. Then M is a normal subgroup of G and $M = L_1(M \cap S)$. Since S normalizes C, $M \subseteq N(C)$. Therefore, $M \subseteq L$. This applies, in particular, when $L_1 = C(C_1)$ for some normal subgroup C_1 of G that contains C. We obtain

$$O_p(G/C(C_1)) \subseteq L/C(C_1). \tag{3.4}$$

Since G is p-stable, we have that $A \subseteq L$ whenever

$$\begin{aligned} &A \in \mathscr{A}(S), \\ &C \subseteq C_1 \subseteq G, \qquad C_1 \text{ is a normal } p\text{-subgroup of } G, \text{ and} \\ &[C_1, A, A] = 1. \end{aligned} \tag{3.5}$$

By Corollary 3.1, there exists $A \in \mathscr{A}(S)$ such that $[B, A, A] = 1$. By (3.5), $A \subseteq L$. Thus

$$d(L \cap S) = d(S), \quad J(L \cap S) \subseteq J(S), \quad \text{and} \quad C \subseteq Z(J(S)) \subseteq Z(J(L \cap S)), \tag{3.6}$$

by Lemma 3.5 (a). Let $X = Z(J(L \cap S))$. By the Frattini argument,

$$G = LN(L \cap S) = LN(X). \tag{3.7}$$

Let V be the (normal) subgroup of G generated by all the conjugates of C in L. By (3.6) and (3.7), (3.8) $$V \subseteq X = Z(J(L \cap S)).$$

Now by (3.3) there exists $A \in \mathscr{A}(S)$ such that $A \not\subseteq L$. Choose A such that $A \cap V$ is maximal. By (3.5),

(3.9) $$[V, A, A] \neq 1.$$

From the Replacement Theorem we obtain $A^* \in \mathscr{A}(S)$ such that

$$[A^*, A, A] = 1 \quad \text{and} \quad A \cap V \subset A^* \cap V.$$

Because of the choice of A, $A^* \subseteq L \cap S$. By (3.8) and Lemma 3.5 (a),

$$[V, A, A] \subseteq [X, A, A] \subseteq [A^*, A, A] = 1,$$

contrary to (3.9). This completes the proof of Theorem 3.2.

COROLLARY 3.4. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite p-stable group G. Then*:

(a) *$Z(J(S))$ is a normal subgroup of G if and only if $Z(J(S))$ is contained in a normal Abelian subgroup of G*;

(b) *If $O_p(G)$ contains an element of $\mathscr{A}(S)$, then $Z(J(S))$ is a normal subgroup of G.*

Proof. (a) One part is trivial. Conversely, if $Z(J(S))$ is contained in a normal Abelian subgroup B of G, we may apply Theorem 3.2.

(b) By Lemma 3.5 (a), $Z(J(S)) \subseteq Z(J(O_p(G)))$. Apply (a).

LEMMA 3.7. *Let S be a Sylow p-subgroup of a finite group G. Let H be a subgroup of $Z(J(S))$. Then*

$$N(H) = C(H)(N(J(S)) \cap N(H)).$$

Proof. Since $C(H)$ is a normal subgroup of $N(H)$, $C(H)(N(J(S)) \cap N(H))$ is a subgroup of $N(H)$. Let T be a Sylow p-subgroup of $C(H)$ that contains $J(S)$. By the Frattini argument,

$$N(H) = C(H)(N(T) \cap N(H)).$$

By Lemma 3.5 (b), $J(S) = J(T)$. Thus $J(S)$ is a characteristic subgroup of T; thus $N(T) \subseteq N(J(S))$. Hence

$$N(H) \subseteq C(H)(N(J(S)) \cap N(H)) \subseteq N(H).$$

COROLLARY 3.5. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite group G. Suppose that G is p-stable and that $C(O_p(G)) \subseteq O_p(G)$. Then*

$$G = C(Z(S))N(J(S)).$$

Proof. Let $P = O_p(G)$. Since $P \subseteq S$, then $Z(S) \subseteq C(P) \subseteq P$. Therefore, $Z(S) \subseteq Z(P)$. By Theorem 3.2, $Z(P) \cap Z(J(S))$ is a normal subgroup of G. Hence, by Lemma 3.7,

$$G = C(Z(P) \cap Z(J(S)))N(J(S)) \subseteq C(Z(S))N(J(S)) \subseteq G.$$

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Remark 3.2. In (**4**), we proved that $G = \langle C(Z(S)), N(J(S))\rangle$ when $p = 2$, $C(O_p(G)) \subseteq O_p(G)$, and the symmetric group of degree four is not involved in G.

4. A replacement theorem for odd primes. In this section we generalize Theorem 3.1 for odd primes and use it to obtain Theorem A.

LEMMA 4.1. *Let G be a group, and let $x, y, z \in G$.*
(a) $[x, y] = [y, x]^{-1}$.
(b) (P. Hall) $[x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x = 1$.
(c) *Let H be the subgroup of G generated by x, y, and z. If $[H, H] \subseteq Z(H)$, then*

$$[x, yz] = [x, y][x, z] \quad \textit{and} \quad [yz, x] = [y, x][z, x].$$

(d) *If $X, Y \subseteq G$, then $[X, Y] = [Y, X]$.*
(e) (Three Subgroups Lemma; P. Hall) *Suppose that $X, Y, Z \subseteq G$ and that*

$$[X, Y, Z] = [Y, Z, X] = 1.$$

Then $[Z, X, Y] = 1$.

Proof. Parts (a) and (b) follow from computation. Part (c) is a consequence of Lemma 3.2 (a). Parts (d) and (e) are obvious applications of (a) and (b).

THEOREM 4.1. *Let S be a finite p-group. Suppose that $A \in \mathscr{A}(S)$, B is a normal subgroup of S of nilpotence class at most two, and $[B, B] \subseteq Z(J(S))$. Assume that $[B, A; 3] \neq 1$ or that p is odd and $[B, A, A] \neq 1$. Then there exists $A^* \in \mathscr{A}(S)$ such that $A \cap B \subset A^* \cap B$ and $[A^*, A, A] = 1$.*

Remark 4.1. We do not know whether A^* exists if $p = 2$ and $[B, A, A] \neq 1$. In the proof of Theorem 4.1, as in that of Theorem 3.1, A^* has the form $[x, A]C_A([x, A])$ for some $x \in B$. Examples show that for $p = 2$ there may be no subgroup A^* of this form.

Proof. We first note that since

$$[B, A \cap B, A] \subseteq [B, B, J(S)] = 1 \quad \text{and} \quad [A \cap B, A, B] \subseteq [A, A, B] = 1,$$

we have that $[A, B, A \subset B] = 1$ by the Three Subgroups Lemma. Thus

$$A \cap B \text{ centralizes } [B, A]. \tag{4.1}$$

Since S is a nilpotent group, then

$$[B, A; r] = 1 \text{ for some least positive integer } r, \tag{4.2}$$

and thus

$$[B, A; n] \text{ is Abelian for some least positive integer } n. \tag{4.3}$$

Suppose that $[B, A; n + 1] \neq 1$. Then $r \geqq n + 2 \geqq 3$. Now, A does not centralize $[B, A; r - 2]$ since $[B, A; r - 1] \neq 1$. Take $x \in [B, A; r - 3]$ such that A does not centralize $[x, A]$. Let $M = [x, A]$. Since $r \geqq n + 2$,

$$M \subseteq [B, A; r - 2] \subseteq [B, A; n]$$

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by Lemma 3.2 (c). By (4.3), M is Abelian. Therefore, the Replacement Theorem yields $MC_A(M) \in \mathscr{A}(S)$. By (4.1), $A \cap B \subseteq C_A(M)$. However, $M \not\subseteq A$, since A does not centralize M. Consequently,

$$MC_A(M) \cap B \supseteq M(A \cap B) \supset A \cap B.$$

By Lemma 3.2,

$$[MC_A(M), A, A] = [M, A, A] \subseteq [B, A; r] = 1.$$

Clearly, we may take $A^* = MC_A(M)$.

Thus we may assume that

$$[B, A; n+1] = 1. \tag{4.4}$$

We may also assume that $S = AB$, and thus that $[B, B] \subseteq Z(S)$. Let $B' = [B, B]$. Let $S_1, S_2, \ldots$ be the terms of the lower central series of S, i.e.,

$$S_i = [S, S; i-1], \qquad i = 1, 2, \ldots .$$

A simple induction argument using Lemma 3.2 shows that, modulo B',

$$S_i \equiv [B, A; i-1], \qquad i = 2, 3, \ldots . \tag{4.5}$$

By (4.4), $S_{n+2} \subseteq [B, A; n+1]B' = B' \subseteq Z(S)$. Therefore,

$$S_{n+3} = 1. \tag{4.6}$$

Let m be the greatest integer not exceeding $\frac{1}{2}(n+4)$. By (**9**, Corollary 10.3.5, p. 156),

$$[S_m, S_m] \subseteq S_{2m} \subseteq S_{n+3} = 1.$$

Therefore $[B, A; m-1]$ is Abelian, thus $m - 1 \geqq n$ by (4.3). We obtain

$$n \leqq m - 1 \leqq \tfrac{1}{2}(n+4) - 1 = \tfrac{1}{2}n + 1;$$

thus $n \leqq 2$. By (4.4) and (4.6),

$$[B, A; 3] \subseteq [B, A; n+1] = 1, \tag{4.7}$$

and

$$S_5 = 1. \tag{4.8}$$

Henceforth, we may assume that p is odd and that $[B, A; 2] \neq 1$. Take $x \in B$ such that A does not centralize $[x, A]$. Let $b, c \in A$. Taking $y = b^{-1}$ and $z = [x, c]$ in Lemma 4.1 (b), we obtain

$$[[x, b], [x, c]]^{b^{-1}}[b^{-1}, [x, c]^{-1}, x]^z[[x, c], x^{-1}, b^{-1}]^x = 1.$$

Since $B' \subseteq Z(S)$, this yields

$$\begin{aligned} [[x, b], [x, c]] &= [b^{-1}, [x, c]^{-1}, x]^{-1} \\ &= [[b^{-1}, [x, c]^{-1}]^{-1}, x] \quad \text{(by Lemma 4.1 (c))} \\ &= [[[x, c]^{-1}, b^{-1}], x] \quad \text{(by Lemma 4.1 (a)).} \end{aligned} \tag{4.9}$$

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Let $\bar{B} = B/B'$, $\bar{A} = AB'/B'$, and $\bar{S} = S/B'$. Since $[B, A; 3] = 1$, we have that $[\bar{B}, \bar{A}, \bar{A}] \subseteq Z(\bar{S})$. By Lemma 4.1 (c) applied to the group $[\bar{B},\bar{A}]\bar{A}$, and by Lemma 3.2 (b),

$$[[x, c]^{-1}, b^{-1}] \equiv [[x, c], b^{-1}]^{-1} \equiv [x, c, b] \equiv [x, b, c] \quad \text{modulo } B'.$$

Since $B' \subseteq Z(S)$, (4.9) yields

$$[[x, b], [x, c]] = [[x, c, b], x] = [[x, b, c], x].$$

By using Lemma 4.1 (a) and by applying symmetry, we obtain

$$[[x, b], [x, c]]^{-1} = [[x, c], [x, b]] = [[x, b, c], x] = [[x, b], [x, c]].$$

Thus $[[x, b], [x, c]]^2 = 1$. Since p is odd, then $[[x, b], [x, c]] = 1$. Therefore, $[x, A]$ is Abelian.

Let $M = [x, A]$. By the Replacement Theorem, $MC_A(M) \in \mathscr{A}(S)$. By (4.1) and (4.7),

$$A \cap B \subseteq C_A(M) \quad \text{and} \quad [MC_A(M), A, A] = 1.$$

Since x was chosen such that A does not centralize $[x, A]$, we have that $M \not\subseteq A \cap B$. Therefore, we may let $A^* = MC_A(M)$. This completes the proof of Theorem 4.1.

We obtain the following straightforward analogues of Corollaries 3.1 and 3.2.

COROLLARY 4.1. *Let p be a prime. Suppose that S is a finite p-group and that B is a normal subgroup of S such that $[B, B] \subseteq Z(B) \cap Z(J(S))$. Then there exists $A \in \mathscr{A}(S)$ such that $[B, A; 3] = 1$, and if p is odd, there exists $A \in \mathscr{A}(S)$ such that $[B, A, A] = 1$.*

COROLLARY 4.2. *Suppose that p is an odd prime and that S is a Sylow p-subgroup of a finite group G. Let $P = O_p(G)$. Assume that $[P, P] \subseteq Z(S)$, that $C(P) \subseteq P$, and that G is p-stable. Then P contains an element of $\mathscr{A}(S)$.*

By substituting Corollary 4.1 for Corollary 3.1 in the proof of Theorem 3.2, we obtain the following theorem.

THEOREM 4.2. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Suppose that B is a normal p-subgroup of G of nilpotence class at most two and that $[B, B] \subseteq Z(J(S))$. If G is p-stable, then $Z(J(S)) \cap B$ is a normal subgroup of G.*

We may remove the restriction on B in Theorem 4.2.

THEOREM 4.3. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Suppose that B is a normal p-subgroup of G. If G is p-stable, then $Z(J(S)) \cap B$ is a normal subgroup of G.*

Proof. Assume the theorem fails for some group G. Suppose that B is a counter-example of least order. Let $Z = Z(J(S))$, and let B_1 be the (normal) subgroup of G generated by all the conjugates of $Z \cap B$ in G. Then $B_1 \subseteq B$ and $Z \cap B_1 = Z \cap B$. Hence $Z \cap B_1$ is not a normal subgroup of G. By the choice of B,

(4.10) $B = B_1$, the smallest normal subgroup of G that contains $Z \cap B$.

Let $B' = [B, B]$. Then $B' \subset B$, thus $Z \cap B'$ is a normal subgroup of G. Since

$$[Z \cap B, B] \subseteq Z \cap B',$$

then $Z \cap B \subseteq C_G(B/(Z \cap B'))$. By (4.10), $B \subseteq C_G(B/(Z \cap B'))$. Thus $B' \subseteq Z \cap B' \subseteq Z$. Therefore, $Z \cap B$ centralizes B' and (4.10) yields $B \subseteq C(B')$. Thus B satisfies the hypothesis of Theorem 4.2. This contradiction completes the proof of Theorem 4.3.

We may now prove Theorem A. Assume the notation and hypothesis of Theorem A, and let $P = O_p(G)$ and $Z = Z(J(S))$. Since Z is an Abelian normal subgroup of S, then $[P, Z, Z] = 1$. Since G is p-stable and $C(P) \subseteq P$, we have that

$$ZC(P)/C(P) \subseteq O_p(G/C(P)) = P/C(P).$$

Therefore, $Z \subseteq P$. Take $B = P$ in the preceding theorem. We find that Z is a normal subgroup of G. Let α be an automorphism of G, and take $g \in G$ such that $S^\alpha = S^g$. Then $Z^\alpha = Z(J(S^g)) = Z^g = Z$.

Remark 4.2. By using Lemma 5.3, we may avoid invoking p-stability to show that $Z \subseteq P$. The preceding results in §§3 and 4 use p-stability only in connection with the replacement theorems. Hence, Theorem A holds if we only require that $AC(P)/C(P) \subseteq O_p(G/C(P))$ whenever $A \in \mathscr{A}(S)$, P is a normal p-subgroup of G, and $[P, A, A] = 1$.

Under the stronger hypothesis that $\mathrm{Qd}(p)$ is not involved in G and that $C(O_p(G)) \subseteq O_p(G)$, we may show by a more complicated argument that $d(O_p(G)) = d(S)$. Then by Lemma 6.3, G is p-stable, thus Corollary 3.4 (b) applies.

To obtain an analogue of Corollary 3.3, we require some concepts of symplectic geometry, as in (**2**, p. 111 and Chapter III). The following result may be verified by straightforward computation.

LEMMA 4.2. *Let V be a finite-dimensional vector space of even dimension over a field F of characteristic different from two. Suppose that f is a non-singular skew-symmetric bilinear form on V and that G is a subgroup of the symplectic group on V with respect to f. Let H be the set of all ordered pairs (x, α), for $x \in V$ and $\alpha \in F$. Define multiplication in H by*

$$(x, \alpha)(y, \beta) = (x + y, \tfrac{1}{2}f(x, y) + \alpha + \beta).$$

For each $g \in G$, define a mapping $A(g)$ of H into itself by

$$(x, \alpha)^{A(g)} = (x^g, \alpha).$$

Then:

(a) *H forms a group under multiplication*;

(b) *For* (x, α) *and* (y, β) *in* H, $[(x, \alpha), (y, \beta)] = (0, f(x, y))$;

(c) *A is an isomorphism of G into the automorphism group of H.*

In discussing a group G of linear transformations of a vector space V, let $[V, G]$ be the subspace of V generated by the elements of the form v^{x-1} for $v \in V$ and $x \in G$. This definition coincides with the previous definition of $[V, G]$ if V and G are embedded in their semi-direct product. As usual, we consider G to be a group of linear transformations on the dual space V^* by defining

$$f^g(v) = f(v^{g^{-1}}) \quad \text{for } f \in V^*, g \in G, \text{ and } v \in V.$$

LEMMA 4.3. *Let H be a group of linear transformations of a finite-dimensional vector space V. Let V^* be the dual space of V. Then $C_{V^*}(H)$ is the subspace of V^* orthogonal to $[V, H]$.*

Proof. Let $v \in V, f \in V^*$, and $h \in H$. If $f \in C_{V^*}(H)$, then

$$f(v^{h-1}) = f(v^h) - f(v) = f^{h^{-1}}(v) - f(v) = f(v) - f(v) = 0.$$

Conversely, if f is orthogonal to $[V, H]$, then

$$f^{h-1}(v) = f^h(v) - f(v) = f(v^{h^{-1}}) - f(v) = f(v^{h^{-1}} - v) = 0.$$

COROLLARY 4.3. *Let p be an odd prime and let A be an Abelian p-group of symplectic transformations of a finite-dimensional vector space V over a finite field F of characteristic p. Let $M = [V, A]$. Then there exists a subgroup B of A such that*:

(a) $[V, B, B] = 1$ *and*

(b) $|A/B| \leqq |M/C_M(A)|^{1/2}$.

Proof. Construct a group H as in Lemma 4.2, and let S be the semi-direct product of H by A. Then $[H, H] = Z(H) = Z(S)$. By Corollary 4.1, there exists $C \in \mathscr{A}(S)$ such that $[H, C, C] = 1$. Let

$$D_0 = \{x : x \in V \text{ and } (x, \alpha) \in C \text{ for some } \alpha \in F\},$$
$$D = \{(x, \alpha) : x \in D_0 \text{ and } \alpha \in F\},$$

and $B = A \cap CH$. Note that $D = C \cap H$ since $Z(S) \subseteq C$. Since C is Abelian, D_0 is an isotropic subspace of V and D is Abelian. However, $B \subseteq CH \subseteq C_S(D_0)$; thus B centralizes D, by the construction of H. Therefore,

$$d(S) = |C| = |C/(C \cap H)||C \cap H| = |CH/H||D| = |A \cap CH||D| = |B||D| = |BD|.$$

Since BD is Abelian, $BD \in \mathscr{A}(S)$. Moreover, since $B = A \cap CH$ and $[H, C, C] = 1$, then

$$[H, B, B] \subseteq [H, CH, CH] \subseteq [H, C, C][H, H] = [H, H].$$

Therefore $[V, B, B] = 1$. We shall show that B satisfies (b).

Since D_0 is isotropic, $|D_0| \leqq |V|^{1/2}$. Thus

$$d(S) = |BD| = |B|\,|D_0|\,|F| \leqq |B|\,|V|^{1/2}|F|. \tag{4.11}$$

Let W be a maximal isotropic subspace of $C_V(A)$.

Since V is a symplectic space, V is A-isomorphic to V^*. Hence, by Lemma 4.3,

$$|V/M| = |V/[V, A]| = |C_V(A)|, \tag{4.12}$$

and the radical of $C_V(A)$ is $C_V(A) \cap [V, A]$, that is, $C_M(A)$. Therefore, a maximal isotropic subspace W of $C_V(A)$ has order

$$|W| = |C_M(A)|\,|C_V(A)/C_M(A)|^{1/2} = |C_V(A)|^{1/2}|C_M(A)|^{1/2}. \tag{4.13}$$

Let

$$W_0 = \{(x, \alpha) \colon x \in W, \alpha \in F\}.$$

Then AW_0 is Abelian. By (4.11), (4.12), and (4.13),

$$|A|\,|F|\,|C_V(A)|^{1/2}|C_M(A)|^{1/2} = |AW_0| \leqq d(S) = |B|\,|M|^{1/2}|C_V(A)|^{1/2}|F|.$$

Thus $|A/B| \leqq |M/C_M(A)|^{1/2}$.

Remark 4.3. For arbitrary primes p and arbitrary Abelian p-groups of linear transformations, Corollary 4.3 holds if we substitute for (b) the condition:

$$|A/B| \leqq |M/C_M(A)|. \tag{b$'$}$$

This may be proved by similar methods that use only Lemmas 4.2 and 4.3 and Thompson's Replacement Theorem.

5. A conjugacy condition. Let $\mathscr{C}$ be the class of all finite p-groups. We define a *characteristic functor* K to be a mapping from $\mathscr{C}$ into $\mathscr{C}$ with the following properties:

(a) $K(P) \subseteq P$ for all $P \in \mathscr{C}$;

(b) If $P \in \mathscr{C}$ and $|P| > 1$, then $|K(P)| > 1$;

(c) If $P, Q \in \mathscr{C}$ and if ϕ is an isomorphism of P onto Q, then

$$\phi(K(P)) = K(Q).$$

Consider the following conditions for a prime p, a finite group G, and a characteristic functor K.

(C_p) *Let S be a Sylow p-subgroup of G. If $W \subseteq S$, $g \in G$, and $W^g \subseteq S$, then there exist $c \in C_G(W)$ and $n \in N_G(K(S))$ such that $g = cn$.*

($\mathrm{C}_p{}^*$) *Let S be a Sylow p-subgroup of G. If W is a p-subgroup of $N_G(K(S))$, $g \in G$, and $W^g \subseteq N_G(K(S))$, then there exist $c \in C_G(W)$ and $n \in N_G(K(S))$ such that $g = cn$.*

Note that in both (C_p) and ($\mathrm{C}_p{}^*$) we obtain $W^g = W^n$; thus W is conjugate to W^g in $N(K(S))$. Suppose that V is a non-empty subset of S and W is the subgroup of S generated by V. If $g \in G$ and V^g is contained in S, then $W^g \subseteq S$.

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Since $C(V) = C(W)$, the conclusion of Theorem B is equivalent to (C_p) for $K(S) = Z(J(S))$.

LEMMA 5.1. *Let G be a finite group, p a prime, and K a characteristic functor. Then* (C_p) *and* $(\mathrm{C}_p{}^*)$ *are equivalent.*

Proof. Obviously, $(\mathrm{C}_p{}^*)$ implies (C_p). Conversely, assume that G satisfies (C_p). Let S be a Sylow p-subgroup of G, and let $N = N_G(K(S))$. Suppose that W is a p-subgroup of N, $g \in G$, and $W^g \subseteq N$. Take k, $m \in N$ such that $W^{gk} \subseteq S$ and $W^m \subseteq S$. Then

$$(W^m)^{m^{-1}gk} = W^{gk} \subseteq S.$$

By (C_p), there exist $c \in C(W^m)$ and $n \in N$ such that $m^{-1}gk = cn$. Now, $g = mcnk^{-1} = (mcm^{-1})(mnk^{-1})$. Since $mnk^{-1} \in N$ and

$$mcm^{-1} = c^{m^{-1}} \in C(W^m)^{m^{-1}} = C(W),$$

G satisfies $(\mathrm{C}_p{}^*)$.

THEOREM 5.1. *Let p be a prime and let S be a Sylow p-subgroup of a finite group G. Let K be the characteristic functor given by $K(P) = Z(J(P))$. Suppose that G has at least one of the following properties*:

(a) *Whenever $Z(S) \subseteq H \subseteq S$, then $N(H)$ satisfies* (C_p);

(b) *Every element of $\mathscr{M}_p(G)$ satisfies* (C_p).

Then G satisfies (C_p).

Proof. Let $J = J(S)$. Clearly, we may assume that $S \neq 1$. Note that if S satisfies (a), then every Sylow p-subgroup of G satisfies (a). We prove the theorem in four steps, the first of which is the most difficult.

(I) *If $g \in G$ and $Z(S)^g \subseteq S$, then $Z(S)^g \subseteq Z(J)$.*

Proof. We require the methods of Burnside (**9**, p. 46) and Thompson (**13**). Suppose some $g \in G$ violates (I). Then $Z(S) \subseteq S^{g^{-1}}$, but $Z(S) \not\subseteq Z(J(S^{g^{-1}}))$. Out of all the Sylow p-subgroups S_1 of G for which

$$Z(S) \subseteq S_1 \quad \text{but} \quad Z(S) \not\subseteq Z(J(S_1)),$$

choose S_1 such that $|N(J) \cap S_1|$ is maximal. Let $D = N(J) \cap S_1$ and let T_0 be a Sylow p-subgroup of $N(J) \cap N(D)$. Then $D \subset S_1$, thus D is not a Sylow p-subgroup of $N(J)$. By Sylow's theorem,

$$(5.1) \qquad |N(D) \cap S_1| > |D| \quad \text{and} \quad |T_0| = |N(J) \cap N(D)|_p > |D|.$$

Let A be a subgroup of $N(D) \cap S_1$ that contains D. Suppose that $Z(S)^a \subseteq Z(J)$ for every $a \in A$. Let V be the subgroup of D generated by all the subgroups of the form $Z(S)^a$ for $a \in A$. Then $J \subseteq C(V)$ and $A \subseteq N(V)$. Let S^* be a Sylow p-subgroup of $N(V)$ that contains A. Then $S^* \cap C(V)$ is a Sylow p-subgroup of $C(V)$. Take $c \in C(V)$ such that

$$J \subseteq (S^* \cap C(V))^c \subseteq S^{*c}.$$

Then $J = J(S^{*c})$ and $A^c \subseteq S^{*c} \subseteq N(J)$. Thus

$$(5.2) \qquad Z(S) = Z(S)^c \subseteq (S_1)^c \quad \text{and} \quad |(S_1)^c \cap N(J)| \geqq |A^c| = |A|.$$

If $D \subset A$, then by (5.2) and the choice of S_1,

$$Z(S) \subseteq Z(J((S_1)^c)) \quad \text{and} \quad Z(S) = Z(S)^{c^{-1}} \subseteq Z(J(S_1)),$$

which is false. Hence $A = D$.

Taking $A = Z(S_1)D$, we obtain $Z(S_1) \subseteq D$. By (5.1), $D \subset N(D) \cap S_1$. Hence we cannot take $A = N(D) \cap S_1$. Thus there exists $a \in N(D) \cap S_1$ such that $Z(S)^a \nsubseteq Z(J)$. Note that $Z(S)^a \subseteq D \subseteq N(J)$.

Let T_1 be a Sylow p-subgroup of $N(D)$ that contains T_0. Then we have

$$(5.3) \qquad \begin{gathered} Z(S) \subseteq D \subseteq T_1, \quad Z(S)^a \subseteq D \subseteq T_1, \\ |N_{T_1}(J)| = |T_0| > |D|, \quad \text{and} \quad Z(S_1) \subseteq D \subseteq S_1. \end{gathered}$$

Let $\mathscr{H}$ be the set of all non-identity p-subgroups H of G with the following properties:

(i) There exists a Sylow p-subgroup T of $N(H)$ and an element g of $N(H)$ such that

$$Z(S) \subseteq T, \quad Z(S)^g = Z(S)^a \subseteq T, \quad \text{and} \quad |N_T(J)| > |D|;$$

(ii) If G has property (a), then $Z(S_2) \subseteq H \subseteq S_2$ for some Sylow p-subgroup S_2 of G.

By (5.3), $D \in \mathscr{H}$; therefore, $\mathscr{H}$ is non-empty. Define a partial ordering $(\leqq)$ on $\mathscr{H}$ as follows: $H \leqq K$ if

(i)′ $|N(H)|_p < |N(K)|_p$, or

(ii)′ $|N(H)|_p = |N(K)|_p$ and $|N(H)| \leqq |N(K)|$.

Let H be a maximal element of $\mathscr{H}$ with respect to $\leqq$. Let $N = N(H)$. Take T and g to satisfy (i). If G has property (a), then N satisfies (C_p) by (ii). Otherwise, G satisfies (b) and, by (i)′ and (ii)′, $N \in \mathscr{M}_p(G)$. Thus, in any case, N satisfies (C_p). By (i), there exists $n \in N_N(Z(J(T)))$ such that $Z(S)^n = Z(S)^a$. Let T^* be a Sylow p-subgroup of $N(Z(J(T)))$ that contains T, and let S_3 be a Sylow p-subgroup of G that contains T^*. Then $H \subseteq T \subseteq S_3$; thus

$$T = S_3 \cap N(H) \supseteq Z(S_3).$$

Clearly, $Z(S_3) \subseteq Z(T) \subseteq Z(J(T))$. Hence $Z(J(T)) \in \mathscr{H}$. By the maximality of H,

$$|N(T)|_p \leqq |N(Z(J(T)))|_p = |T^*| \leqq |N(H)|_p = |T|.$$

Therefore, T is a Sylow p-subgroup of $N(T)$ and thus of G. Since $Z(S) \subseteq T$ and $|N_T(J)| > |D|$, our choice of S_1 insures that $Z(S) \subseteq Z(J(T))$.

Take $b \in G$ such that $S = T^b$. Since S and $J(T)$ are contained in $C(Z(S))$, there exists $c \in C(Z(S))$ such that $J(T) \subseteq S^c$. Clearly, $J(T) = J(S)^c = J^c$. Thus,

$$J^{cb} = J(T)^b = J(S) = J.$$

Let $h = cb$. Then $b = c^{-1}h$ and $h \in N(J)$. Take n as in the above paragraph, and let $k = n^b$. Then

$$k = n^b \in N(J(T))^b = N(J)$$

and

$$n = bkb^{-1} = c^{-1}hkh^{-1}c.$$

Let $Y = Z(S)^{hkh^{-1}}$ Since $hkh^{-1} \in N(J)$, $Y \subseteq Z(J)$. Moreover,

$$Z(S)^a = Z(S)^n = Z(S)^{c^{-1}hkh^{-1}c} = Z(S)^{hkh^{-1}c} = Y^c.$$

By (5.3), $Y^c \subseteq D \subseteq N(J) \subseteq N(Z(J))$.

Let $X = N(Z(S))$. If G satisfies (a), let $M = X$. Otherwise, let M be an element of $\mathscr{M}_p(G)$ that contains X. In either case, M satisfies (C_p) by the hypothesis of the theorem. Therefore, M satisfies (C_p^*) by Lemma 5.1. Moreover, S is a Sylow p-subgroup of M. Since $c \in \mathrm{C}(Z(S)) \subseteq M$ and $Y \subseteq Z(J) \subseteq S$, Y and Y^c are p-subgroups of $N_M(Z(J))$ that are conjugate in M. By (C_p^*), there exists $m \in N_M(Z(J))$ such that $Y^c = Y^m$. Since $Y \subseteq Z(J)$,

$$Z(S)^a = Y^c = Y^m \subseteq Z(J),$$

contrary to the choice of a. This completes the proof of step (I).

(II) *G satisfies* (C_p) *for* $W = Z(S)$.

Proof. Suppose that $g \in G$ and that $Z(S)^g \subseteq S$. By (I), $Z(S)^g \subseteq Z(J)$. Therefore, $C(Z(S))$ contains $J^{g^{-1}}$ and S. Take $c \in C(Z(S))$ such that $(J^{g^{-1}})^c \subseteq S$, and let $n = g^{-1}c$. Then $g = cn^{-1}$. Clearly, $J^n = J(S) = J$; thus $n \in N(J) \subseteq N(Z(J))$.

(III) *G satisfies* (C_p) *when* $C_S(W)$ *is a Sylow* p*-subgroup of* $C_G(W)$.

Proof. Let $Z = Z(J)$. Suppose that $g \in G$ and that $W^g \subseteq S$. Then $Z(S)$ centralizes W^g. Therefore, $Z(S)^{g^{-1}} \subseteq C(W)$. Take $a \in C(W)$ such that $Z(S)^{g^{-1}a} \subseteq C_S(W)$ By (II), there exists $c \in C(Z(S))$ and $n \in N(Z)$ such that $g^{-1}a = cn$. Then

$$W^g = W^{an^{-1}c^{-1}} = W^{n^{-1}c^{-1}}.$$

Let $M = N(Z(S))$ if G satisfies (a); otherwise, let M be an element of $\mathscr{M}_p(G)$ that contains $N(Z(S))$. Then S is a Sylow p-subgroup of M, and M satisfies (C_p) and (C_p^*). Since $W^g \subseteq S$, $W^g \subseteq N_M(Z)$. Consequently,

$$W^{n^{-1}} \subseteq N(Z) \quad \text{and} \quad W^{n^{-1}} = W^{gc} \subseteq M^c = M.$$

Thus W^g and $W^{n^{-1}}$ are p-subgroups of $N_M(Z)$ that are conjugate in M. By (C_p^*), there exist $d \in C_M(W^g)$ and $m \in N_M(Z)$ such that $c = dm$. Then for all $w \in W$,

$$w^{n^{-1}} = w^{an^{-1}} = w^{gc} = w^{gdm} = ((w^g)^d)^m = (w^g)^m = w^{gm};$$

hence $w = w^{gmn}$. Thus

$$gmn \in C(W), \quad mn \in N(Z), \quad \text{and} \quad g = (gmn)(mn)^{-1}.$$

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(IV) *G satisfies* (C_p) *for arbitrary* W.

Proof. Let $Z = Z(J)$. Suppose that $W^g \subseteq S$. Let S_1 be a Sylow p-subgroup of G that contains some Sylow p-subgroup of $N(W)$. Take $x \in G$ such that $S^x = S_1$ and let $V = W^{x^{-1}}$. Then $W \subseteq S_1$, $V = W^{x^{-1}} \subseteq S$, and $N_S(V)$ is a Sylow p-subgroup of $N(V)$. Now, $C_S(V) = N_S(V) \cap C(V)$, which is a Sylow p-subgroup of $C(V)$. Also, $V^x = W \subseteq S$ and $V^{xg} = W^g \subseteq S$. By (III), there exist $c, d \in C(V)$ and $m, n \in N(Z)$ such that $c^{-1}m = x$ and $dn = xg$. Since $W = V^x = V^m$, we have $m^{-1}cdm = (cd)^m \in C(V)^m = C(W)$. Therefore,

$$m^{-1}cdm \in C(W), \quad m^{-1}n \in N(Z),$$

and

$$g = x^{-1}dn = m^{-1}cdn = (m^{-1}cdm)(m^{-1}n).$$

This completes the proof of Theorem 5.1.

LEMMA 5.2. *Let p be a prime and let S be a Sylow p-subgroup of a finite group G. Suppose that $P = S \cap O_{p',p}(G)$ and $\bar{G} = G/O_{p'}(G)$. If $C_S(P) \subseteq P$, then $C(O_p(\bar{G})) \subseteq O_p(\bar{G})$.*

Proof. Let $C = C(P)$. Since P is a normal subgroup of S, $C_S(P)$ is a Sylow p-subgroup of C. But

$$C_S(P) = P \cap C(P) = Z(P) \quad \text{and} \quad Z(P) \subseteq Z(C).$$

Thus $Z(P)$ is a Sylow p-subgroup of C that is contained in the centre of its normalizer in C. By a theorem of Burnside (**8**, p. 203), C has a normal p-complement. Thus $C = O_{p',p}(C)$. Since $O_{p'}(C) \subseteq O_{p'}(G)$, then

$$C(P) = C \subseteq O_{p',p}(G).$$

Let $L = O_{p',p}(G)$ and $M = O_{p'}(G)$. Since L/M is a p-group, then $L = MP$. By the Frattini argument,

$$G = LN(P) = MPN(P) = MN(P).$$

Suppose that

$$\bar{x} \in C_{\bar{G}}(O_p(\bar{G})) = C_{\bar{G}}(MP/M).$$

Take $x \in N(P)$ such that x lies in the coset $\bar{x}$. Then

$$[x, P] \subseteq M \cap P = 1.$$

Thus $x \in C(P) \subseteq L$, and $\bar{x} \in L/M = O_p(\bar{G})$.

LEMMA 5.3. *Let P be a normal subgroup of a finite p-group S, and let A be the automorphism group of S. Then $C_A(P) \cap C_A(S/P)$ is a p-group.*

Proof. Let $\alpha \in C_A(P) \cap C_A(S/P)$ and $n = |P|$. We will show that $\alpha^n = 1$. Suppose that $g \in S$ and that $g^\alpha = gh$. Then $h \in P$ and

$$g^\alpha = gh, \quad g^{\alpha^2} = (gh)^\alpha = g^\alpha h = gh^2, \ldots, g^{\alpha^n} = gh^n = g.$$

THEOREM 5.2. *Let G be a finite group, p a prime, and K a characteristic functor. Assume that every subquotient Q of G has the following properties*:

(a) *If $C(O_p(Q)) \subseteq O_p(Q)$, then $K(Q_p)$ is a normal subgroup of Q for every Sylow p-subgroup Q_p of Q*;

(b) *If $|Q| < |G|$, then Q satisfies* (C_p).

Then, if $O_p(G) \neq 1$, G satisfies (C_p).

Note. For p odd, Theorem A and Example 10.1 show that G satisfies (a) for some K if and only if $\mathrm{Qd}(p)$ is not involved in G. In that case we may let $K(P) = Z(J(P))$.

Proof. Let $P = O_p(G)$, $N = N(K(S))$, and $R = PC(P) \cap S$. Since $PC(P)$ is a normal subgroup of G, the Frattini argument yields

$$G = PC(P)N(R) = C(P)PN(R) = C(P)N(R). \tag{5.4}$$

Let $M = O_{p'}(N(R))$, $P_1 = S \cap O_{p',p}(N(R))$, and $Q = N(R)/M$. Then

$$C_S(P_1) \subseteq C_S(R) \subseteq C_S(P) \subseteq R \subseteq P_1.$$

By Lemma 5.2, $C_Q(O_p(Q)) \subseteq O_p(Q)$. By (a), $K(SM/M)$ is a normal subgroup of Q. By the definition of a characteristic functor, $K(SM/M) = K(S)M/M$. Thus $K(S)M$ is a normal subgroup of $N(R)$. Since $K(S)$ is a Sylow p-subgroup of $K(S)M$, we have that

$$N(R) = M(N(K(S)) \cap N(R)) = MN_N(R) \tag{5.5}$$

by the Frattini argument. However,

$$[P, M] \subseteq P \cap M = 1.$$

Hence, by (5.4) and (5.5),

$$G = C(P)N(R) = C(P)MN_N(R) = C(P)N_N(R) = C(P)N.$$

Suppose that $C(P)S \subset G$. Take $W \subseteq S$ and $g \in G$ such that $W^g \subseteq S$. Choose $d \in C(P)$ and $n \in N$ such that $dn = g$. Then

$$W^d = W^{gn^{-1}} \subseteq N^{n^{-1}} = N.$$

Since $d \in C(P)S \subset G$ and $W \subseteq C(P)S$, there exist $c \in C(W)$ and $m \in N$ such that $cm = d$. Hence $g = dn = c(mn)$. Thus G satisfies (C_p).

We assume henceforth that $C(P)S = G$. Suppose that $W \subseteq S$, $g \in G$, and $W^g \subseteq S$. Take $c_0 \in C(P)$ and $n_0 \in S$ such that $c_0 n_0 = g$. We have $n_0 \in N$, and if $W \subseteq P$, then $c_0 \in C(W)$. We may assume that $W \not\subseteq P$. Then $1 \subset P \subset S$. Let $\bar{G} = G/P$, $\bar{S} = S/P$, and $\bar{W} = W/P$, and let $\bar{g}$ be the coset of P that contains g. Then

$$\bar{W}^{\bar{g}} \subseteq \bar{S}.$$

By (b),

$$\bar{g} = \bar{c}\bar{n} \quad \text{for some} \quad \bar{c} \in C_{\bar{G}}(\bar{W}) \quad \text{and} \quad \bar{n} \in N_{\bar{G}}(K(\bar{S})). \tag{5.6}$$

Take $T \subseteq G$ such that $P \subseteq T$ and $T/P = K(\bar{S})$. Since $P \subset S$, we have that

$$(5.7) \qquad \bar{S} \neq 1, \quad K(\bar{S}) \neq 1, \quad P \subset T, \quad \text{and} \quad S \subseteq N(T) \subset G.$$

Moreover,

$$\bar{W}^{\bar{n}} = \bar{W}^{\bar{c}\bar{n}} = \bar{W}^{\bar{g}} \subseteq \bar{S}.$$

Let n be an element of the coset $\bar{n}$. Then $n \in N(T)$ and $W^n \subseteq S$. By (5.7) there exist $d \in C(W)$ and $m \in N$ such that $dm = n$. Obviously, $d \in C_G(\bar{W})$. By (5.6), $gm^{-1} \in C_G(\bar{W})$. Thus

$$(5.8) \qquad g \in C_G(\bar{W})N.$$

Let x be a p'-element of $C_G(\bar{W})$. Since $G = C(P)S$, $G/C(P)$ is a p-group. Hence $x \in C(P)$. Thus x centralizes WP/P and P. By Lemma 5.3, x centralizes WP. Thus $C_G(\bar{W})/C_G(WP)$ is a p-group. When $C_S(\bar{W})$ is a Sylow p-subgroup of $C_G(\bar{W})$, (5.8) yields

$$g \in C_G(\bar{W})N = C_G(WP)C_S(\bar{W})N \subseteq C_G(W)N,$$

as desired. The general case now follows by the method used in step (IV) of the proof of Theorem 5.1.

Remark 5.1. After Theorem 5.1 (a) had been proved, it was generalized in several ways by Alperin and Gorenstein (1) to the case where K is an arbitrary characteristic functor. Using these generalizations, we may prove a stronger form of Theorem 5.2, which we will not require in this paper.

THEOREM 5.2′. *Let G be a finite group, p a prime, and K a characteristic functor. Assume that whenever Q is a subquotient of G and $C(O_p(Q)) \subseteq O_p(Q)$, then $K(Q_p)$ is a normal subgroup of Q for every Sylow p-subgroup Q_p of Q. Then G satisfies* (C_p).

6. Proof of Theorem B.

In this section we show that a finite group in which $\mathrm{Qd}(p)$ is not involved must be p-stable. This result then yields Theorem B.

LEMMA 6.1 (Baer). *Let g be a p-element of a finite group G. Suppose that for every $x \in G$, g and g^x generate a p-group. Then $g \in O_p(G)$.*

Proof. Let c be the nilpotence class of a Sylow p-subgroup of G. Then

$$[x, g; c+2] = [(g^x)^{-1}g, g; c+1] = 1 \quad \text{for all } x \in G.$$

Thus g is a (bounded) left Engel element of G (**12**, p. 207). By a theorem of Baer (**12**, p. 212), g lies in a normal nilpotent subgroup N of G. Clearly, $g \in O_p(N) \subseteq O_p(G)$.

LEMMA 6.2. *Let p be an odd prime and let $F = \mathrm{GF}(p)$. Assume that H is a group of linear transformations that acts irreducibly and faithfully on a vector space V over F. Suppose that H is generated by two p-elements, each with minimal*

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polynomial $(x - 1)^2$. *Then there exists a field* K *of endomorphisms of* V *such that*

(a) F *is contained in* K;

(b) V *has dimension two over* K; *and*

(c) *either* G *is the special linear group of* V *over* K, *or* $|K| = 9$ *and* G *is isomorphic to* $\mathrm{SL}(2, 5)$.

Proof. A similar but incorrect statement appears in (**8**, Lemma 4.1, p. 136). In the proof of that lemma, the authors showed that the dimension m of V over F is even and that for a suitable basis of V, the elements x and y are represented by matrices of the form

$$\begin{pmatrix} I & R \\ O & I \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} I & O \\ I & I \end{pmatrix}.$$

(Here, the submatrices are square matrices of degree $\frac{1}{2}m$, I being the identity matrix.) An easy argument shows that R is the matrix of a non-singular irreducible transformation. Let K_1 be the algebra over F that is generated by R. By Schur's lemma, K_1 is a field. Let K be the field of endomorphisms of V represented by the matrices of the form

$$\begin{pmatrix} S & O \\ O & S \end{pmatrix}, \qquad S \in K_1.$$

Then K satisfies (a) and (b). Now, (c) follows from a result of Dickson (**6**, Theorem 2.8.4).

Remark. The author thanks Professors J. Alperin and D. Gorenstein for informing him of the error in (**8**) and of the appearance of Dickson's result in (**6**).

LEMMA 6.3. *Let* p *be an odd prime, and let* G *be a finite group. The following are equivalent*:

(a) $\mathrm{Qd}(p)$ *is not involved in* G;

(b) *Every subquotient of* G *is* p-*stable.*

Proof. It was pointed out in §2 that $\mathrm{Qd}(p)$ is not p-stable and thus that (b) implies (a). Conversely, assume that some subquotient Q of G is not p-stable. Take Q of minimal order. We will prove that Q is isomorphic to $\mathrm{Qd}(p)$.

Clearly, we may assume that $Q = G/1 = G$. Then by the definition of p-stability, $O_p(G) \neq 1$. Let H be a p-subgroup of G of least order subject to the following conditions:

(i) $O_{p'}(G)H$ is a normal subgroup of G;

(ii) $N(H)$ contains an element x such that $[H, x, x] = 1$ and such that the coset of $C(H)$ containing x lies outside $O_p(N(H)/C(H))$.

Let $M = O_{p'}(G)$ and $C = C_G(MH/M)$. By the Frattini argument,

$$G = MHN(H) = MN(H). \tag{6.1}$$

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Hence $M \subseteq C \subseteq MN(H)$. Therefore,

$$C = M(N(H) \cap C) = MC(H). \tag{6.2}$$

Denote by a bar the image of an element or subgroup of G in the natural mapping of G onto G/M. By (6.1) and (6.2),

$$\overline{N(H)} = \bar{G} = N_{\bar{G}}(\bar{H}), \qquad \overline{C(H)} = \bar{C} = C_{\bar{G}}(\bar{H}).$$

Since $N(H) \cap M \subseteq N(H) \cap C = C(H)$, we obtain the natural isomorphisms

$$N(H)/C(H) \cong (N(H)/(C(H) \cap M))/(C(H)/(C(H) \cap M)) \cong \overline{N(H)}/\overline{C(H)} \cong N_{\bar{G}}(\bar{H})/C_{\bar{G}}(\bar{H}).$$

Therefore, $[\bar{H}, \bar{x}, \bar{x}] = 1$, but the coset of $C_{\bar{G}}(\bar{H})$ containing $\bar{x}$ does not lie in $O_p(N_{\bar{G}}(\bar{H})/C_{\bar{G}}(\bar{H}))$. Therefore $\bar{G}$ is not p-stable. Thus $M = 1$, $C = C(H)$, and H is a normal subgroup of G.

Let R be a Sylow r-subgroup of C for any prime r. By the Frattini argument, $G = CN(R)$. Take $c \in C$ and $y \in N(R)$ such that $cy = x$; then $[H, y, y] = 1$. In the natural homomorphism of G onto G/C, y maps onto the coset of x and $N(R)$ maps onto G/C. Moreover, $H \subseteq C(R) \subseteq N(R)$. Thus, $N(R)$ is not p-stable. Hence $G = N(R)$. Since $O_{p'}(G) = 1$ and r is an arbitrary prime, we have that C is a p-group.

Suppose that G contains a normal subgroup K such that $1 \subset K \subset H$. Take K to be a minimal normal subgroup of G. Then K is an elementary Abelian group and G is irreducible on K. By Lemma 3.4,

$$O_p(G/C(K)) = 1. \tag{6.3}$$

Since $[K, x, x] \subseteq [H, x, x] = 1$, we obtain $x \in C(K)$ by (6.3) and our choice of H. Take $N \subseteq G$ such that

$$C_G(H/K) \subseteq N \quad \text{and} \quad N/C_G(H/K) = O_p(G/C_G(H/K)).$$

Since G/K is p-stable and $[H, x, x] = 1 \subseteq K$, we have $x \in N$. Therefore, $x \in C_N(K)$. Let $D = C_G(H/K)$. Since N/D is a p-group, so are $C_N(K)D/D$ and $C_N(K)/(C_N(K) \cap D)$. Now,

$$(C_N(K) \cap D)/C = (C_G(K) \cap C_G(H/K))/C_G(H),$$

which is a p-group by Lemma 5.3. Since C is a p-group, so are $C_N(K) \cap D$ and $C_N(K)$. Thus $x \in C_N(K) \subseteq O_p(G)$. This is impossible since

$$O_p(G)C/C \subseteq O_p(G/C).$$

Hence K does not exist. Consequently,

(6.4) $\quad H$ is an elementary Abelian group, and G is irreducible on H.

Let $w \in G$, and let G^* be the subgroup of G generated by x, x^w, and C. We choose w such that G^*/C is not a p-group; this is possible by Lemma 6.1. Then $O_p(G^*/C) \subset G^*/C$. Since

$$[H, x, x] = 1 \quad \text{and} \quad [H, x^w, x^w] = [H^w, x^w, x^w] = [H, x, x]^w = 1,$$

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G^* is not p-stable. Therefore,

$$G = G^*. \tag{6.5}$$

We consider H as a vector space over $\mathrm{GF}(p)$. Let y be the linear transformation of H given by conjugation by x. For each $h \in H$,

$$h^{(y-1)^2} = [h, x, x] = 1.$$

Thus $(y-1)^2 = 0$, and $y^p - 1 = (y-1)^p = (y-1)^2(y-1)^{p-2} = 0$. The same is true of the automorphism corresponding to x^w. Hence by (6.4), (6.5), and Lemma 6.2, there exists a finite field K_0 such that H is a two-dimensional vector space over K_0 and G/C corresponds either to the special linear group of H over K_0 or possibly (if $|K_0| = 9$) to $\mathrm{SL}(2, 5)$. Then the element y introduced above is contained in a subgroup L of G/C that isomorphic to $\mathrm{SL}(2, p)$. Since $y \notin O_p(L)$ and since every proper subgroup of G is p-stable, we have that $L = G/C$. Thus $K_0 = \mathrm{GF}(p)$.

Thus G/C is isomorphic to $\mathrm{SL}(2, p)$. Therefore, G/C contains a unique subgroup T/C of order two, and the non-identity element of T/C corresponds to the automorphism $h \to h^{-1}$ of H. Let T_2 be a Sylow 2-subgroup of T. By the Frattini argument,

$$G = TN(T_2) = CT_2N(T_2) = CN(T_2).$$

Therefore, $N(T_2)H$ is not p-stable; thus

$$G = N(T_2)H. \tag{6.6}$$

Let $E = C \cap N(T_2)$. Since C is a normal subgroup of G, E is normalized by $N(T_2)$. As it is obviously centralized by H, E is a normal subgroup of G. Moreover,

$$[E, T_2] \subseteq C \cap T_2 \subseteq O_p(G) \cap T_2 = 1;$$

thus $E \cap H \subseteq C(T_2) \cap H = 1$. By (6.6), $C = EH$. Consequently, $C = E \times H$. Hence G/E is not p-stable. Thus $E = 1$, and $C = H$. This completes the proof of Lemma 6.3.

We may now prove Theorem B by induction on $|G|$. Assume the theorem holds for all groups of order less that $|G|$. Obviously, we may assume that p divides $|G|$. By Theorem 5.1 (a), we may assume that $O_p(G) \neq 1$. Let $K(R) = Z(J(R))$ for every subquotient R of G that is a p-group. Then K is a characteristic functor. By Lemma 6.3 and Theorem A, K satisfies condition (a) of Theorem 5.2. Since $\mathrm{Qd}(p)$ is not involved in any subquotient of G, condition (b) holds by the induction hypothesis. Hence Theorem B follows from Theorem 5.2.

COROLLARY 6.1. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite group G. Assume that $\mathrm{Qd}(p)$ is not involved in G. For every non-empty subset H of S,*

$$N(H) = C(H)(N(H) \cap N(Z(J(S)))).$$

7. Proof of Theorem C.

LEMMA 7.1. *Let G be a finite group and p a prime. Suppose that $H \subseteq G$ and $G = O_{p'}(G)H$. Let W be a p-subgroup of H. If $g \in G$ and $W^g \subseteq H$, then there exist $c \in O_{p'}(G) \cap C(W)$ and $h \in H$ such that $g = ch$.*

Proof. Take $b \in O_{p'}(G)$ and $k \in H$ such that $g = bk$. Then

$$W^b = (W^g)^{k^{-1}} \subseteq H.$$

Let R be the subgroup of G generated by W and W^b. Then $R \subseteq H$, and since $W \subseteq R \subseteq WO_{p'}(G)$, we have that

$$R = W(O_{p'}(G) \cap R).$$

Therefore, W and W^b are Sylow p-subgroups of R and there exists $d \in O_{p'}(G) \cap R$ such that $(W^b)^d = W$. For every $x \in W$, $x^{-1}x^{bd} \in R$ and $x \equiv x^{bd}$ (modulo $O_{p'}(G)$). Thus $x^{-1}x^{bd} \in R \cap O_{p'}(G) = 1$ for all $x \in W$. Let $c = bd$ and $h = d^{-1}k$.

LEMMA 7.2. *Let p be an odd prime and let G be a finite p-stable group such that $O_p(G) \neq 1$. Then $G/O_{p'}(G)$ is p-stable.*

Proof. Let $M = O_{p'}(G)$ and $\bar{G} = G/M$. Then $O_{p'}(\bar{G}) = 1$ and $O_p(\bar{G}) \neq 1$. Let $\bar{P}$ be a normal p-subgroup of $\bar{G}$. Suppose $\bar{x} \in \bar{G}$ and $[\bar{P}, \bar{x}, \bar{x}] = 1$. Take $R \subseteq G$ such that $M \subseteq R$ and $R/M = \bar{P}$. Let P be a Sylow p-subgroup of R. By the Frattini argument, $G = MN(P)$. Take $x \in N(P)$ and $L \subseteq N(P)$ such that x lies in the coset $\bar{x}$, $C(P) \subseteq L$, and $L/C(P) = O_p(N(P)/C(P))$. Then

$$[P, x, x] \in P \cap M = 1.$$

Since G is p-stable, $x \in L$. Therefore, $\bar{x} \in LM/M$. Since $C(P)M/M \subseteq C_{\bar{G}}(\bar{P})$ and $L/C(P)$ is a p-group, we conclude that $(LM/M)/C_{\bar{G}}(\bar{P})$ is a p-group. Thus $\bar{G}$ is p-stable.

Consider the following condition for a prime p, a finite group G, and a characteristic functor K:

(F_p) *Let S be a Sylow p-subgroup of G. Then*

$$G = O_{p'}(G)N_G(K(S)).$$

THEOREM 7.1. *Let G be a finite group, p a prime, and K a characteristic functor. Suppose that S is a Sylow p-subgroup of G. Assume that G satisfies (C_p) and that every element of $\mathscr{M}_p(G)$ satisfies (F_p). Then, for every non-identity subgroup W of S,*

$$N(W) = O_{p'}(N(W))(N(W) \cap N(K(S))).$$

Proof. Suppose that $1 \subset W \subseteq S$. Let $N = N(W)$ and $L = O_{p'}(N)$. Then N is contained in some element M of $\mathscr{M}_p(G)$. Obviously, for any such M,

$$W \subseteq M \quad \text{and} \quad N = L(N \cap M). \tag{7.1}$$

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Out of all the elements of $\mathscr{M}_p(G)$ that satisfy (7.1), choose M such that $|M|_p$ is maximal. Let T be a Sylow p-subgroup of M that contains W. By (F_p) (for M) and Lemma 7.1,

$$(7.2)\qquad \begin{aligned} M &= O_{p'}(M)N_M(K(T)) \quad \text{and} \\ N \cap M &= (O_{p'}(M) \cap C(W))N_{N\cap M}(K(T)). \end{aligned}$$

Now, $O_{p'}(M) \cap C(W)$ is a normal subgroup of $N \cap M$. Therefore, by (7.1), $L(O_{p'}(M) \cap C(W))$ is a normal subgroup of N. Since it is a p'-group and contains L, it must coincide with L. By (7.1) and (7.2),

$$(7.3)\qquad N = LN_{N\cap M}(K(T)) = LN_N(K(T)).$$

Let M_1 be an element of $\mathscr{M}_p(G)$ that contains $N(K(T))$. By (7.3), $N = L(N \cap M_1)$. By our choice of M,

$$|T| = |M|_p \geqq |M_1|_p \geqq |N(K(T))|_p \geqq |N(T)|_p \geqq |T|.$$

Therefore, T is a Sylow p-subgroup of its normalizer. By Sylow's theorem, T is a Sylow p-subgroup of G. Take $g \in G$ such that $T^g = S$. Then $W^g \subseteq S$. By (C_p), there exist $c \in C(W)$ and $n \in N(K(S))$ such that $g = cn$. Now,

$$K(T)^c = K(T^c) = K(T^{gn^{-1}}) = K(T^g)^{n^{-1}} = K(S)^{n^{-1}} = K(S)$$

and $N = N(W) = N(W^c) = N(W)^c = N^c$. Therefore, by (7.3),

$$N = L^cN_N(K(T))^c = O_{p'}(N^c)N_N(K(T)^c) = LN_N(K(S)).$$

This completes the proof of Theorem 7.1.

We may now prove Theorem C. Let $K(P) = Z(J(P))$ for every finite p-group P. Assume that G is p-stable and p-constrained. We will first verify the hypothesis of Theorem 7.1.

Suppose that $M \in \mathscr{M}_p(G)$ and T is a Sylow p-subgroup of M. Let $\bar{M} = M/O_{p'}(M)$ and $\bar{T} = TO_{p'}(M)/O_{p'}(M)$. Since M is p-stable and p-constrained, $\bar{M}$ is p-stable and $C(O_p(\bar{M})) \subseteq O_p(\bar{M})$, by Lemmas 7.2 and 5.2. By Theorem A, $Z(J(\bar{T}))$ is a normal subgroup of $\bar{M}$. Since

$$Z(J(\bar{T})) = Z(J(T))O_{p'}(M)/O_{p'}(M),$$

then

$$M = O_{p'}(M)N_M(Z(J(T)))$$

by the Frattini argument. Thus M satisfies (F_p). By Lemma 7.1, M satisfies (C_p). Hence by Theorem 5.1 (b), G satisfies (C_p). Thus, G satisfies the hypothesis of Theorem 7.1.

Suppose that V is a non-empty subset of S, $g \in G$, and V^g is contained in S. Let W be the subgroup of S generated by V. Then $W^g \subseteq S$. By (C_p), there exist $c \in C(W)$ and $n \in N(Z(J(S)))$ such that $g = cn$. By Theorem 7.1, there exist $d \in O_{p'}(N(W))$ and $m \in N(W) \cap N(Z(J(S)))$ such that $dm = c$.

Now

$$[W, d] \subseteq W \cap O_{p'}(N(W));$$

thus $d \in C(W)$. Hence

$$d \in O_{p'}(N(W)) \cap C(W) \subseteq O_{p'}(C(W)) = O_{p'}(C(V)) \subseteq O_{p'}(N(V)).$$

Since $g = d(mn)$ and $mn \in N(Z(J(S)))$, this completes the proof of Theorem C.

COROLLARY 7.1. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite group G. Assume that G is p-stable and p-constrained. Then for every non-empty subset H of S,*

$$N(H) = O_{p'}(C(H))(N(H) \cap N(Z(J(S)))).$$

8. Proof of Theorem D. We prove Theorem D by induction on $|G|$. Assume that the theorem is false and that G is a counter-example of least order. Using the method of Thompson, as in (**13**, pp. 43–44), we find that $C(O_p(G)) \subseteq O_p(G)$, that G is solvable, and that there exists a prime q with the following properties:

(i) the Sylow q-subgroups of G are Abelian;

(ii) $q \neq p$; and

(iii) p and q are the only prime divisors of $|G|$.

Since p is odd, G has an Abelian Sylow 2-subgroup. Hence $\mathrm{Qd}(p)$ is not involved in G. By Lemmas 6.3, G is p-stable. (This also follows from Theorem B of Hall and Higman (**10**).) Since $C(O_p(G)) \subseteq O_p(G)$, $Z(J(S))$ is a normal subgroup of G, by Theorem A. Thus $G = N(Z(J(S)))$. This contradiction completes the proof of Theorem D.

9. A self-centralizing subgroup. Let S be an arbitrary finite p-group. Define two sequences of characteristic subgroups of S in the following manner. Let $K_0 = 1$ and $S_0 = S$. Given $K_0, \ldots, K_i$, and $S_0, \ldots, S_i$, let K_{i+1} and S_{i+1} be the subgroups of S that contain K_i and satisfy

$$K_{i+1}/K_i = Z(J(S_i/K_i)) \quad \text{and} \quad S_{i+1}/K_i = C_{S_i/K_i}(K_{i+1}/K_i).$$

We let $ZJ_i(S) = K_i$ for all i. Thus $ZJ_0(S) = 1$, $ZJ_1(S) = Z(J(S))$, and

$$ZJ_0(S) \subseteq ZJ_1(S) \subseteq \ldots \quad \text{and} \quad S_0 \supseteq S_1 \supseteq \ldots.$$

Moreover,

$$ZJ_{i+1}(S)/ZJ_1(S) = ZJ_i(S_1/ZJ_1(S)) \quad \text{for } i = 0, 1, 2, \ldots. \tag{9.1}$$

Let n be the smallest integer such that $ZJ_n(S) = ZJ_{n+1}(S) = \ldots$, and let $ZJ^*(S) = ZJ_n(S)$. Then

$$1 = K_{n+1}/K_n = Z(J(S_n/K_n)),$$

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and therefore $S_n = K_n$. We note that if $x \in C(K_n)$, then $x \in S_i$ for all i. As a result,

$$C(ZJ^*(S)) \subseteq ZJ^*(S). \tag{9.2}$$

Clearly, ZJ^* and ZJ_0, ZJ_1, ... determine characteristic functors, and ZJ^* has the additional property (9.2). We shall prove that ZJ^* satisfies some analogues of Theorems A through D.

LEMMA 9.1. *Let p be a prime and let S be a Sylow p-subgroup of a finite group G. Suppose that N is a normal subgroup of G and that $C(O_p(G)) \subseteq O_p(G)$. Then:*
(a) $C_N(O_p(N)) \subseteq O_p(N)$;
(b) *If $N \subseteq Z(G)$, then $C_{G/N}(O_p(G/N)) \subseteq O_p(G/N)$.*

Proof. (a) Let $x \in C_N(O_p(N))$ and $P = O_p(G)$. Then

$$[P, x] \subseteq P \cap N \subseteq O_p(N).$$

Thus x centralizes $P/O_p(N)$ and $O_p(N)$. Since $C(P)$ is a p-group, x is a p-element by Lemma 5.3. Thus $C_N(O_p(N))$ is a normal p-subgroup of N, and therefore $C_N(O_p(N)) \subseteq O_p(N)$.

(b) Let $P = O_p(G)$. Since $C(P) \subseteq P$, we have that $N \subseteq P$. Thus

$$O_p(G/N) = P/N.$$

Now

$$C_G(P/N) = C_G(P/N) \cap C_G(N).$$

As in the proof of (a), $C_G(P/N)$ is a p-group. Hence $C_{G/N}(P/N) \subseteq P/N$.

LEMMA 9.2. *Let p be a prime and let S be a Sylow p-subgroup of a finite group G. Suppose N is a normal subgroup of G and $C_S(N) \subseteq N \subseteq S$. Then*

$$C(N) = Z(N) \times O_{p'}(G).$$

Proof. Let $C = C(N)$ and $\bar{C} = C/O_{p'}(C)$. Then $Z(N) = S \cap C$, which is a Sylow p-subgroup of C. Since $Z(N) \subseteq Z(C)$, Burnside's transfer theorem (**9**, p. 203) implies that

$$C = Z(N)O_{p'}(C) = Z(N) \times O_{p'}(C).$$

Now, $O_{p'}(C) \subseteq O_{p'}(G)$. On the other hand,

$$[N, O_{p'}(G)] \subseteq N \cap O_{p'}(G) = 1.$$

Since $C/O_{p'}(C)$ is a p-group, $O_{p'}(G) \subseteq O_{p'}(C)$. Thus $O_{p'}(G) = O_{p'}(C)$.

THEOREM A′. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite group G. Suppose that $C(O_p(G)) \subseteq O_p(G)$ and that $\mathrm{Qd}(p)$ is not involved in G. Then, $ZJ^*(S)$ is a characteristic subgroup of G and $C(ZJ^*(S)) \subseteq ZJ^*(S)$.*

Proof. Let $K = ZJ^*(S)$. Since $[O_{p'}(G), O_p(G)] \subseteq O_{p'}(G) \cap O_p(G) = 1$, we have that $O_{p'}(G) = 1$. Suppose that K is a normal subgroup of G. From the

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last step of the proof of Theorem A, we see that K is a characteristic subgroup of G. Since $O_{p'}(G) = 1$, we have that $C(K) \subseteq K$ by Lemma 9.2.

Thus, it suffices to show that K is a normal subgroup of G. Take $K_1, K_2, \ldots$ and $S_1, S_2, \ldots$ as above. We shall prove by induction on $|G|$ that K_i is a normal subgroup of G for $i = 1, 2, \ldots$. We may assume that p divides $|G|$.

By Lemma 6.3, G is p-stable. Since $K_1 = Z(J(S))$, K_1 is a normal subgroup of G by Theorem A. Let $C = C(K_1)$ and $\bar{C} = C/K_1$. Then $S_1 = S \cap C$, which is a Sylow p-subgroup of C. Therefore

$$S_1/K_1 \text{ is a Sylow } p\text{-subgroup of } \bar{C}. \tag{9.3}$$

Let $P = O_p(C)$. By Lemma 9.1 (a), $C_C(P) \subseteq P$. By Lemma 9.1 (b),

$$C_{\bar{C}}(O_p(\bar{C})) \subseteq O_p(\bar{C}).$$

Obviously, $\mathrm{Qd}(p)$ is not involved in $\bar{C}$. Suppose that i is an integer and $i \geqq 2$. By the induction hypothesis, $(ZJ_{i-1}(S_1/K_1))/K_1$ is a normal, and therefore a characteristic, subgroup of $\bar{C}$. By (9.1),

$$K_i/K_1 = (ZJ_{i-1}(S_1/K_1))/K_1.$$

Thus K_i/K_1 is a characteristic subgroup of $\bar{C}$. Hence K_i/K_1 is a normal subgroup of G/K_1, and K_i is a normal subgroup of G.

Since $K = K_n$ for some n, we have that K is a normal subgroup of G.

THEOREM B′. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite group G. Assume that* $\mathrm{Qd}(p)$ *is not involved in G. Suppose that W is a non-empty subset of S, $g \in G$, and W^g is contained in S. Then there exist $c \in C(W)$ and $n \in N(ZJ^*(S))$ such that $g = cn$.*

Proof. We use induction on $|G|$. Assume the theorem holds for all groups of order less than $|G|$. By Lemma 6.3, G is p-stable. Clearly, we may assume that p divides $|G|$. If $O_p(G) \neq 1$, the result follows from Theorem A′ and Theorem 5.2. Assume $O_p(G) = 1$. By Theorem B, there exist $d \in C(W)$ and $m \in N(Z(J(S)))$ such that $dm = g$. Since $O_p(G) = 1$, $N(Z(J(S))) \subset G$. By the induction hypothesis, there exist $d' \in C(W)$ and $m' \in N(ZJ^*(S))$ such that $d'm' = m$. Then we may let $c = dd'$ and $n = m'$ to complete the proof.

THEOREM C′. *Let p be an odd prime and let S be a Sylow p-subgroup of a finite group G. Assume that* $\mathrm{Qd}(p)$ *is not involved in G and that G is p-constrained. Suppose that W is a non-empty subset of S, $g \in G$, and W^g is contained in S. Then there exist $c \in O_{p'}(C(W))$ and $n \in N(ZJ^*(S))$ such that $g = cn$.*

Proof. Let $K(P) = ZJ^*(P)$ for every finite p-group P. Then G satisfies (C_p) by Theorem B′. The remainder of the proof is parallel to that of Theorem C.

THEOREM D′. *Let p be an odd prime, and let S be a Sylow p-subgroup of a finite group G. Then G has a normal p-complement if and only if*

$$N(ZJ^*(S))/C(ZJ^*(S))$$

is a p-group.

Proof. By imitating the proof of Theorem D, we find that G has a normal p-complement if and only if $N(ZJ^*(S))$ has a normal p-complement. Let $Z = ZJ^*(S)$ and $L = N(Z)$. By Lemma 9.2, $C(Z) = Z(Z) \times O_{p'}(L)$. Therefore, L has a normal p-complement if and only if $L/C(Z)$ is a p-group.

10. Variations and counter-examples. The following lemma can be proved by straightforward computation.

LEMMA 10.1. *Let V and W be finite-dimensional vector spaces over a field F. Suppose that G is a faithful operator group of linear transformations on V and also on W, and suppose that f is a bilinear G-mapping of V and W into F. Let H be the set of all ordered triples (v, w, α) for $v \in V$, $w \in W$, and $\alpha \in F$. Define multiplication on H by*

$$(v_1, w_1, \alpha_1)(v_2, w_2, \alpha_2) = (v_1 + v_2, w_1 + w_2, \alpha_1 + \alpha_2 - f(v_2, w_1)).$$

For each $g \in G$, define a mapping $A(g)$ of H into itself by

$$(v, w, \alpha)^{A(g)} = (v^g, w^g, \alpha).$$

Then:

(a) *H forms a group under multiplication*;

(b) *For (v, w, α), (v_1, w_1, α_1), and (v_2, w_2, α_2) in H,*

$$(v, w, \alpha)^{-1} = (-v, -w, -f(v, w) - \alpha)$$

and

$$[(v_1, w_1, \alpha_1), (v_2, w_2, \alpha_2)] = (0, 0, f(v_1, w_2) - f(v_2, w_1));$$

(c) *A is an isomorphism of G into the automorphism group of H.*

Example 10.1. The hypothesis of p-stability, or some similar condition, seems necessary for Theorems A, B, and C and analogous statements. Similarly, Theorem D and possible analogues fail for $p = 2$. For example, let p be a prime, and let $G = \mathrm{Qd}(p)$. A Sylow p-subgroup S of G is a dihedral group of order eight if $p = 2$ and a non-Abelian group of order p^3 and exponent p if p is odd. Then $Z(J(S)) = Z(S)$. The only characteristic subgroups are $Z(S)$, S, and, if $p = 2$, the cyclic subgroup T of order four in S. Although $C(O_p(G)) = O_p(G)$, none of these characteristic subgroups is a normal subgroup of G. Thus G does not satisfy the conclusions of Theorem A, B, or C, and these conclusions are also invalid if we replace $Z(J(S))$ by any characteristic subgroup of S. If $p = 2$, then G is isomorphic to the symmetric group of degree four and does not have a normal p-complement, although

$$N(Z(S)) = N(T) = N(S) = S,$$

which has a trivial normal p-complement. Thus Theorem D fails for $p = 2$.

Example 10.2. Let S be a Sylow 2-subgroup of a finite group G in which $C(O_2(G)) \subseteq O_2(G)$. If Qd(2) is not involved in G, there may be a characteristic subgroup $K(S)$ of S that is normal in G (and depends only on S, not on G). However, we will show that $Z(J(S))$ is not that subgroup.

Let q be an odd prime and let F be a finite field of characteristic 2 that contains a primitive qth root of unity, say, ω. Suppose that V and W are two-dimensional vector spaces over F. Let D be a dihedral group of order $2q$ with generators τ and π such that

$$\tau^2 = \pi^q = 1, \qquad \tau^{-1}\pi\tau = \pi^{-1}.$$

Let $\{x_1, x_2\}$ and $\{y_1, y_2\}$ be bases of V and W, respectively. We consider D as a (faithful) operator group of linear transformations on V and W by defining

$$x_1^\tau = x_2, \quad x_2^\tau = x_1, \quad y_1^\tau = y_2, \quad y_2^\tau = y_1,$$

and

$$x_1^\pi = \omega x_1, \quad x_2^\pi = \omega^{-1}x_2, \quad y_1^\pi = \omega^{-1}y_1, \quad y_2^\pi = \omega y_2.$$

Let f be the bilinear mapping of V and W into F determined by

$$f(x_1, y_1) = f(x_2, y_2) = 1 \quad \text{and} \quad f(x_1, y_2) = f(x_2, y_1) = 0.$$

Then f is a D-mapping, and we may construct a corresponding group H as in Lemma 10.1. By Lemma 10.1 (c), we may consider D to be a group of automorphisms of H. Let G be the semi-direct product of H by D.

Let S be the subgroup of G that is generated by H and τ. Then S is a Sylow 2-subgroup of G and

$$Z(S) = Z(H) = \{(0, 0, \alpha): \alpha \in F\}.$$

Moreover,

$$C_H(\tau) = \{(\beta x_1 + \beta x_2, \gamma y_1 + \gamma y_2, \alpha): \alpha, \beta, \gamma \in F\}$$

and

$$C_{H/Z(H)}(\tau) = Z(S/Z(H)) = C_H(\tau)/Z(H).$$

Thus $C_S(\tau)$ is an elementary Abelian group of order $2|F|^3$. We claim that $\mathscr{A}(S) = \{C_S(\tau)\}$.

Suppose that A is an Abelian subgroup of S and $|A| \geqq 2|F|^3$. Since $|S/H| = 2$, $|A \cap H| \geqq |F|^3$. Now, the commutator map from $H \times H$ into $Z(H)$ corresponds to a non-singular skew-symmetric bilinear form on the internal direct sum of V and W. This direct sum has order $|F|^4$ and has maximal isotropic subspaces of order $|F|^2$. Therefore, $d(H) = |F|^2|Z(H)| = |F|^3$, and

$|A \cap H| \leqq |F|^3$. Hence $A \cap H \in \mathscr{A}(H)$, $|A/(A \cap H)| = 2$, and $S = AH$. Since $H/Z(H)$ is Abelian,

$$(A \cap H)/Z(H) \subseteq Z(S/Z(H)) = C_H(\tau)/Z(H).$$

Since $|C_H(\tau)| = |F|^3$, $A \cap H = C_H(\tau)$. By the definition of H,

$$C_H(C_H(\tau)) = C_H(\tau).$$

Hence

$$|C_S(C_H(\tau))/C_H(\tau)| = 2 \quad \text{and} \quad C_S(\tau) = C_S(C_H(\tau)) = C_S(A \cap H) \supseteq A.$$

Thus $\mathscr{A}(S) = \{C_S(\tau)\}$. Consequently, $J(S) = Z(J(S)) = C_S(\tau)$. However, $C_S(\tau)$ is not a normal subgroup of G, although $C(O_2(G)) = C(H) \subseteq H$. In fact, since $C_S(\tau)$ is an elementary Abelian group, no characteristic subgroup of $J(S)$ is normal in G.

A characteristic subgroup of S that may satisfy an analogue of Theorem A for $p = 2$ is given by Thompson in (**15**).

Example 10.3. In Remark 3.1, we pointed out that the proof of Corollary 3.2 requires p-stability only for the case of an *Abelian* subgroup P and an element x such that $[P, x, x] = 1$. In general, these cases are not sufficient to prove Theorem A.

Let p be an odd prime and let V be a two-dimensional vector space over $\mathrm{GF}(p)$. As is well known (**2**, p. 174), there exists a non-singular skew-symmetric bilinear form f on V, and the corresponding symplectic group is just the special linear group $\mathrm{SL}(V)$. By using Lemma 4.2, construct a group H and consider $\mathrm{SL}(V)$ to be a group of automorphisms of H. Let G be the semi-direct product of H by $\mathrm{SL}(V)$. Then $O_p(G) = H$ and $C(H) = Z(H) = Z(G)$, the only Abelian normal subgroup of G. Let x be an element of order p in $\mathrm{SL}(V)$, and let S be the subgroup of G that is generated by H and x. Then S is a Sylow p-subgroup of G, $C_S(x)$ is an elementary Abelian group of order p^3, and $J(S) = Z(J(S)) = C_S(x)$. But $C_S(x)$ is not a normal subgroup of G.

Remark 10.1. By a result of Feit and Thompson (**3**, Lemma 8.2, p. 795), we need only consider subgroups P of nilpotence class at most two in the definition of p-stability.

Example 10.4. Let p be an odd prime and let S be a finite p-group. We showed that $ZJ^*(S)$ has the property that

$$C_S(ZJ^*(S)) \subseteq ZJ^*(S)$$

and satisfies analogues of Theorems A through D. The same property is possessed by $J(S)$, but these analogues are false for $J(S)$.

Let L be the linear fractional (projective special linear) group of degree two over $\mathrm{GF}(p)$. Then L may be considered (**2**, p. 200) to be a group of orthogonal transformations with respect to a non-singular symmetric bilinear

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form f on a three-dimensional vector space V over $\mathrm{GF}(p)$. Let $W = V$. We construct a group H and consider L to be a group of automorphisms of H by using Lemma 10.1. Let G be the semi-direct product of H by L.

Clearly, $H = O_p(G)$ and $C(H) \subseteq H$. Since we observe that

$$|G| = |L|\,|H| = \tfrac{1}{2}p(p^2 - 1)p^7,$$

$|\mathrm{Qd}(p)| = p(p^2 - 1)p^2$, and p is odd ,we see that $|\mathrm{Qd}(p)|_2 = 2|G|_2$. Therefore, $\mathrm{Qd}(p)$ is not involved in G. Similarly, $\mathrm{Qd}(p)$ is not involved in the semi-direct product LV. By Lemma 6.3, both of these groups are p-stable. Let x be an element of order p in L. Since $|V| = p^3$ and LV is p-stable, x must have a minimal polynomial of the form $(z - 1)^3$ on V. Thus

$$C_V(x) = V^{(x-1)^2} \quad \text{and} \quad |C_V(x)| = p.$$

By Lemma 4.3, $C_V(x)$ is isotropic. By Lemma 10.1 (b), the subgroup

$$\{(v, w, \alpha)\colon v, w \in C_V(x), \alpha \in F\}$$

is Abelian. But this subgroup is just $C_H(x)$. Let S be the subgroup of G generated by H and x. Then S is a Sylow p-subgroup of G, $C_S(x)$ is generated by $C_H(x)$ and x, and

$$|C_S(x)| = p|C_H(x)| = p^4.$$

Now, $H/Z(H)$ has order p^6. As in the discussion of Example 10.2, we may show that every maximal Abelian subgroup of H has order p^4. But every element of H generates a cyclic group that is contained in a maximal Abelian subgroup of H. Thus $J(S) = S$, and consequently $J(S)$ is not a normal subgroup of G. Hence we cannot replace $ZJ^*(S)$ by $J(S)$ in Theorem A′.

Suppose that we assume $p = 3$ in the above example. Then $|L| = 12$, and L is isomorphic to the alternating group on four symbols. Therefore,

$$N_G(J(S)) = N_G(S) = S \quad \text{and} \quad N(J(S))/C(J(S)) = S/Z(J(S)),$$

which is a p-group. Since G does not have a normal p-complement, Theorem D′ is false for $p = 3$ if we replace $ZJ^*(S)$ by $J(S)$. However, in some unpublished work, Thompson has proved that it is true for $p > 3$.

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