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# Conformally invariant complete metrics†

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*Abstract*

For a domain $G$ in the one-point compactification $\overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}$ of $\mathbb{R}^n$, $n \geqslant 2$, we characterise the completeness of the modulus metric $\mu_G$ in terms of a potential-theoretic thickness condition of $\partial G$, Martio's $M$-condition [35]. Next, we prove that $\partial G$ is uniformly perfect if and only if $\mu_G$ admits a minorant in terms of a Möbius invariant metric. Several applications to quasiconformal maps are given.

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## 1. *Introduction*

Conformal invariance is one of the key notions in the geometric theory of conformal and quasiconformal maps both in the plane $\mathbb{R}^2 = \mathbb{C}$ and in the Euclidean space $\mathbb{R}^n$, $n \geqslant 3$. Most clearly this is visible in the study of metrics: the uniformisation theorem [6] and the hyperbolic (Poincaré) metric of the unit disk in $\mathbb{C}$ provide a way to define the hyperbolic metric in any plane domain $G$ with card $(\mathbb{C} \setminus G) \geqslant 2$. This method fails for $n \geqslant 3$ because by Liouville's theorem [19, 45] conformal maps in dimensions $n \geqslant 3$ are Möbius transformations. A widely studied natural question is whether some other methods would work and whether there are counterparts of the hyperbolic metric in subdomains $G$ of $\mathbb{R}^n$ and what sort of invariance or quasi-invariance properties, if any, such metrics might have in higher dimensions $n \geqslant 3$. From the vast literature we mention A. F. Beardon [4, 5], J. Ferrand

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[12, 13, 14, 15, 31], F. W. Gehring [18, 20, 21], D.A. Herron [11, 24, 25, 26, 27], M. Vuorinen [23, 53, 55]. The recent extensive research on metrics in geometric function theory has many faces: two examples are the monograph [28] of M. Jarnicki and P. Pflug which provides an encyclopedic treatise on invariant metrics of complex manifolds and the monograph of A. Papadopoulos which lists twelve metrics recurrent in geometric function theory [40, pp.42–48].

Our main aim is to study one of these metrics, *the modulus metric* of a domain $G \subset \overline{\mathbb{R}}^n = \mathbb{R}^n \cup \{\infty\}, n \geqslant 2$, denoted by $\mu_G(x, y), x, y \in G$, see Sections 3 and 4 for definitions. In the special case of the unit ball, the modulus metric $\mu_{\mathbb{B}^n}(x, y)$ has an explicit formula in terms of the hyperbolic metric of the unit ball $\mathbb{B}^n$; the case of $\mu_{\mathbb{B}^2}(x, y)$ has already been studied by H. Grötzsch [1, p.72]. The conformal invariant $\mu_G(x, y)$ has found numerous applications [23, 55], but still many fundamental questions remain open. Very recently a problem due to J. Ferrand [15], [23, pp.294–295] was solved as follows.

THEOREM A ([8, 44, 56]). *A homeomorphism $f : G \to G'$, where $G$ and $G'$ are domains in $\mathbb{R}^n, n \geqslant 2$, is an isometry between $(G, \mu_G)$ and $(G', \mu_{G'})$ if and only if $f$ is conformal.*

As pointed out above, $\mu_{\mathbb{B}^n}(x, y)$ is closely related to the hyperbolic metric of $\mathbb{B}^n$. We next study conditions on the domain $G$ under which $\mu_G$ defines an intrinsic metric of $G$ having properties similar to the hyperbolic metric. It turns out that the geometry of this metric significantly depends on the "potential theoretic thickness" of the boundary, measured in terms of the conformal capacity. As is well known, the conformal capacity is very closely connected with the moduli of curve families [19, theorem 5·2·3, p.164], [23, theorem 9·6, p.152].

If the boundary $\partial G$ is polar, i.e. if it has null conformal capacity $\operatorname{cap}(\partial G) = 0$, then $\mu_G \equiv 0$; otherwise $\mu_G$ is a conformally invariant metric. Even if $\operatorname{cap}(\partial G) > 0$, the modulus metric $\mu_G$ might not reflect the intrinsic geometry of $G$ very precisely. For instance, a polar compact set $N \subset G$ is invisible for the modulus metric in the sense that if $\operatorname{cap} N = 0$, then $\mu_G(x, y) = \mu_{G\setminus N}(x, y)$ for $x, y \in G \setminus N$. Therefore, it is meaningful to look for a condition on $G$ so as to guarantee that $\mu_G$ is a complete metric. We remark that a similar problem for the Kobayashi metric on domains in $\mathbb{C}^n$ is rather difficult (see, e.g., [17, 41]).

In connection with this completeness property, we recall another notion on metric spaces. A metric space $(X, m)$ is called *proper* [10] if the closed metric ball $\{x \in X : m(x, a) \leqslant r\}$ is compact whenever $a \in X$ and $r > 0$. This is equivalent to say that the open metric ball $\{x \in X : m(x, a) < r\}$ is relatively compact for $a \in X$ and $r > 0$. Note that a proper metric space is locally compact and complete. However, the converse is not true in general. (Consider, e.g., $(X, m/(1 + m))$ for a locally compact but non-compact complete metric space $(X, m)$ such as $\mathbb{R}^n$ with the Euclidean metric.)

Our first result characterizes domains $G$ for which the metrics $\mu_G$ are complete.

THEOREM 1·1. *Let $G$ be a domain in $\overline{\mathbb{R}}^n$ with $\partial G \neq \emptyset$. Then the following conditions are equivalent*:

(i) *$(G, \mu_G)$ is a proper metric space*;

(ii) *$(G, \mu_G)$ is a complete metric space*;

(iii) *$G$ is an M-domain. That is to say, each boundary point $x$ of $G$ satisfies the M-condition.*

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The M-condition for $x \in \partial G$ was introduced by O. Martio [35][1] in his study of potential theoretic regularity of the domain. If this condition holds for all $x \in \partial G$, the complement $\overline{\mathbb{R}}^n \setminus G$ of $G$ is "thick enough" at every point of $\partial G$ [35, 37]. See Section 3 for definitions of those concepts and related properties.

Our second result refines further the case when $\mu_G$ is complete. We assume now that the boundary of a domain is uniformly perfect in the sense of Ch. Pommerenke [42, 43] — in this case the M-condition is valid, see Corollary 1·5. This notion was introduced by A. F. Beardon and Ch. Pommerenke [7] for unbounded closed sets in $\mathbb{C}$, but about the same time an equivalent concept was studied by P. Tukia and J. Väisälä [51] under the name "homogeneously dense sets" in the setting of general metric spaces. By definition, a compact set $E$ in $\overline{\mathbb{R}}^n$ with card $(E) \geqslant 2$ is called *uniformly perfect* if there exists a constant $c \in (0, 1)$ such that $E$ meets the closed annulus $cr \leqslant |x - a| \leqslant r$ whenever $a \in E \setminus \{\infty\}$ and $r \in (0, \mathrm{diam}(E))$, where $\mathrm{diam}(E)$ denotes the Euclidean diameter of $E$ and set $\mathrm{diam}(E) = +\infty$ when $\infty \in E$. In the planar case when $G \subset \mathbb{R}^2 = \mathbb{C}$, A. F. Beardon and Ch. Pommerenke [7] gave another characterisation in terms of the hyperbolic and quasihyperbolic metrics $h_G(x, y)$ and $k_G(x, y)$, resp. (see Section 2), and proved that $\partial G$ is uniformly perfect if and only if there is a constant $b > 0$ such that

$$h_G(x, y) \geqslant b k_G(x, y) \quad \text{for } all \ x, y \in G\,.$$

Here we give an alternative characterisation of uniform perfectness of $\partial G$ in terms of intrinsic metrics which is valid in higher dimensions as well and, moreover, is applicable to subsets of the Möbius space. This characterisation requires that the modulus metric be minorised by a Möbius invariant metric $\delta_G$, defined in terms of the absolute ratio 2.10 for all domains $G \subset \overline{\mathbb{R}}^n$ with $\mathrm{card}(\partial G) \geqslant 2$. This metric was first introduced in [55, pp.115–116] and, later on, studied by P. Seittenranta in his PhD thesis [47] where he also suggested the name "Möbius metric".

THEOREM 1·2. *Let $G \subset \overline{\mathbb{R}}^n$ be a domain with* card $(\partial G) \geqslant 2$. *Then $\partial G$ is uniformly perfect if and only if there exists a constant $b > 0$ such that for all $x, y \in G$ the inequality*

$$\mu_G(x, y) \geqslant b\, \delta_G(x, y) \tag{1·3}$$

*holds, where $\mu_G$ is the modulus metric and $\delta_G$ is the Möbius metric.*

For a proper subdomain $G$ of $\mathbb{R}^n$, the lower bound 1·3 can be expressed in terms of a similarity invariant metric, the *distance-ratio metric* of $G$ as follows. For $x, y \in G$ define

$$j_G(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d_G(x), d_G(y)\}}\right), \tag{1·4}$$

which is a metric on $G$, where $d_G(x)$ denotes the Euclidean distance from $x$ to the boundary $\partial G$ [23, lemma 4·6, p.59]. When $G \subset \mathbb{R}^n$, the above condition (1·3) is equivalent to the requirement that for some constant $b' > 0$

$$\mu_G(x, y) \geqslant b' j_G(x, y)$$

for all $x, y \in G$. Since $(G, \delta_G)$ is a proper metric space (see Lemma 2·14 below), we have the following result as a corollary of Theorems 1·1 and 1·2.

[1] The M-condition $\mathsf{M}(x, \overline{\mathbb{R}}^n \setminus G) = \infty$ was denoted by $M_x = \infty$ in Martio's paper [35].

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COROLLARY 1·5. *Let $G \subset \overline{\mathbb{R}}^n$ be a domain with* card $(\partial G) \geqslant 2$. *If $\partial G$ is uniformly perfect, then $G$ is an M-domain.*

The converse is not true in general. A counterexample will be given in Section 3.

The proof of Theorem 1·2 is based, in part, on a potential theoretic thickness characterisation of uniform perfectness [**29, 54**]. Many authors have contributed to the research of uniformly perfect sets and related thickness conditions, see [**3, 9**], [**16**, pp.343–345], [**22, 30, 32, 33, 34**] and the survey of T. Sugawa [**48**] on uniform perfectness.

Uniform domains play an important role in geometric function theory. See [**20**] and the recent monograph [**18**] for details. For convenience of the reader, we will provide a brief account on this notion in the next section.

THEOREM 1·6. *Suppose that $G \subset \overline{\mathbb{R}}^n$ is a uniform domain. Then there exist constants $d_1, d_2$ depending only on $n$ and the uniformity parameters such that*

$$\mu_G(x, y) \leqslant d_1\, \delta_G(x, y) + d_2 \qquad x, y \in G. \tag{1·7}$$

*Conversely, suppose that a domain $G$ in $\overline{\mathbb{R}}^2$ with continuum as its boundary satisfies (1·7). Then $G$ is uniform.*

Note that the boundary of a domain $G$ in $\overline{\mathbb{R}}^2 = \overline{\mathbb{C}}$ is a continuum; that is, a non-degenerate connected compact set, if and only if $G$ is a simply connected hyperbolic domain. It is known that such a domain $G$ is uniform precisely when $G$ is a quasidisk, that is to say, $G$ is the image of the unit disk $\mathbb{B}^2$ under a quasiconformal homeomorphism of $\overline{\mathbb{C}}$ onto itself [**18**]. Therefore, as a corollary, we have the following characterisation of quasidisks.

COROLLARY 1·8. *Let $G$ be a simply connected domain in the Riemann sphere $\overline{\mathbb{C}}$ with* card $(\overline{\mathbb{C}} \setminus G) \geqslant 2$. *Then $G$ is a quasidisk if and only if there are positive constants $d_1$ and $d_2$ such that the inequality*

$$\mu_G(z, w) \leqslant d_1\, \delta_G(z, w) + d_2$$

*holds for all $z, w \in G$.*

In this corollary, we may replace the modulus metric $\mu_G$ by Ferrand's modulus metric $\lambda_G^{-1}$ (see Lemma 4·5 below). We remark that for $G \subset \mathbb{C}$ the above condition is also equivalent to the condition

$$\mu_G(z, w) \leqslant d_1'\, j_G(z, w) + d_2' \qquad \text{for } z, w \in G\,.$$

As we will see later, the constant $d_2$ in Corollary 1·8 cannot be dropped. We expect that the converse would be true for all dimensions $n \geqslant 2$ under a weaker assumption on the boundary such as uniform perfectness of the boundary. These observations lead to the following problem.

1·9. *Open problem*

Let $n \geqslant 2$. Find a geometric condition $(*)$ on the boundaries of domains $G$ in $\overline{\mathbb{R}}^n$ with the following property: If a domain $G$ in $\overline{\mathbb{R}}^n$ satisfies the condition $(*)$ and the inequality (1·7) for some constants $d_1 > 0$ and $d_2 > 0$, then $G$ is uniform.

Finally, we consider the hyperbolic metric $h_G$ and the Ferrand metric $\sigma_G$, see (2·7), in planar domains $G$. It is well known [**7**] that if $\partial G$ is uniformly perfect, then the distances in the $h_G$ metric are comparable to those in the quasihyperbolic metric $k_G$. Furthermore, this comparison property fails to hold if the domain $G$ has isolated boundary points. Indeed, the following asymptotic formulae hold.

LEMMA 1·10. *Let $G$ be a hyperbolic domain in $\overline{\mathbb{C}}$ and suppose that $G$ has an isolated boundary point $a$ with $a \neq \infty$. Then, for a fixed $z_0 \in G$, as $z \to a$*

$$\sigma_G(z, z_0) = \log \frac{1}{|z-a|} + O(1) \quad \textit{and} \quad \delta_G(z, z_0) = \log \frac{1}{|z-a|} + O(1), \tag{1·11}$$

*while*

$$h_G(z, z_0) = \log\log \frac{1}{|z-a|} + O(1). \tag{1·12}$$

It is a challenging task, studied in [**49, 50**], to give concrete bounds for the $h_G$ distances in domains $G$ whose boundary consists only of isolated points. Since $\log(1+x)$ is a subadditive function on $0 \leqslant x < +\infty$, we can easily see that $\log(1 + m(x, y))$ is a distance function on $X$ whenever $m(x, y)$ is a distance function on $X$ [**2**, 7·42(1)]. In view of the above behaviour of the hyperbolic distance around isolated boundary points, we are led to the introduction of the *logarithmic Möbius metric* $\Delta_G(x, y)$ and the *logarithmic Ferrand metric* $\Sigma_G(x, y)$ for a domain $G \subset \overline{\mathbb{R}}^n$ with card $(\overline{\mathbb{R}}^n \setminus G) \geqslant 2$ as follows:

$$\Delta_G(x, y) = \log(1 + \delta_G(x, y)), \quad x, y \in G, \tag{1·13}$$

$$\Sigma_G(x, y) = \log(1 + \sigma_G(x, y)), \quad x, y \in G. \tag{1·14}$$

Because $\delta_G$ and $\sigma_G$ are Möbius invariant, $\Delta_G$ and $\Sigma_G$ are Möbius invariant metrics, too. We also have $\Delta_G(x, y) \leqslant \Sigma_G(x, y)$ (see Lemma 2·12 below). When the complement of $G$ in $\overline{\mathbb{C}}$ is a finite set, the hyperbolic distance $h_G$ is majorised by $\Delta_G$. However, $h_G$ is never minorised by it for any domain with a puncture; namely, with an isolated boundary point. In fact, we prove a slightly stronger result.

THEOREM 1·15. *Let $A$ be a finite set in $\overline{\mathbb{C}}$ with* card $(A) \geqslant 3$ *and let $G = \overline{\mathbb{C}} \setminus A$. Then there exists a positive constant $c = c(A)$ such that for all $z, w \in G$,*

$$h_G(z, w) \leqslant c\, \Delta_G(z, w) = c \log(1 + \delta_G(z, w)).$$

*On the other hand, for an arbitrary hyperbolic domain $G$ in $\overline{\mathbb{C}}$ with a puncture, there is no non-decreasing function $\Phi : [0, +\infty) \to [0, +\infty)$ with $\Phi(t) > 0$ for $t > 0$ such that for all $z, w \in G$,*

$$\Phi(\delta_G(z, w)) \leqslant h_G(z, w).$$

All the results here will be proved in the subsequent sections. More precisely, this paper is organised as follows. Section 2 is devoted to definitions and basic properties of the metrics involved, with the exception of the modulus metric, which will be defined in Section 4. In Section 3, we recall the notion of the (conformal) modulus of a curve family and its fundamental properties. We also introduce the notion of M-domains defined in terms of the continuum criterion of Martio [**35**]. The modulus metric is defined and related results are established in Section 4. We give some applications of the above results to quasiconformal or quasiregular mappings in Section 5. Theorem 1·15 is proved in the last section. Two open problems are pointed out, namely 3·12 and 4·15.

## 2. *Preliminary notation and results*

We follow standard notation. See e.g. [4, 52] for more details. We write

$$\begin{aligned} B^n(x,r) &= \{z \in \mathbb{R}^n : |z-x| < r\}, \\ \overline{B}^n(x,r) &= \{z \in \mathbb{R}^n : |z-x| \leqslant r\}, \\ S^{n-1}(x,r) &= \{z \in \mathbb{R}^n : |z-x| = r\}, \end{aligned}$$

for balls and spheres, respectively, and

$$\mathbb{B}^n = B^n(0,1), \quad \mathbb{H}^n = \{(x_1,\ldots,x_n) \in \mathbb{R}^n : x_n > 0\}.$$

First we recall the definition of the *chordal (spherical) distance* $q(x,y)$ on $\overline{\mathbb{R}}^n$:

$$\begin{cases} q(x,y) = \dfrac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}\,, & x, y \neq \infty\,, \\ q(x,\infty) = q(\infty,x) = \dfrac{1}{\sqrt{1+|x|^2}}\,, & x \neq \infty\,. \end{cases} \tag{2·1}$$

For distinct points $a, b, c, d \in \overline{\mathbb{R}}^n$, the *absolute (cross) ratio* is defined by

$$|a,b,c,d| = \frac{q(a,c)q(b,d)}{q(a,b)q(c,d)}\,.$$

When none of the points is $\infty$, we see that

$$|a,b,c,d| = \frac{|a-c||b-d|}{|a-b||c-d|}\,.$$

### 2·2. *Hyperbolic metric*

The hyperbolic metrics $2|dx|/(1-|x|^2)$ on $\mathbb{B}^n$ and $|dx|/x_n$ on $\mathbb{H}^n$ induce the hyperbolic distances $h_{\mathbb{B}^n}(x,y)$ and $h_{\mathbb{H}^n}(x,y)$ respectively. When $n=2$, any domain $G$ of $\overline{\mathbb{R}}^2 = \overline{\mathbb{C}}$ with $\operatorname{card}(\partial G) \geqslant 3$ is known to have a holomorphic universal covering projection $p$ of the unit disk $\mathbb{B}^2$ onto $G$. Thus the hyperbolic distance $h_G$ of $G$ can be defined by

$$h_G(z_1,z_2) = \min_{\zeta_1 \in p^{-1}(z_1), \zeta_2 \in p^{-1}(z_2)} h_{\mathbb{B}^2}(\zeta_1,\zeta_2) = \inf_{\gamma \in \Gamma} \int_\gamma \rho_G(z)|dz|,$$

where $\Gamma$ is the set of all rectifiable curves joining $z_1$ and $z_2$ in $G$ and $\rho_G(z)$ denotes the hyperbolic density determined by the relation $2/(1-|\zeta|^2) = \rho(p(\zeta))|p'(\zeta)|$, $\zeta \in \mathbb{B}^2$ (see [6, 30] for details).

### 2·3. *Quasihyperbolic metric*

For higher dimensions, however, we cannot define hyperbolic metric for general domains. Quasihyperbolic metrics were introduced by F.W. Gehring and B. Palka [21] as a substitute for it. For a domain $G \subsetneq \mathbb{R}^n$, the *quasihyperbolic metric* $k_G$ is defined by

$$k_G(x, y) = \inf_{\gamma \in \Gamma} \int_\gamma \frac{|dt|}{d_G(t)}, \quad x, y \in G,$$

where $\Gamma$ is the family of all rectifiable curves in $G$ joining $x$ and $y$. Note here that the inequality

$$j_G(x, y) \leqslant k_G(x, y)$$

holds for an arbitrary $G \subsetneq \mathbb{R}^n$ and all $x, y \in G$ [21, lemma 2·1].

2·4. *Uniform domains*

A proper subdomain $G$ of $\mathbb{R}^n$ is called *uniform* if there exist positive constants $a$ and $b$ with the following property [20, 37]: for every pair of points $x_1, x_2 \in G$, there is a rectifiable curve $\gamma$ joining $x_1$ and $x_2$ in $G$ in such a way that $\ell(\gamma) \leqslant a|x_1 - x_2|$ and that $\min\{\ell(\gamma_1), \ell(\gamma_2)\} \leqslant b\, d_G(x)$ for each $x \in \gamma$, where $\gamma_j$ is the part of $\gamma$ between $x_j$ and $x$ for each $j = 1, 2$, $\ell(\gamma)$ denotes the length of the curve $\gamma$ and $d_G(x)$ is the Euclidean distance to the boundary of $G$ from $x$. The class of uniform domains can also be defined in terms of a comparison inequality between two metrics [20, 55][2] a subdomain $G$ of $\mathbb{R}^n$ with non-empty boundary is uniform if and only if there exists a constant $c \geqslant 1$ such that

$$k_G(x, y) \leqslant c\, j_G(x, y) \tag{2·5}$$

for all $x, y \in G$, where $k_G$ and $j_G$ are the quasihyperbolic and distance-ratio metrics, respectively. Note that $j_G(x, y) \leqslant k_G(x, y)$ holds for every domain $G$ and all $x, y \in G$ by [21, lemma 2·1].

2·6. *Ferrand's metric*

Since the definition of the quasihyperbolic metric relies on the Euclidean metric, it is not defined for all subdomains of the Möbius space and therefore it is not Möbius invariant. To overcome this shortcoming, Ferrand [12] modified the definition as follows. For a subdomain $G$ of $\overline{\mathbb{R}}^n$ with $\mathrm{card}\,(\partial G) \geqslant 2$, define a density function

$$w_G(x) = \sup_{a,b \in \partial G} \frac{|a - b|}{|x - a|\,|x - b|}, \quad x \in G \setminus \{\infty\},$$

and the metric $\sigma_G$ in $G$,

$$\sigma_G(x, y) = \inf_{\gamma \in \Gamma} \int_\gamma w_G(t)|dt|, \tag{2·7}$$

where $\Gamma$ is the family of all rectifiable curves in $G$ joining $x$ and $y$. The following result is due to Ferrand [12, p.122] and $\sigma_G(x, y)$ is now called the *Ferrand metric* [23, Chapter 5].

LEMMA 2·8. *Let $G \subset \overline{\mathbb{R}}^n$ be a domain with* $\mathrm{card}\,(\partial G) \geqslant 2$. *The Ferrand metric $\sigma_G$ has the following properties:*

[2] In [20], condition (2·5) was given in the slightly different form $k_G(x, y) \leqslant a\, j_G(x, y) + b$ for some constants $a, b$. We easily see that we can take $b = 0$ by letting $a$ be larger if necessary. See [53, 2·50 (2)].

(i) $\sigma_G$ is a Möbius invariant metric;

(ii) When $G$ is either $\mathbb{B}^n$ or $\mathbb{H}^n$, $\sigma_G$ coincides with the hyperbolic metric $h_G$;

(iii) $k_G \leqslant \sigma_G \leqslant 2k_G$ for every domain $G \subsetneq \mathbb{R}^n$.

We remark that the metric $\sigma_G$ was recently studied by D. A. Herron and P. K. Julian [26].

2·9. *Möbius metric*

Let $G \subset \overline{\mathbb{R}}^n$ be an open set with $\operatorname{card}(\partial G) \geqslant 2$. The *Möbius metric* on $G$ is defined as follows ([55, pp.115–116], Seittenranta [47]):

$$\delta_G(x, y) := \log\left(1 + m_G(x, y)\right), \quad m_G(x, y) := \sup_{a,b\in\partial G} |a, x, b, y| . \tag{2·10}$$

Note that the Möbius metric $\delta_G$ coincides with the hyperbolic metric $h_G$ when $G$ is either $\mathbb{B}^n$ or $\mathbb{H}^n$ [55, lemma 8·39]. A metric very similar to the Möbius metric is the Apollonian metric of Beardon [5].

2·11. *Chordal distance-ratio metric*

For a proper subdomain $G$ of $\overline{\mathbb{R}}^n$ we define the *chordal (spherical) distance-ratio metric* by

$$\hat{j}_G(x, y) = \log\left(1 + \frac{q(x, y)}{\min\{\hat{d}_G(x), \hat{d}_G(y)\}}\right),$$

where

$$\hat{d}_G(x) = \inf_{a\in\partial G} q(x, a).$$

The triangle inequality for this metric follows from [47, lemma 2·2].

The following results are due to Seittenranta [47].

LEMMA 2·12. *Let $G$ be an open subset of $\overline{\mathbb{R}}^n$ with* $\operatorname{card}(\partial G) \geqslant 2$ . *Then $\delta_G$ is a Möbius invariant metric and the following hold*:

(i) $\delta_G \leqslant \sigma_G$;

(ii) $\delta_G \leqslant 2\hat{j}_G$

(iii) *if $G \subsetneq \mathbb{R}^n$ , then $j_G \leqslant \delta_G \leqslant 2j_G$.*

*Proof.* The fact that $\delta_G$ satisfies the triangle inequality, assertions (i) and (iii) follow from theorems 3·3, 3·4 and 3·12 in [47], respectively. In order to show assertion (ii), we introduce the auxiliary metric

$$j^*_G(x, y) = \log\left(1 + \frac{q(x, y)}{\hat{d}_G(x)}\right) + \log\left(1 + \frac{q(x, y)}{\hat{d}_G(y)}\right).$$

Theorem 3·6 in [47] means the inequality $\delta_G(x, y) \leqslant j^*_G(x, y)$ for $x, y \in G$. It is easy to verify the inequalities $\hat{j}_G(x, y) \leqslant j^*_G(x, y) \leqslant 2\hat{j}_G(x, y)$. Thus assertion (ii) now follows.

As a consequence of the previous lemma, we have the following inequality, which will be used in the proof of Theorem 1·2 later:

$$j_G(x,y) \leqslant 2\,\hat{j}_G(x,y), \quad x,y \in G \subsetneq \mathbb{R}^n. \tag{2·13}$$

We note that there is no constant $c = c(n) > 0$ depending only on $n$ such that $j_G(x,y) \geqslant c\,\hat{j}_G(x,y)$, $x,y \in G$, holds for all proper subdomains $G$ of $\mathbb{R}^n$. The following result follows also from the previous lemma.

LEMMA 2·14. *The metric space $(G, \delta_G)$ is proper for $G \subset \overline{\mathbb{R}}^n$ with* card $(\partial G) \geqslant 2$.

*Proof.* By the Möbius invariance, we may assume that $G \subset \mathbb{R}^n$. Then $j_G \leqslant \delta_G$ by Lemma 2·12 (iii). Therefore, it is enough to show that $(G, j_G)$ is proper in this case. For $a \in G$ and $0 < r$, we have to show that the set $B = \{x \in G : j_G(x,a) < r\}$ is relatively compact. It is enough to show that $B$ is bounded and $\mathrm{dist}(B, \partial G) > 0$. The inequality $\log(1 + |x-a|/d_G(a)) \leqslant r$ holds for $x \in B$ and thus $|x-a| \leqslant d_G(a)(e^r - 1)$, which proves that $B$ is bounded. On the other hand, the inequality $\log(1 + |x-a|/d_G(x)) \leqslant r$ holds for $x \in B$. Note that $d_G(x) \geqslant d_G(a)/2$ if $|x-a| \leqslant d_G(a)/2$. For $x \in B$ with $|x-a| \geqslant d_G(a)/2$, we thus have $d_G(x) \geqslant |x-a|/(e^r-1) \geqslant d_G(a)/(e^r-1)$. Therefore, we have shown $\mathrm{dist}(B, \partial G) \geqslant \min\{d_G(a)/2, d_G(a)/(e^r-1)\} > 0$ as required.

2·15. *Möbius uniform domains*

We now consider a Möbius invariant characterisation of uniform domains. As we saw above, uniform domains in $\mathbb{R}^n$ are characterised by the condition (2·5) in terms of quasihyperbolic and distance-ratio metrics. These two metrics are invariant under similarity transformations but unfortunately not under Möbius transformations. To overcome this lack of invariance we apply Ferrand's Möbius invariant metric $\sigma_G$ and the Möbius metric $\delta_G$ .

*Definition 2·16* ([**47**]). We say that a domain $G \subset \overline{\mathbb{R}}^n$ with card $(\overline{\mathbb{R}}^n \setminus G) \geqslant 2$ is *Möbius uniform*, if there exists a constant $c \geqslant 1$ such that for all $x, y \in G$

$$\sigma_G(x,y) \leqslant c\,\delta_G(x,y)\,.$$

Note that Definition 2·5 only applies to subdomains of $\mathbb{R}^n$ whereas Definition 2.16 applies to subdomains of $\overline{\mathbb{R}}^n$. Indeed, we have the following result.

PROPOSITION 2·17. *Let $G \subset \mathbb{R}^n$ be a domain with* card $(\partial G) \geqslant 2$. *Then $G$ is Möbius uniform if and only if it is uniform in the sense of* (2·5).

*Proof.* From Lemmas 2·8 and 2·12 it follows that if $G$ is Möbius uniform with a constant $c_1$, then it is uniform in the sense of (2·5) with the constant $2c_1$ . Conversely, from Lemmas 2·8 and 2·12 it follows that if $G$ is uniform in the sense of (2·5) with a constant $c_2$, then it is Möbius uniform with the the constant $2c_2$ .

Therefore, we will use the shorter term "uniform" below for both uniform domains and Möbius uniform domains unless we want to emphasise which definition is intended.

We end this section with a proof of Lemma 1·10.

*Proof of Lemma 1·10.* By assumption, there is a number $r > 0$ such that the punctured disk $0 < |z-a| < r$ is contained in $G$. It is enough to prove the assertions for $a = 0$ and $r = 1$.

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By assumption, we can find a finite boundary point $b$ of $G$ so that

$$m_G(z,z_0)\geqslant |0,z,b,z_0| = \frac{|b||z-z_0|}{|z||b-z_0|}\geqslant \frac{|b||z_0|}{2|z||b-z_0|} =: \frac{C}{|z|}$$

for $z\in G$ with $0<|z|<|z_0|/2$. Hence,

$$\delta_G(z,z_0)=\log\left(1+m_G(z,z_0)\right)\geqslant \log\left(1+C/|z|\right)=\log\frac{1}{|z|}+O(1)$$

as $z\to 0$. Next we estimate $w_G(z)$ from above for $0<|z|\leqslant 1/4$. For $b\in\partial G\setminus\{0\}$, we have $|z-b|/|b|\leqslant 1+|z|/|b|\leqslant 1+|z|$ and $|z-b|/|b|\geqslant 1-|z|/|b|\geqslant 1-|z|$ and thus

$$\frac{16}{5}\leqslant \frac{1}{|z|(1+|z|)}\leqslant \frac{|b|}{|z||z-b|}\leqslant \frac{1}{|z|(1-|z|)}=\frac{1}{|z|}+\frac{1}{1-|z|}\leqslant \frac{1}{|z|}+\frac{4}{3}$$

for $0<|z|\leqslant 1/2$. For $b_1,b_2\in\partial G\setminus\{0\}$, we have $|z-b_j|\geqslant |b_j|-|z|\geqslant 3|b_j|/4\geqslant 3/4$ and

$$\frac{|b_1-b_2|}{|z-b_1||z-b_2|}\leqslant \frac{|z-b_2|+|z-b_1|}{|z-b_1||z-b_2|}=\frac{1}{|z-b_1|}+\frac{1}{|z-b_2|}\leqslant \frac{8}{3}$$

as $z\to 0$. Hence, we obtain $w_G(z)\leqslant 1/|z|+4/3$ for $0<|z|\leqslant 1/4$. For a given $z_0$, we take a point $z_1\in G$ so that $|z_1|\leqslant \min\{|z_0|,1/4\}$. Then, for $0<|z|<|z_1|$, we have

$$\sigma_G(z,z_0)\leqslant \sigma_G(z,z_1)+\sigma_G(z_1,z_0)\leqslant \int_\gamma \frac{|dt|}{|t|}+O(1)=\log\frac{1}{|z|}+O(1),$$

where $\gamma$ is the curve going from $z_1$ to the point $(|z_1|/|z|)z$ along the circle $|t|=|z_1|$ and then going to $z$ radially. Since $\delta_G(z,z_0)\leqslant \sigma_G(z,z_0)$, (1·11) follows.

Secondly, we prove (1·12). For simplicity, we further assume that $1,\infty\in\partial G$. (For the general case, we may use a suitable Möbius transformation to reduce to this case.) Then

$$\mathbb{D}^*=\{z\in\mathbb{C}:0<|z|<1\}\subset G\subset \mathbb{C}\setminus\{0,1\}$$

and therefore

$$\rho_{\mathbb{D}^*}(z)\geqslant \rho_G(z)\geqslant \rho_{\mathbb{C}\setminus\{0,1\}}(z)$$

for $0<|z|<1$. Since

$$\rho_{\mathbb{D}^*}(z)=\frac{1}{|z|\log(1/|z|)}\quad\text{and}\quad \rho_{\mathbb{C}\setminus\{0,1\}}(z)=\frac{1}{|z|(C_0+\log(1/|z|))},$$

where $C_0=1/\rho_{\mathbb{C}\setminus\{0,1\}}(-1)$ (see [30] for instance), we have

$$\rho_G(z)=\frac{1}{|z|\log(1/|z|)}+O\left(\frac{1}{|z|\log^2(1/|z|)}\right)$$

as $z\to 0$. Noting the fact that the real function $1/[t\log^2 t]$ is integrable over $(0,1/2]$, we obtain the required asymptotics (1·12) as required.

*Remark 2·18.* As the above proof shows, (1·11) is valid also in dimensions $n\geqslant 2$.

## 3. *Modulus and M-domains*

We recapitulate some of the basic facts about moduli of curve families and quasiconformal maps, following [19, 52]. Let $\Gamma$ be a family of curves in $\overline{\mathbb{R}}^n$. We say that a non-negative

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Borel-measurable function $\rho : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is an admissible function for $\Gamma$, if $\int_\gamma \rho ds \geqslant 1$ for each locally rectifiable curve $\gamma$ in $\Gamma$. The (conformal) modulus of $\Gamma$ is

$$\mathsf{M}(\Gamma) = \inf_{\rho \in \mathcal{F}(\Gamma)} \int_{\mathbb{R}^n} \rho^n dm,$$

where $\mathcal{F}(\Gamma)$ is the family of admissible functions for $\Gamma$ and $m$ stands for the $n$-dimensional Lebesgue measure. We set $\mathsf{M}(\Gamma) = \infty$ when $\mathcal{F}(\Gamma)$ is empty. The most important property of the modulus is a quasi-invariance; that is, a homeomorphism $f : G \to G'$ between domains in $\overline{\mathbb{R}}^n$ is $K$-quasiconformal if and only if

$$\mathsf{M}(\Gamma)/K \leqslant \mathsf{M}(f(\Gamma)) \leqslant K\mathsf{M}(\Gamma)$$

for all families of curves $\Gamma$ in $G$. In particular, $\mathsf{M}(f(\Gamma)) = \mathsf{M}(\Gamma)$ for a conformal homeomorphism $f$.

For two curve families $\Gamma_1$ and $\Gamma_2$ in $\overline{\mathbb{R}}^n$, we say that $\Gamma_2$ is minorised by $\Gamma_1$ and denote $\Gamma_2 > \Gamma_1$ if every $\gamma \in \Gamma_2$ has a subcurve which belongs to $\Gamma_1$. A collection of curve families $\Gamma_j$ $(j = 1, 2, \ldots)$ is said to be disjointly supported if there are Borel sets $\Omega_j$ $(j = 1, 2, \ldots)$ such that all curves in $\Gamma_j$ are contained in $\Omega_j$ and that $m(\Omega_j \cap \Omega_{j'}) = 0$ for $j \neq j'$. Then the following properties of the conformal modulus are fundamental (see [52] or [19]).

LEMMA 3·1.

(1) *If* $\Gamma_1 < \Gamma_2$, *then* $\mathsf{M}(\Gamma_1) \geqslant \mathsf{M}(\Gamma_2)$. *In particular,* $\mathsf{M}(\Gamma_2) \leqslant \mathsf{M}(\Gamma_1)$ *for* $\Gamma_2 \subset \Gamma_1$.

(2) *For a collection of curve families* $\Gamma_j$ $(j = 1, 2, \ldots)$,

$$\mathsf{M}\left(\bigcup_j \Gamma_j\right) \leqslant \sum_j \mathsf{M}(\Gamma_j).$$

*Moreover, equality holds if the collection is disjointly supported.*

A pair $(G, E)$ of a domain $G$ in $\overline{\mathbb{R}}^n$ and a compact set $E$ in $G$ is called a *condenser*. The *capacity of the condenser* $(G, E)$ is

$$\operatorname{cap}(G, E) = \mathsf{M}(\Delta(E, \partial G; G)). \tag{3·2}$$

Another equivalent definition makes use of Dirichlet integral minimisation property [19, theorem 5·2·3]. Here and hereafter, for sets $E, F, G \subset \overline{\mathbb{R}}^n$, let $\Delta(E, F; G)$ denote the family of all curves joining the sets $E$ and $F$ in $G$, and let $\Delta(E, F) = \Delta(E, F; \overline{\mathbb{R}}^n)$. Here, a curve $\gamma : [a, b] \to \overline{\mathbb{R}}^n$ is said to join $E$ and $F$ in $G$ if $\gamma(a) \in E, \gamma(b) \in F$ and if $\gamma((a, b)) \subset G$. For a compact set $E$ in $\overline{\mathbb{R}}^n$, we write $\operatorname{cap} E = 0$ ($\operatorname{cap} E > 0$) if $\operatorname{cap}(G, E) = 0$ ($\operatorname{cap}(G, E) > 0$) for some bounded domain $G$ containing $E$ cf. [55, 7·12]. Note that $\operatorname{cap}(G', E) = 0$ for any domain $G'$ containing $E$ if $\operatorname{cap} E = 0$. It is known that $E$ is totally disconnected and has Hausdorff dimension 0 if $\operatorname{cap} E = 0$, see [45, p.120, corollary 2], [46, p.166, theorem VII·1·15].

A domain $R$ in $\overline{\mathbb{R}}^n$ is called a *ring* if the complement $\overline{\mathbb{R}}^n \setminus R$ consists of exactly two connected components, say, $E$ and $F$, and $R$ is often denoted by $R(E, F)$. In particular, $R_{G,n}(s) := R(\overline{\mathbb{B}}^n, [se_1, \infty])$, $s > 1$, is called the *Grötzsch ring* and $R_{T,n}(t) := R([-e_1, 0], [te_1, \infty])$, $t > 0$, is called the *Teichmüller ring*, where $e_1$ is the unit vector $(1, 0, \ldots, 0)$ in $\mathbb{R}^n$. The

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capacity of the ring $R(E,F)$ is $\operatorname{cap} R(E,F) = \operatorname{cap}(\overline{\mathbb{R}}^n \setminus F, E)$ and its modulus is

$$\operatorname{mod} R(E,F) = \left( \frac{\omega_{n-1}}{\operatorname{cap} R(E,F)} \right)^{1/(n-1)}.$$

When $R = R(E,F)$ is the standard ring $\{x \in \mathbb{R}^n : a < |x| < b\}$, one has $\operatorname{mod} R = \log(b/a)$. The capacities of $R_{T,n}(t)$ and $R_{G,n}(s)$ are denoted by $\tau_n(t)$ and $\gamma_n(s)$, respectively. By [55, lemma 5·53], $\tau_n : (0,+\infty) \to (0,+\infty)$ and $\gamma_n : (1,+\infty) \to (0,+\infty)$ are decreasing homeomorphisms and they satisfy the functional identity

$$\gamma_n(s) = 2^{n-1}\tau_n(s^2-1), \quad s > 1. \tag{3·3}$$

Here we state a couple of fundamental properties of uniformly perfect sets. Recall that a ring $R = R(E_1, E_2)$ is said to separate a set $A$ in $\overline{\mathbb{R}}^n$ if $A \subset E_1 \cup E_2$ and $A \cap E_j \neq \emptyset$ for $j = 1, 2$. Then the following characterization of uniformly perfect sets is well known (see, for instance, [3] for planar case and [22] for general case).

LEMMA 3·4. *Let $A$ be a compact set in $\overline{\mathbb{R}}^n$ with* $\operatorname{card}(A) \geqslant 2$. *Then $A$ is uniformly perfect precisely when there exists a constant $M > 0$ such that $\operatorname{mod} R \leqslant M$ for every ring $R$ separating $A$.*

We also note the following simple fact.

LEMMA 3·5. *Let $G$ be a domain in $\overline{\mathbb{R}}^n$ for which the complement $C = \overline{\mathbb{R}}^n \setminus G$ contains at least two points. Then $\partial G$ is uniformly perfect if and only if so is $C$.*

*Proof.* By the previous lemma, it is enough to show that a ring $R$ separates $C$ if and only if $R$ separates $\partial G$. Indeed, if a ring $R = R(E_1, E_2)$ separates $C$ then $R \subset G$ and each $E_j$ meets $C$. Note that $\overline{\mathbb{R}}^n \setminus E_2 = R \cup E_1$ is a domain. Choose a point $a$ from $E_1 \cap C$ and $z_0$ from $R$ and take a curve $\gamma : [0,1] \to \overline{\mathbb{R}}^n \setminus E_2$ with $\gamma(0) = z_0$ and $\gamma(1) = a$. Then there is a $t \in (0,1]$ such that $\gamma(t) \in \partial G$. Obviously, $\gamma(t) \in E_1$, which implies that $E_1 \cap \partial G \neq \emptyset$. Likewise we have $E_2 \cap \partial G \neq \emptyset$. We now conclude that $R$ separates $\partial G$.

Conversely, suppose that a ring $R = R(E_1, E_2)$ separates $\partial G$. Then $R \subset G$ or $R \cap G = \emptyset$. If the latter occurs, one component of $\overline{\mathbb{R}}^n \setminus R$, say $E_1$, contains $G$. Then $E_2 \cap \partial G = \emptyset$, which contradicts the choice of $R$. Hence the latter case cannot occur. Therefore, we have shown that $R$ separates $C$.

For the study of the geometry of the modulus metric below, we now introduce a new class of conformally invariant domains, M-domains. The definition of this class makes use of the continuum criterion introduced and studied by O. Martio [35]. The continuum criterion is closely connected with the potential theoretic boundary regularity of a domain [36].

3·6. *Definition*

We say that a closed set $C \subset \mathbb{R}^n$ satisfies the *continuum criterion* at $x \in C$ if there exists a continuum $K \subset \{x\} \cup \left(\overline{\mathbb{R}}^n \setminus C\right)$ such that

$$\mathsf{M}(\Delta(K, C; \overline{\mathbb{R}}^n \setminus C)) < \infty.$$

We write $\mathsf{M}(x, C) < \infty$ if this holds, and otherwise we write $\mathsf{M}(x, C) = \infty$.

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We now recall that a continuum is a compact connected set in $\overline{\mathbb{R}}^n$ containing at least two points. We note that $\mathsf{M}(x_0,\ C)=\infty$ if a continuum $C_0\subset C$ contains $x_0$. In fact, the sphere $|x-x_0|=r$ meets both $K$ and $C$ for all small enough $r>0$ in this case. A simple application of the following lemma implies that

$$\mathsf{M}(\Delta(K,C;\overline{\mathbb{R}}^n\setminus C))\geqslant \mathsf{M}(\Delta(K,C;\overline{\mathbb{R}}^n))=\infty$$

for every continuum $K$ with $x_0\in K\subset(\overline{\mathbb{R}}^n\setminus C)\cup\{x_0\}$. Here we have used the relation $\Delta(K,C;\overline{\mathbb{R}}^n\setminus C)<\Delta(K,C;\overline{\mathbb{R}}^n)$ and Lemma 3·1.

LEMMA 3·7 (Vaisala [52, theorem 10·12]). *Let $0<a<b<+\infty$. Let $E$ and $F$ be closed sets in $\overline{\mathbb{R}}^n$ and suppose that the sphere $|x|=t$ meets both $E$ and $F$ for every $t$ with $a<t<b$. Then $\mathsf{M}(\Delta(E,F;\overline{\mathbb{R}}^n))\geqslant c_n\log(b/a)$, where $c_n$ is a positive constant depending only on $n$.*

*We now define the notion of M-domains.*

*Definition 3·8.* A boundary point $x$ of a domain $G\subset\overline{\mathbb{R}}^n$ is said to satisfy the *M-condition* (relative to $G$) if $\mathsf{M}(x,\overline{\mathbb{R}}^n\setminus G)=\infty$; in other words, the complement $\overline{\mathbb{R}}^n\setminus G$ does not satisfy the continuum criterion at $x$. The domain $G$ is called an *M-domain* if every boundary point $x\in\partial G$ satisfies the M-condition relative to $G$.

By the above observation, a point $x\in\partial G$ satisfies the condition $\mathsf{M}(x,\overline{\mathbb{R}}^n\setminus G)<\infty$ only if the singleton $\{x\}$ is a connected component of $\partial G$. On the other hand, any isolated point $x$ of $\partial G$ satisfies $M(x,\overline{\mathbb{R}}^n\setminus G)<\infty$.

We need the following result in the proof of Theorem 1·1. Our proof is similar to that of [35, lemma 3·5].

LEMMA 3·9. *Let $G$ be a domain in $\overline{\mathbb{R}}^n$. Suppose that a point $x_0\in\partial G\setminus\{\infty\}$ and a continuum $K$ in $G\cup\{x_0\}$ with $x_0\in K$ satisfy the condition $\mathsf{M}(\Delta(K,\partial G;G))<\infty$. Then*

$$\lim_{r\to 0}\mathsf{M}(\Delta(K\cap\overline{B}^n(x_0,r),\partial G;G))=0.$$

*Proof.* If $\partial G=\{x_0\}$, the assertion trivially holds. Thus we may assume that $\partial G$ contains at least two points. By the conformal invariance of the capacity, we may assume that $\infty\in\partial G$. For brevity, we write $\overline{B}(r)=\overline{B}^n(x_0,r)$ and $S(r)=\partial B(r)$ throughout the proof. Let $M_0=\mathsf{M}(\Delta(K,\partial G;G))<\infty$ and choose $r_0>0$ large enough so that $K\subset B(r_0)$. For a decreasing sequence $r_j\,(j=0,1,2,\ldots)$ with $r_j\to 0\,(j\to\infty)$, consider the ring $R_j=\{x\in\mathbb{R}^n: r_{j+1}<|x-x_0|<r_j\}$. We can choose such a sequence so that

$$c_j:=\operatorname{cap}R_j=\left(\frac{\omega_{n-1}}{\log(r_j/r_{j+1})}\right)^{1/(n-1)}\qquad\text{satisfies}\qquad\sum_{j=0}^{\infty}c_j<\infty\,.$$

For instance, for $c_j=2^{-j}$, we define $r_j$ recursively by the formula

$$r_{j+1}=r_j\exp\left(-\omega_{n-1}\,c_j^{1-n}\right)=r_j\exp\left(-\omega_{n-1}2^{(n-1)j}\right)$$

for $j=0,1,2,\ldots$. It is obvious that $r_j\to 0$ as $j\to\infty$ for this choice. Let $K_j=K\cap\overline{R_j}$ and denote by $\Delta_j$ the family of curves joining $K_j$ and $\partial G$ in the set $\{x\in G: r_{j+2}<|x-x_0|<r_{j-1}\}$ for $j=1,2,\ldots$. Then the families $\Delta_{N+3j}\,(j=0,1,2,\ldots)$ are disjointly supported and

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contained in the family $\Delta(K,\partial G;G)$ for $N=1,2,3,\ldots$. By Lemma 3·1 (ii) we obtain

$$\sum_{j=0}^{\infty}\mathsf{M}(\Delta_{N+3j})\leqslant\mathsf{M}(\Delta(K,\partial G;G))=M_0\quad(N=1,2,3,\ldots)$$

and hence

$$\sum_{j=1}^{\infty}\mathsf{M}(\Delta_j)\leqslant 3M_0.$$

For a given number $\eta>0$, take a large enough integer $N>0$ so that

$$\sum_{j=N}^{\infty}\mathsf{M}(\Delta_j)<\eta\quad\text{and}\quad\sum_{j=N-1}^{\infty}c_j<\eta.$$

By construction, we easily see that the curve family $\Delta(K_j,\partial G;G)\setminus\Delta_j$ is minorised by the family

$$\Delta(S(r_j),S(r_{j-1});R_{j-1})\cup\Delta(S(r_{j+2}),S(r_{j+1});R_{j+1}).$$

Thus, by Lemma 3·1 (i), we obtain

$$\begin{aligned}\mathsf{M}(\Delta(K_j,&\partial G;G))\\ &\leqslant\mathsf{M}(\Delta_j)+\mathsf{M}(\Delta(K_j,\partial G;G)\setminus\Delta_j))\\ &\leqslant\mathsf{M}(\Delta_j)+\mathsf{M}(\Delta(S(r_j),S(r_{j-1});R_{j-1}))+\mathsf{M}(\Delta(S(r_{j+2}),S(r_{j+1});R_{j+1}))\\ &=\mathsf{M}(\Delta_j)+\operatorname{cap}R_{j-1}+\operatorname{cap}R_{j+1}.\end{aligned}$$

Therefore, we finally have

$$\begin{aligned}\mathsf{M}(\Delta(K\cap\overline{B}(r_N),\partial G;G))&\leqslant\mathsf{M}(\Delta(\{x_0\},\partial G;G))+\sum_{j=N}^{\infty}\Big[\mathsf{M}(\Delta_j)+c_{j-1}+c_{j+1}\Big]\\ &<0+\eta+\eta+\eta=3\eta.\end{aligned}$$

Hence we obtain $\mathsf{M}(\Delta(K\cap\overline{B}^n(x_0,r),\partial G;G))<3\eta$ for $0<r\leqslant r_N$.

The next theorem due to Martio [**35**, theorem 3·4] will also be used in Section 4.

LEMMA 3·10. *Let $G$ be a proper subdomain of $\overline{\mathbb{R}}^n$ and fix a point $a\in G$. For a boundary point $x_0$ of $G$ with $x_0\neq\infty$, set*

$$L(\varepsilon)=\inf_K\mathsf{M}(\Delta(K,\partial G;G)),$$

*where the infimum is taken over all continua $K$ joining $a$ and the sphere $S^{n-1}(x_0,\varepsilon)$ in $G$. Then $\mathsf{M}(x_0,\overline{\mathbb{R}}^n\setminus G)=\infty$ if and only if $L(\varepsilon)\to\infty$ as $\varepsilon\to 0^+$.*

It is clear that M-domains are invariant under Möbius transformations and conformal mappings. We next give an example of an M-domain which does not have uniformly perfect boundary.

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3·11. *Example*

Let $\{s_k\}$ and $\{r_k\}$ $(k = 1, 2, 3, \ldots)$ be two sequences of positive numbers converging to 0 monotonically with the following property:

$$(*) \quad \alpha_k := s_k - r_k - (s_{k+1} + r_{k+1}) > 0.$$

Then the closed balls $\overline{B}_k = \overline{B}^n(s_k e_1, r_k)$, $k = 1, 2, \ldots$, are disjoint because $\operatorname{dist}(\overline{B}_k, \overline{B}_{k+1}) = \alpha_k > 0$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Let $C = \{0\} \cup \bigcup_{k=1}^{\infty} \overline{B}_k$ and $K_0 = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leqslant 0\} \cup \{\infty\}$. Note that the ring $R_k = \{x : r_k < |x - s_k e_1| < r_k{}'\}$ separates $C$, where $r_k{}' = r_k + \min\{\alpha_{k-1}, \alpha_k\}$. Observe that $\alpha_{k-1} \geqslant \alpha_k$ if and only if $2s_k - s_{k-1} - s_{k+1} \leqslant r_{k+1} - r_{k-1}$. This condition is fulfilled when $\{s_k\}$ is convex.

(1) The domain $G = \overline{\mathbb{R}}^n \setminus (K_0 \cup C)$ is an M-domain because every connected component of $K_0 \cup C$ is a continuum. However, $\partial G$ is not uniformly perfect when $\limsup_{k\to\infty} (r_k{}'/r_k) = \infty$. For instance, we can choose a convex sequence $\{s_k\}$ with $2s_{k+1} \leqslant s_k$ (such as $s_k = 2^{-k}$) and let $r_k = 2^{-k} s_k$ for $k \geqslant 1$. Then

$$r_{k+1}/r_k = s_{k+1}/(2s_k) \leqslant 1/4, \quad r_k{}' = 2^k r_k - (2^{k+1} + 1) r_{k+1}$$

and thus

$$\frac{r_k{}'}{r_k} \geqslant 2^k - \frac{1}{4}(2^{k+1} + 1) = 2^{k-1} - 2^{-2} \to +\infty$$

as $k \to \infty$.

(2) Let $G = \overline{\mathbb{R}}^n \setminus C$. Suppose that the sequence of rings $A_k = \{x : s_k - r_k < |x| < s_k + r_k\}$ satisfies the condition $\limsup_{k\to\infty} \operatorname{mod} A_k = \infty$. For instance, we can take $s_k = 2^{-k^2}$, $r_k = s_k - 2s_{k+1}$. Then $\mathsf{M}(0, C) = \infty$. Indeed, for each $k$ and $t \in (s_k - r_k, s_k + r_k)$, the sphere $|x| = t$ intersects $C$ by definition. Hence, for any continuum $K$ with $0 \in K \subset G \cup \{0\}$, Lemma 3.7 now yields

$$\mathsf{M}(\Delta(K, \partial G; G)) \geqslant \mathsf{M}(\Delta(K, C; \overline{\mathbb{R}}^n)) \geqslant c_n \log \frac{s_k + r_k}{s_k - r_k}$$

for sufficiently large $k$. By the assumption, we have $\mathsf{M}(\Delta(K, \partial G; G)) = \infty$. In this case, the singleton $\{0\}$ is a connected component of $\partial G$ but the condition $\mathsf{M}(0, \overline{\mathbb{R}}^n \setminus G) = \infty$ is satisfied.

(3) Let $G = \overline{\mathbb{R}}^n \setminus C$ again. Then

$$\Delta(K_0, C; G) \subset \bigcup_{k=0}^{\infty} \Delta_k,$$

where $\Delta_k = \Delta(K_0, \overline{B}_k; \overline{\mathbb{R}}^n)$ for $k \geqslant 1$ and $\Delta_0 = \Delta(K_0, \{0\}; \overline{\mathbb{R}}^n)$. Note that $\beta_0 := \mathsf{M}(\Delta_0) = 0$. Since the ring $R(K_0, \overline{B}_k)$ contains $R_k$ as a subring, we have

$$\mathsf{M}(\Delta_k) = \operatorname{cap} R(K_0, \overline{B}_k) \leqslant \operatorname{cap} R_k = \omega_{n-1} (\operatorname{mod} R_k)^{1-n} = \omega_{n-1} \left( \log \frac{r_k{}'}{r_k} \right)^{1-n}.$$

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Let $D_k = \{x : |x - s_k| < s_k\}$ for $k \geqslant 1$ and $H = \{x : x_1 > 0\} = \overline{\mathbb{R}}^n \setminus K_0$. Then

$$\mathsf{M}(\Delta_k) = \operatorname{cap}(H, B_k) \leqslant \operatorname{cap}(D_k, B_k) = \omega_{n-1}\left(\log \frac{s_k}{r_k}\right)^{1-n} =: \beta_k$$

for $k \geqslant 1$. If $\sum_k \beta_k < +\infty$, we have

$$\mathsf{M}(\Delta(K_0, C; G)) \leqslant \sum_{k=0}^{\infty} \mathsf{M}(\Delta_k) \leqslant \sum_{k=0}^{\infty} \beta_k < +\infty.$$

Hence $\mathsf{M}(0, \partial G) < \infty$ in this case. For instance, if we choose $s_k$ and $r_k$ so that $r_k = s_k e^{-k^2}$ then $\beta_k = \omega_{n-1} k^{2-2n}$ satisfies the above condition. Hence, $\mathsf{M}(0, \overline{\mathbb{R}}^n \setminus G) < \infty$. This gives an example of a non-isolated boundary point of a domain which does not satisfy the M-condition.

3·12. *Open problem*

It is well known that the Hausdorff dimension of the boundary of a domain with uniformly perfect boundary is positive [29]. We do not know whether the boundary of an $M$-domain has positive Hausdorff dimension.

## 4. *Modulus metric*

In this section, we first give a definition of the modulus metric $\mu_G(x, y)$ and its dual quantity $\lambda_G(x, y)$. After that, we will prove Theorems 1·1 and 1·2. For further results, we refer to [8, 12-15, 23, 31, 38, 39, 44, 56].

*Definition 4·1* ([55, Chapter 8]). Let $G$ be a proper subdomain of $\overline{\mathbb{R}}^n$ and $x, y \in G$. Then we define

$$\mu_G(x, y) = \inf_{C_{xy}} \mathsf{M}(\Delta(C_{xy}, \partial G; G)),$$

where the infimum runs over all curves $C_{xy}$ in $G$ joining $x$ and $y$. We also define

$$\lambda_G(x, y) = \inf_{C_x, C_y} \mathsf{M}(\Delta(C_x, C_y; G)),$$

where the infimum runs over all curves $C_x$ and $C_y$ in $G$ joining $x$ (respectively $y$) and $\partial G$.

In some special cases, the extremal configurations for the curve families defining $\mu_G(x, y)$ and $\lambda_G(x, y)$ are known. Indeed, for the case when $G = \mathbb{B}^n$ and $0 \neq x \in \mathbb{B}^n$, $y = 0$, we have

$$\mu_{\mathbb{B}^n}(x, 0) = \mathsf{M}(\Delta([0, x], \partial\mathbb{B}^n; \mathbb{B}^n)) = \gamma_n(1/|x|)\,, \tag{4·2}$$

and, by the symmetry principle [19, theorem 4·3·5], with $e = x/|x|$,

$$\begin{aligned}\lambda_{\mathbb{B}^n}(x, 0) &= \mathsf{M}(\Delta((-e, 0], [x, e); \mathbb{B}^n)) = 2^{1-n}\mathsf{M}(\Delta([-\infty, 0], [x, e/|x|]; \overline{\mathbb{R}}^n)) \\ &= 2^{1-n}\mathsf{M}(\Delta([-e, 0], [\frac{|x|^2}{1-|x|^2}e, +\infty]; \overline{\mathbb{R}}^n)) = 2^{1-n}\tau_n(|x|^2/(1-|x|^2))\,,\end{aligned} \tag{4·3}$$

see [23, theorem 10·4] for details. Here, we recall that the Grötzsch capacity function $\gamma_n(s)$ and the Teichmüller capacity function $\tau_n(t)$ are defined by

$$\gamma_n(s) = \mathsf{M}(\Delta([0, se_1], \partial\mathbb{B}^n; \mathbb{B}^n)) \quad \text{and} \quad \tau_n(t) = \mathsf{M}(\Delta([-e_1, 0], [te_1, \infty]; \overline{\mathbb{R}}^n))\,,$$

for $0 < s < 1$ and $t > 0$.

Next we look at the case when $G = \mathbb{R}^n \setminus \{0\}$. By the definition of $\lambda_G(te_1, -e_1)$, $t > 0$, there are two natural choices to connect $te_1$ and $-e_1$ with the boundary $\{0, \infty\}$ of the domain $G$, either the pair $[te_1, 0), [-e_1, -\infty)$ or the pair $[te_1, \infty), [-e_1, 0)$. Therefore

$$\lambda_G(te_1, -e_1) = \min\{\tau_n(1/t), \tau_n(t)\}$$

and, because $\tau_n : (0, \infty) \to (0, \infty)$ is a strictly decreasing homeomorphism, for $t > 1$, we have $\tau_n(t) < \tau_n(1) < \tau_n(1/t)$ and thus

$$\lambda_{\mathbb{R}^n \setminus \{0\}}(te_1, -e_1) = \tau_n(t) = \mathsf{M}(\Delta([-e_1, 0), [te_1, \infty); \mathbb{R}^n \setminus \{0\})), \quad t > 1.$$

See [**1**, p.72] and [**23**, pp.178–181] for more details.

Suppose that $G_1$ and $G_2$ are proper subdomains of $\overline{\mathbb{R}}^n$ with $G_1 \subset G_2$. Then for a continuum $C_{xy}$ joining $x$ and $y$ in $G_1$ we have $\Delta(C_{xy}, \partial G_2; G_2) > \Delta(C_{xy}, \partial G_1; G_1)$. By Lemma 3·1 (i), we further obtain for all $x, y \in G_1$

$$\mu_{G_2}(x, y) \leqslant \mathsf{M}(\Delta(C_{xy}, \partial G_2; G_2)) \leqslant \mathsf{M}(\Delta(C_{xy}, \partial G_1; G_1)).$$

Hence $\mu_{G_2}(x, y) \leqslant \mu_{G_1}(x, y)$. By definition, the quantities $\mu_G(x, y)$ and $\lambda_G(x, y)$ are both conformally invariant. Ferrand [**14**] proved that $\lambda_G(x, y)^{1/(1-n)}$ is a distance function of $G$. Thus $\lambda_G(x, y)^{1/(1-n)}$ is often called *Ferrand's modulus metric*. When $n = 2$ and $G$ is a simply connected domain in $\overline{\mathbb{R}}^n$ with card $(\partial G) \geqslant 2$, Ferrand's modulus metric is the same as the modulus metric (up to a constant multiple). Moreover, for $n \geqslant 2$ there exists [**23**, (9·12), theorem 10·4] a constant $c_n > 0$ depending only on $n$ such that for all $x, y \in \mathbb{B}^n$

$$\mu_{\mathbb{B}^n}(x, y) \geqslant 2^{n-1} c_n\, h_{\mathbb{B}^n}(x, y)\,. \tag{4·4}$$

LEMMA 4·5. *Let $G$ be a simply connected hyperbolic domain in $\overline{\mathbb{R}}^2 = \overline{\mathbb{C}}$. Then $\mu_G(x, y) = 4\lambda_G(x, y)^{-1}$.*

*Proof.* Fix a pair of distinct points $x, y \in G$. The Riemann mapping theorem asserts that there is a conformal homeomorphism $f : G \to \mathbb{B}^2 = \{z \in \mathbb{C} : |z| < 1\}$ such that $f(x) = 0$ and $f(y) = u \in (0, 1)$. Since the modulus metric and Ferrand's modulus metric are conformally invariant, we have $\mu_G(x, y) = \mu_{\mathbb{B}^2}(0, u)$ and $\lambda_G(x, y) = \lambda_{\mathbb{B}^2}(0, u)$. By (4·2) and (4·3) together with (3·3), we can write

$$\mu_{\mathbb{B}^2}(0, u) = \gamma_2(1/u) = 2\tau_2(u^{-2} - 1) \quad \text{and} \quad \lambda_{\mathbb{B}^2}(0, u) = \tau_2(1/(u^{-2} - 1))/2\,.$$

In view of the formula $\tau_2(t)\tau_2(1/t) = 4$ [**2**, 5·19 (7)], we obtain $\mu_{\mathbb{B}^2}(0, u)\lambda_{\mathbb{B}^2}(0, u) = 4$ and thus the assertion.

We take this opportunity to state the following plausible fact with a short proof.

LEMMA 4·6. *Let $G$ be a domain in $\overline{\mathbb{R}}^n$ such that the complement $F = \overline{\mathbb{R}}^n \setminus G$ is of positive capacity. Then there is a positive constant $c(F)$ such that the inequality*

$$\mu_G(x, y) \geqslant d_0 \min\{q(x, y), c(F)\} \tag{4·7}$$

*holds for $x, y \in G$, where $d_0 > 0$ is a constant depending only on $n$. In particular, the modulus metric $\mu_G$ induces the same topology on $G$ as the relative topology on $G$ induced by $\overline{\mathbb{R}}^n$ with the spherical metric $q$.*

*Proof.* The inequality (4·7) follows from [55, theorem 6·1] and implies the inclusion map $(G,\mu_G)\to(\overline{\mathbb{R}}^n,q)$ is continuous. In order to show the other inclusion map $(G,q)\to(G,\mu_G)$ is continuous, we may assume that $G\subset\mathbb{R}^n$ and replace $q$ by the Euclidean metric. Take an arbitrary point $x\in G$ and choose a small enough number $r>0$ so that $B:=B^n(x,r)\subset G$. By the domain monotonicity of the modulus metric, we obtain

$$\mu_G(x,y)\leqslant\mu_B(x,y)=\gamma_n(r/|y-x|),\qquad y\in B,$$

by (4·2). Since $\gamma(t)\to 0$ as $t\to+\infty$, we see that $\mu_G(x,y)\to 0$ as $|y-x|\to 0$, which proves the required assertion.

We are now in a position to prove the first main result.

4·8. *Proof of Theorem 1·1*

The part (i) $\Rightarrow$ (ii) is obvious. We show now that (ii) implies (iii) by contradiction. Suppose that $G$ is not an M-domain, namely, $\mathsf{M}(x_0,\overline{\mathbb{R}}^n\setminus G)<\infty$ for some $x_0\in\partial G$. By the conformal invariance, we may assume that $x_0\neq\infty$. We write $B(r)=B^n(x_0,r)$ and $\overline{B}(r)=\overline{B}^n(x_0,r)$ for brevity. By definition, there is a continuum $K$ with $x_0\in K\subset G\cup\{x_0\}$ such that $M_0:=\mathsf{M}(\Delta(K,\partial G;G))<\infty$. Take a point $x_1$ from $K\cap G$ and fix it. Let $r_1=|x_1-x_0|$ and $K_1=K$. For each $x\in K\cap B(r_1)$ and $r\in(0,|x-x_0|)$, let $K_1(x,r)$ be the connected component of $K_1\setminus B(r)$ containing $x$. Note that $K_1(x,r)$ is a continuum. By construction, $K_1(x,r)\subset K_1(x,r')$ for $0<r'<r<|x-x_0|$. We set

$$C_1=C(x_1,K_1):=\bigcup_{0<r<r_1}K_1(x_1,r).$$

Then, $C_1$ is connected and, for $x,y\in C_1$, we have $x,y\in K_1(x_1,r)$ for some $0<r<r_0$. In particular, for such a pair of points $x$, $y$ and $r$,

$$\mu_G(x,y)\leqslant\mathsf{M}(\Delta(K_1(x_1,r),\partial G;G))\leqslant\mathsf{M}(\Delta(K_1,\partial G;G)).$$

We also see that $x_0\in\overline{C_1}$. Indeed, otherwise $\overline{C_1}$ would be a continuum in $K\setminus\overline{B}(\varepsilon)$ for small enough $\varepsilon>0$ and thus $K_1(x_1,\varepsilon)\supset\overline{C_1}\supset C_1$. Since $K_1(x_1,\varepsilon)\subset C_1$, the set $C_1$ would be closed and have a positive distance to $K\setminus C_1$, which would violate connectedness of $K$.

Let $K_2$ be the connected component of the compact set $K_1\cap\overline{B}(r_1/2)$ containing $x_0$. Since $x_0\in\overline{C_1}$, we have $C_1\cap K_2\neq\emptyset$. Take a point $x_2$ from $C_1\cap K_2$ and fix it. As before, set $C_2=C(x_2,K_2)$. Then $C_2\subset C_1\cap K_2$. Repeating this procedure, we define sequences of points $x_j$, continua $K_j$ and connected sets $C_j$ inductively with the following properties:

(i) $K_j\subset\overline{B}(r_12^{1-j})$;

(ii) $x_j\in C_j\subset C_{j-1}\cap K_j$

(iii) $x_0\in\overline{C_j}\subset K_j$ and

(iv) $\mu_G(x,y)\leqslant\mathsf{M}(\Delta(K_j,\partial G;G))$ for all $x,y\in C_j$.

In particular, we observe that

$$\mu_G(x_j,x_k)\leqslant\mathsf{M}(\Delta(K_j,\partial G;G)),\qquad j\leqslant k.$$

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By Lemma 3·9, we have

$$\mathsf{M}(\Delta(K_j, \partial G; G)) \leqslant \mathsf{M}(\Delta(K \cap \overline{B}(r_1 2^{1-j}), \partial G; G)) \to 0 \quad (j \to \infty).$$

Hence, we conclude that $\{x_j\}$ is a Cauchy sequence in $(G, \mu_G)$. Suppose that this sequence is convergent; that is, $\mu_G(x_j, x_\infty) \to 0$ as $j \to \infty$ for some $x_\infty \in G$. On the other hand, since $|x_j - x_0| \leqslant r_1 2^{1-j}$, we have $x_j \to x_0$ in $\overline{\mathbb{R}}^n$. Lemma 4·6 now implies that $x_\infty = x_0 \in \partial G$, which is a contradiction. Therefore, $(G, \mu_G)$ is not complete.

Finally, we prove that (iii) implies (i). If cap $\partial G = 0$, then

$$\mathsf{M}(\Delta(K, \overline{\mathbb{R}}^n \setminus G; G)) = \mathsf{M}(\Delta(K, \partial G; G)) = 0,$$

which is not allowed by condition (iii). Therefore, $(G, \mu_G)$ is a metric space under the assumption (iii). Suppose next that the set $X = \{x \in G : \mu_G(x, a) \leqslant r_0\}$ is not compact for some $a \in G$ and $r_0 > 0$. Then there is a point $x_0 \in \partial X \cap (\partial G)$. We may assume that $x_0 \neq \infty$. For every $\varepsilon > 0$, there exists a point $x \in X \cap B^n(x_0, \varepsilon)$. By definition of $X$, $\mathsf{M}(\Delta(K, \partial G; G)) \leqslant r_0$ for a continuum $K$ in $G \cup \{x_0\}$ with $a, x \in K$. Therefore, under the notation in Lemma 3·10, we obtain $L(\varepsilon) \leqslant r_0$. However, the lemma implies that $\mathsf{M}(x_0, \overline{\mathbb{R}}^n \setminus G) < \infty$. By contradiction, we have shown that (iii) implies (i).

Next we prove our second result.

4·9. *Proof of Theorem 1·2*

Since the uniform perfectness is Möbius invariant (Lemma 3·4), we may assume that $\infty \in \partial G$ and thus $G \subset \mathbb{R}^n$ and $\operatorname{diam}(\partial G) = +\infty$.

First suppose that the boundary $\partial G$ of $G$ is uniformly perfect. Lemma 3·5 implies that the complement $E = \overline{\mathbb{R}}^n \setminus G$ is also uniformly perfect. By a theorem of Järvi and Vuorinen [**29**], $E$ satisfies the metric thickness condition. Vuorinen [**54**] proved that for such a domain $G$ there exists a constant $b_1 > 0$ such that for all $x, y \in G$

$$\mu_G(x, y) \geqslant b_1 \,\hat{j}_G(x, y).$$

Applying (2·13), we obtain (1·3) with $b = b_1/4$.

We next suppose (1·3). Then by Lemma 2·12 (iii), we have $\mu_G(x, y) \geqslant b\, j_G(x, y)$. Let $E = \overline{\mathbb{R}}^n \setminus G$ and

$$0 < c < c_0 := \exp\left[-2\left(\frac{2\omega_{n-1}}{b \log 3}\right)^{1/(n-1)}\right].$$

We prove now that $\{x : cr \leqslant |x - a| \leqslant r\} \cap E \neq \emptyset$ for every $a \in E \setminus \{\infty\}$ and $r > 0$. Suppose, to the contrary, that $\{x : cr \leqslant |x - a| \leqslant r\} \cap E = \emptyset$ for some $a \in E$, $a \neq \infty$, and $r > 0$. Set $C_1 = \{x \in \mathbb{R}^n : |x - a| \leqslant cr\}$ and $C_2 = \{x \in \overline{\mathbb{R}}^n : |x - a| \geqslant r\}$. Then the assumption implies that the set $E$ decomposes into the two non-empty sets $E_1 = E \cap C_1$ and $E_2 = E \cap C_2$. Pick two points $x, y$ from the sphere $S = S^{n-1}(a, \rho)$ so that $|x - y| = 2\rho$, where $\rho = \sqrt{c}\, r$. We take a curve $C^0_{xy}$ joining $x$ and $y$ in $S$. Then, by the subadditivity and monotonicity of the modulus (Lemma 3·1), we obtain

$$\begin{aligned}\mu_G(x, y) &\leqslant \mathsf{M}(\Delta(C^0_{xy}, E))\\ &\leqslant \mathsf{M}(\Delta(C^0_{xy}, E_1)) + \mathsf{M}(\Delta(C^0_{xy}, E_2))\\ &\leqslant \mathsf{M}(\Delta(S, C_1; G_1)) + \mathsf{M}(\Delta(S, C_2; G_2)),\end{aligned}$$

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where $G_1 = \{x : |x-a| < \rho\}$ and $G_2 = \{x : |x-a| > \rho\}$. As is well known [55, (5·10), (5·14)],

$$\mathsf{M}(\Delta(S, C_1; G_1)) = \mathsf{M}(\Delta(S, C_2; G_2)) = \frac{\omega_{n-1}}{(\log 1/\sqrt{c})^{n-1}},$$

we have

$$\mu_G(x, y) \leqslant \frac{2\omega_{n-1}}{(-\log \sqrt{c})^{n-1}},$$

where $\omega_{n-1}$ is the $(n-1)$-dimensional area of $\mathbb{S}^{n-1}$. On the other hand, since $d_G(x) \leqslant |x - a| = \rho$ and $d_G(y) \leqslant |y-a| = \rho$, we obtain

$$j_G(x, y) = \log\left(1 + \frac{|x-y|}{\min\{d_G(x), d_G(y)\}}\right) \geqslant \log\left(1 + \frac{2\rho}{\rho}\right) = \log 3.$$

Thus we have $b \log 3 \leqslant 2\omega_{n-1}/(-\log \sqrt{c})^{n-1}$, that is,

$$\left[c \geqslant \exp\left[-2(2\omega_{n-1}/b \log 3)^{1/(n-1)}\right] = c_0,\right]$$

a contradiction.

In the case when $G$ is either $\mathbb{B}^n$ of $\mathbb{H}^n$, the metric $\mu_G(x, y)$ has the explicit expression in terms of the hyperbolic metric $h_G$ [55, theorem 8·6]

$$\mu_G(x, y) = 2^{n-1}\,\tau_n\left(\frac{1}{\sinh^2\left(\frac{1}{2}h_G(x, y)\right)}\right) = \gamma_n\left(\coth^2\left(\frac{h_G(x, y)}{2}\right)\right). \tag{4·10}$$

The decreasing homeomorphism $\mu : (0, 1] \Longrightarrow [0, \infty)$ is defined by

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}\left(\sqrt{1-r^2}\right)}{\mathcal{K}(r)}, \quad \mathcal{K}(r) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2 t}},$$

for $r \in (0, 1)$, $\mu(1) = 0$. Now the Grötzsch capacity for $n = 2$ can be expressed as follows

$$\gamma_2(s) = \frac{2\pi}{\mu(1/s)}, \quad s > 1\,. \tag{4·11}$$

In conjunction with the above relations (4·10), (4·11), when $G$ is the unit disk $\mathbb{B}^2 = \mathbb{D}$ in $\mathbb{C}$, we obtain the expression

$$\mu_{\mathbb{D}}(z, w) = \gamma_2\left(\frac{1}{\tanh \frac{1}{2}h_{\mathbb{D}}(z, w)}\right) = \frac{2\pi}{\mu\left(\tanh \frac{1}{2}h_{\mathbb{D}}(z, w)\right)}, \quad z, w \in \mathbb{D}.$$

The following estimate will be used later.

LEMMA 4·13.

$$\mu(\tanh x) < \frac{\pi^2}{4x}, \quad x > 0.$$

*Proof.* From [2, (5·29)], we note the inequality

$$\mu(r) < \frac{\pi^2}{4\,\mathrm{artanh}\,\sqrt[4]{r}}$$

for $0<r<1$. Let $v=(\tanh x)^{1/4}\in(0,1)$ for $x>0$. Since $0<\tanh x=v^4<v<1$, we obtain $x<\operatorname{artanh} v$. Hence,

$$\mu(\tanh x)=\mu\left(v^4\right)<\frac{\pi^2}{4\operatorname{artanh} v}<\frac{\pi^2}{4x}.$$

We are now ready to show our third result.

4·14. *Proof of Theorem 1·6*

Assume that $G$ is a Möbius uniform domain in $\overline{\mathbb{R}}^n$. By Möbius invariance of Definition 2·16, we may assume that $G\subset\mathbb{R}^n$. By virtue of Lemmas 2·8 and 2·12, the uniformity assumption reads

$$k_G(x,y)\leqslant c\, j_G(x,y),\quad x,y\in G$$

for a positive constant $c$. By [**55**, lemma 8·32 (ii)] (see also [**23**, lemma 10·7]) there are positive constants $b_1, b_2$ depending only on $n$ such that

$$\mu_G(x,y)\leqslant b_1 k_G(x,y)+b_2$$

for all $x,y\in G$. In view of Lemma 2·12, we have the required inequality with $d_j=cb_j$ $(j=1,2)$.

Next we assume that the inequality (1·7) holds for a simply connected domain $G$ in $\overline{\mathbb{C}}$ with non-degenerate boundary. We can also assume that $G\subset\mathbb{C}$. Then, as is well known, the Koebe one-quarter theorem leads to the inequality $k_G(x,y)\leqslant 2h_G(x,y)$. By the Riemann mapping theorem, there is a conformal homeomorphism $f:G\to\mathbb{B}^2=\mathbb{D}$. Since $\mu_G$ and $h_G$ are conformally invariant, we obtain the formula

$$\mu_G(x,y)=\mu_{\mathbb{D}}(f(x),f(y))=\frac{2\pi}{\mu\left(\tanh\frac{1}{2}h_{\mathbb{D}}(f(x),f(y))\right)}=\frac{2\pi}{\mu\left(\tanh\frac{1}{2}h_G(x,y)\right)}.$$

We now apply Lemma 4·13 to get

$$\mu_G(x,y)\geqslant\frac{4}{\pi}\,h_G(x,y)\geqslant\frac{2}{\pi}k_G(x,y).$$

Combining this with (1·7) and Lemma 2·12, we have

$$k_G(x,y)\leqslant\frac{\pi}{2}\mu_G(x,y)\leqslant\frac{\pi}{2}(2d_1 j_G(x,y)+d_2).$$

Now a result of Gehring and Osgood [**20**] implies that $G$ is uniform.

4·15. *Open problem*

As pointed out above, in the case of planar simply connected domains the modulus metric can be expressed as a function of the hyperbolic metric. We do not know, whether for a general hyperbolic planar domain, the hyperbolic metric has a minorant in terms of the modulus metric.

## 5. *Application to quasimeromorphic maps*

The modulus of a curve family is one of the most important conformal invariants of geometric function theory which provides a bridge connecting geometry and potential theory.

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The modulus is the main tool of the theory of quasiconformal, quasiregular and quasimeromorphic mappings in $\mathbb{R}^n$ [2, 19, 23, 45, 46, 52]. These mappings are the higher dimensional counterparts of the classes of conformal, analytic, and meromorphic functions of classical function theory, respectively. We will now apply our results to prove a Möbius invariant counterpart of a result of Gehring and Osgood [20] for quasimeromorphic mappings.

We make use of some basic facts of the theory of quasiconformal, quasiregular, and quasimeromorphic mappings which are readily available in [45, 46, 52, 55]. The first result shows a Lipschitz type property of quasimeromorphic mappings with respect to the modulus metric. Note that these mappings are locally Hölder-continuous with respect to the Euclidean metric as some basic examples show [52, 16·2].

THEOREM 5·1 [55, theorem 10·18]. *Let $f:G_1\to G_2$ be a non-constant $K$-quasimeromorphic mapping where $G_1, G_2 \subset \overline{\mathbb{R}}^n$. Then for all $x, y\in G_1$,*

$$\mu_{G_2}(f(x),f(y)) \leqslant K\,\mu_{G_1}(x,y)\,.$$

*In particular, $f:(G_1,\mu_{G_1})\to(G_2,\mu_{G_2})$ is Lipschitz continuous.*

D. Betsakos and S. Pouliasis [8] have recently proved that if $f$ is an isometric homeomorphism between the metric spaces

$$f:(G_1,\mu_{G_1})\Longrightarrow(G_2,\mu_{G_2}),$$

then $f$ is quasiconformal and it is conformal if $n=2$. This result gives a solution to a question of Ferrand– Martin– Vuorinen [15] when $n=2$. Very recently this result was strengthened by Pouliasis and Yu. Solynin [44] and independently by Zhang [56]: $\mu$-isometries are conformal in all dimensions $n\geqslant 2$.

We next prove a Harnack-type inequality.

THEOREM 5·2. *Let $f:G_1\to G_2$ be a $K$-quasiregular mapping where $G_1$, $G_2$ are subdomains of $\mathbb{R}^n$, $n\geqslant 2$. If the boundary $\partial G_2$ is uniformly perfect, then the function*

$$u_f(x) := d_{G_2}(f(x)) = \inf\{|f(x)-z| : z\in\partial G_2\}$$

*satisfies the Harnack inequality, i.e. there exists a constant $D_1$ such that for all $x\in G_1$, and all $y\in\bar{B}^n(x,d_{G_1}(x)/2)$,*

$$u_f(x)\leqslant D_1\,u_f(y). \tag{1}$$

*Moreover, there exists a constant $D_2$ such that for all $x,y\in G_1$*

$$k_{G_2}(f(x),f(y))\leqslant D_2\,\max\{k_{G_1}(x,y)^{\alpha},k_{G_1}(x,y)\}\,,\quad \alpha=K^{1/(1-n)}. \tag{2}$$

*Proof.* Fix $x\in G_1$ and $y\in\bar{B}^n(x,d/2)$, where $d=d_{G_1}(x)$. Then the ring $R=\{z: d/2<|z-x|<d\}$ separates $\{x,y\}$ from $\partial G_1$ and $\mathrm{mod}\,R=\log 2$. Therefore, by the definitions of $\mu_{G_1}$,

$$\mu_{G_1}(x,y)\leqslant \mathsf{M}(\Delta([x,y],G_1))\leqslant \mathrm{cap}\,R=\omega_{n-1}(\log 2)^{1/(n-1)} =: M,$$

where we used the relation $\Delta([x,y],G_1)>\Delta(S^{n-1}(x,d/2),S^{n-1}(x,d);R)$ and Lemma 3·1 (ii). (A similar estimate is found at [55, 8·8].) Because $\partial G_2$ is uniformly perfect, it follows

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from Theorem 1·2 and Lemma 2·12 that

$$\mu_{G_2}(f(x),f(y)) \geqslant c\delta_{G_2}(f(x),f(y)) \geqslant cj_{G_2}(f(x),f(y))\,.$$

Next, by Theorem 5·1

$$\mu_{G_2}(f(x),f(y)) \leqslant K\,\mu_{G_1}(x,y) \leqslant K\,M\,.$$

The Harnack inequality (1) with the constant $D_1 = \exp\,(KM/2)$ then follows, because for all $z \in \partial G_2$ [**55**, (2·39)]

$$j_{G_2}(f(x),f(y)) \geqslant \log \frac{|f(x)-z|}{|f(y)-z|}\,.$$

The proof of (2) follows now from [**55**, theorem 12·5].

We are next going to prove the following theorem, which extends a result of Gehring and Osgood [**20**, theorem 3] for quasiconformal mappings. This proof is based on the above Harnack inequality.

THEOREM 5·3. *Let $f{:}G_1 \to G_2$ be a $K$-quasimeromorphic mapping where $G_1\,, G_2 \subset \overline{\mathbb{R}}^n\,, n \geqslant 2$. If the boundary $\partial G_2$ is uniformly perfect, then there exists a constant $d_3 > 0$ such that for all $x, y \in G_1$*

$$\sigma_{G_2}(f(x),f(y)) \leqslant d_3\ \max\{\sigma_{G_1}(x,y)^{\alpha}, \sigma_{G_1}(x,y)\}\,, \quad \alpha = K^{1/(1-n)}\,.$$

We prove below in Example (5·5) that the uniform perfectness of $G_2$ cannot be dropped from Theorem 5·3 and the same example also shows that a similar remark applies to Theorem 5·2. In this example, the image domain $G_2$ has one isolated boundary point and cannot therefore be uniformly perfect.

5·4. *Proof of Theorem 5·3*

Choose Möbius transformations $f_1, f_2$ such that $0, \infty \in \partial f_1(G_1)$ and $0, \infty \in \partial f_2(G_2)$. Then

$$g = f_2 \circ f \circ f_1^{-1}{:}\, f_1(G_1) \longrightarrow f_2(G_2)$$

is $K$-quasiregular and by Theorem 5·2 we have

$$k_{f_2(G_2)}(g(x), g(y)) \leqslant d_3\ \max\{k_{f_1(G_1)}(x,y)^{\alpha}, k_{G_1}(x,y)\}\,, \quad \alpha = K^{1/(1-n)}\,.$$

Because $f_1(G_1), f_2(G_2) \subset \mathbb{R}^n$, we obtain by Lemma 2·8 (iii) a similar inequality for the $\sigma$ metric, with a bit different constants.

5·5. *Example*

To show that the condition $\partial G_2$ be uniformly perfect cannot be dropped from Theorem 5·3, we consider the analytic function $g(z) = \exp\,((z+1)/(z-1))$ which maps the unit disk $\mathbb{B}^2$ onto $\mathbb{B}^2 \setminus \{0\}$. Let $G_1 = \mathbb{B}^2$ and $G_2 = \mathbb{B}^2 \setminus \{0\}$, and let $x_j = (e^j - 1)/(e^j + 1)$ for $j = 1, 2, \ldots$. Then $u_j = g(x_j) = \exp\,(-e^j)$. The standard formula for the hyperbolic distance

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[4, pp.38–40], [55, (2·17)] shows that

$$h_{G_1}(x_j, x_{j+1}) = \int_{x_j}^{x_{j+1}} \frac{2dx}{1-x^2} = 2\,\mathrm{artanh}\,(x_{j+1}) - 2\,\mathrm{artanh}\,(x_j) = 1$$

where as

$$k_{G_2}(g(x_j), g(x_{j+1})) = \int_{u_{j+1}}^{u_j} \frac{du}{u} = e^{j+1} - e^j = (e-1)e^j \to +\infty$$

as $j \to \infty$. Thus by (i) and (ii) of Lemma 2·8, when $j \to \infty$, $\sigma_{G_2}(g(x_j), g(x_{j+1})) \to +\infty$ while $\sigma_{G_1}(x_j, x_{j+1}) = h_{G_1}(x_j, x_{j+1}) = 1$. This demonstrates that uniform perfectness is needed in Theorem 5·3.

## 6. *Logarithmic Möbius metric*

In this section we study the logarithmic Möbius metric

$$\Delta_G(z,w) = \log\left(1 + \delta_G(z,w)\right), \qquad z, w \in G,$$

on a planar domain $G$ in $\overline{\mathbb{C}} = \overline{\mathbb{R}^2}$ and prove Theorem 1·15. Though the hyperbolic metric $h_G(z,w)$ is majorized by twice the Möbius metric $2\delta_G(z,w)$ for an arbitrary hyperbolic domain $G \subset \overline{\mathbb{C}}$ (see [47]), the logarithmic Möbius metric $\Delta_G(z,w)$ is not expected to majorize $h_G(z,w)$ in general. Indeed, $\delta_G(z,w)$ is Lipschitz equivalent to $h_G(z,w)$ if $\partial G$ is uniformly perfect as we noted in Introduction. However, the situation is different when $\partial G$ consists of finitely many points. We now prove the first part of Theorem 1·15. By using the results from [50] or [49], we could obtain more explicit estimates for the bound $c = c(A)$. However, for brevity, we shall be content with existence of $c > 0$ only.

*Proof of the first part of Theorem 1·15.* Let $A$ be a finite set in $\overline{\mathbb{C}}$ with $\mathrm{card}\,(A) \geqslant 3$ and $G = \overline{\mathbb{C}} \setminus A$. Since both metrics are Möbius invariant, we may assume that $\infty \in A$ so that $G \subset \mathbb{C}$. We now consider the function

$$F(z,w) = \begin{cases} \dfrac{h_G(z,w)}{\Delta_G(z,w)} & (z \neq w) \\[2ex] \dfrac{\rho_G(z)}{w_G(z)} & (z = w) \end{cases}$$

on $G \times G$. Here, $\rho_G(z)$ is the density of the hyperbolic metric on $G$ and $w_G(z)$ is defined in (2·7). Our aim is to find an upper bound of $F(z,w)$. Since the hyperbolic distance is induced by the Riemannian metric $\rho_G(z)|dz|$, we have

$$\lim_{w \to z} \frac{h_G(z,w)}{|z-w|} = \rho_G(z)$$

for $z \in G$. On the other hand, by definition of the metric $\delta_G(z,w)$ and the property $\log(1 + x) = x + O(x^2)$ $(x \to 0)$, we have

$$\begin{aligned} \lim_{w \to z} \frac{\Delta_G(z,w)}{|z-w|} &= \lim_{w \to z} \frac{\delta_G(z,w)}{|z-w|} \\ &= \lim_{w \to z} \frac{m_G(z,w)}{|z-w|} \\ &= w_G(z) \end{aligned}$$

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for $z \in G$. Therefore, we see that the function $F(z, w)$ is continuous on $G \times G$. Since $\overline{\mathbb{C}} \times \overline{\mathbb{C}}$ is compact, in order to prove that $\sup_{(z,w)\in G\times G} F(z, w) < +\infty$, it is enough to prove that

$$\hat{F}(\zeta, \omega) := \limsup_{(z,w)\to(\zeta,\omega)} F(z, w) < +\infty$$

for each $(\zeta, \omega) \in \partial(G \times G)$. Note that $\partial(G \times G) = (\partial G \times G) \cup (G \times \partial G) \cup (\partial G \times \partial G)$. When $(a, z_0) \in \partial G \times G = A \times G$, by Lemma 1·10, we have $\hat{F}(a, z_0) = 1$. (If $a = \infty$, with the Möbius invariance of $F(z, w)$ in mind, we may consider the inversion $1/z$ to reduce to the finite case.) Likewise, we can see that $\hat{F}(z_0, a) = 1$.

The remaining case is when $(a, b) \in \partial G \times \partial G$. We may further assume that $a \neq \infty \neq b$. If $a \neq b$, letting $C > |a - b|^2$ be a suitable constant, we have

$$m_G(z, w) = |a, z, b, w| = \frac{|a - b||z - w|}{|a - z||b - w|} \leqslant \frac{C}{|a - z||b - w|}$$

for $z, w$ with $|z - a| < \varepsilon$ and $|w - b| < \varepsilon$, where $\varepsilon > 0$ is a small enough number. Therefore, taking a fixed point $z_0 \in G$, we have for the same $z, w$,

$$\begin{aligned} F(z, w) &\leqslant \frac{h_G(z, z_0) + h_G(z_0, w)}{\Delta_G(z, w)} \\ &\leqslant \frac{h_G(z, z_0)}{\log\left[1 + \log\left(1 + C'/|a - z|\right)\right]} + \frac{h_G(z_0, w)}{\log\left[1 + \log\left(1 + C'/|b - w|\right)\right]}, \end{aligned}$$

where $C' = C/\varepsilon$. Taking the upper limit as $z \to a$ and $w \to b$, with the help of (1·12), we finally get $\hat{F}(a, b) \leqslant 2$.

If $a = b$, assuming $a = 0$ and $\mathbb{D}^* \subset G \subset \mathbb{C} \setminus \{0, 1\}$ as before, we have the estimates $h_G(z, w) \leqslant h_{\mathbb{D}^*}(z, w)$ and $m_G(z, w) \geqslant m_{\mathbb{C}\setminus\{0,1\}}(z, w)$ for $z, w \in \mathbb{D}^*$. Hence, $F(z, w) \leqslant h_{\mathbb{D}^*}(z, w)/\Delta_{\mathbb{C}\setminus\{0,1\}}(z, w)$. The expected claim is now implied by (6·4), which is a consequence of the following lemma.

Let $E^* := \{z : 0 < |z| \leqslant e^{-1}\}$. For $z_1, z_2 \in E^*$, define

$$D(z_1, z_2) = \frac{2\sin(\theta/2)}{\max\{\tau_1,\ \tau_2\}} + |\log\tau_2 - \log\tau_1|\,, \tag{6·1}$$

where $\tau_1 = \log(1/|z_1|)$, $\tau_2 = \log(1/|z_2|)$, $\theta = |\arg(z_2/z_1)| \in [0, \pi]$. It is known that $D(z_1, z_2)$ is a distance function on $E^*$ (see [50, lemma 3·1]).

LEMMA 6·2. *Let* $\Omega = \mathbb{C} \setminus \{0, 1\}$.

(i) $h_{\mathbb{D}^*}(z_1, z_2) \leqslant (\pi/4)D(z_1, z_2)$ *for* $z_1, z_2 \in E^*$.

(ii) $D(z_1, z_2) \leqslant M_0\Delta_\Omega(z_1, z_2)$ *for* $z_1, z_2 \in E^*$, *where* $M_0 = 2/\log(1 + \log 3) = 2.6980\ldots$.

The constants $\pi/4$ and $M_0$ are sharp, respectively.

*Proof.* Part (i) is contained in theorem 3·2 of [50]. The sharpness is observed for $z_1 = e^{-\tau}, z_2 = -e^{-\tau}$ as $\tau \to +\infty$. We prove only part (ii). Let $z_1, z_2 \in E^*$. We may assume that $|z_1| \leqslant |z_2|$ by relabeling if necessary. Then $|z_j| = e^{-\tau_j}$ $(j = 1, 2)$ for some $1 \leqslant \tau_2 \leqslant \tau_1 < +\infty$. We put $\tau = \tau_2$, $s = \tau_1/\tau$ and $\varphi = \sin(\theta/2)$, where $\theta = |\arg(z_2/z_1)| \in [0, \pi]$. Then $s \geqslant 1$,

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$0 \leqslant \varphi \leqslant 1$. By definition, we have

$$m_\Omega(z_1, z_2) \geqslant \frac{|z_1 - z_2|}{|z_1|} = \sqrt{(e^{\tau(s-1)} - 1)^2 + 4\varphi^2 e^{\tau(s-1)}}.$$

Let $x := e^{s-1} \geqslant 1$. Then

$$\Delta_\Omega(z_1, z_2) \geqslant \log\left[1 + \log\left(1 + \sqrt{(x^\tau - 1)^2 + 4\varphi^2 x^\tau}\right)\right] =: f_1(\tau, \varphi, x), \quad \text{and}$$
$$D(z_1, z_2) = \frac{2\varphi}{s\tau} + \log(1 + \log x) =: f_2(\tau, \varphi, x).$$

Further let

$$f_3(\tau, \varphi, x) := f_2(\tau, \varphi, x) - M_0 f_1(\tau, \varphi, x).$$

Then $f_3(\tau, \varphi, x)$ is decreasing in $1 \leqslant \tau < +\infty$, and thus $f_3(\tau, \varphi, x) \leqslant f_3(1, \varphi, x)$ for $\tau \geqslant 1$. By straightforward computations, we have

$$\frac{\partial^2}{\partial \varphi^2} f_1(1, \varphi, x) \leqslant 0 \quad \text{and} \quad \frac{\partial^2}{\partial \varphi^2} f_2(1, \varphi, x) = 0.$$

Therefore $f_3(1, \varphi, x)$ is convex in $0 \leqslant \varphi \leqslant 1$. Since

$$f_3(1, 1, x) = \frac{2}{1 + \log x} + \log(1 + \log x) - M_0 \log(1 + \log(x + 2)),$$

it is easy to verify that $f_3(1, 1, x)$ is decreasing in $1 \leqslant x$, which leads to $f_3(1, 1, x) \leqslant f_3(1, 1, 1) = 0$. Noting that $f_3(1, 0, x) = (1 - M_0)\log(1 + \log x) < 0$, we have $f_3(1, \varphi, x) \leqslant 0$ from convexity, and thus $f_3(\tau, \varphi, x) \leqslant f_3(1, \varphi, x) \leqslant 0$. This completes the proof of the required inequality. To show its sharpness, it is enough to put $z_1 = e^{-1}$ and $z_2 = -e^{-1}$.

*Remark 6·3.* As an immediate consequence of the lemma, we have the inequality

$$h_{\mathbb{D}^*}(z_1, z_2) \leqslant \frac{\pi}{2\log(1 + \log 3)} \Delta_{\mathbb{C}\setminus\{0,1\}}(z_1, z_2), \quad 0 < |z_1|, |z_2| \leqslant e^{-1}. \tag{6·4}$$

As the reader can observe in the proof, this constant $(\pi/4)M_0 \approx 2.11904$ is not sharp.

We now complete the proof of Theorem 1·15.

*Proof of the second part of Theorem 1·15.* Let $G$ be a hyperbolic domain in $\overline{\mathbb{C}}$ with a puncture at the point $a$. Suppose that $\Phi(\delta_G(z, w)) \leqslant h_G(z, w)$ for $z, w \in G$. By the Möbius invariance of $\delta_G$ and $h_G$, we may assume that $a = 0$ and that $\mathbb{D}^* \subset G \subset \mathbb{C}$. Then $m_G(x, -x) \geqslant |0, x, \infty, -x| = 2$ and thus $\delta_G(x, -x) \geqslant \log 3$ for $0 < x < 1$. Therefore, we would have $\Phi(\log 3) \leqslant h_G(x, -x)$. On the other hand, letting $\gamma$ be the upper half of the circle $|z| = x$, we obtain

$$h_G(x, -x) \leqslant h_{\mathbb{D}^*}(x, -x) \leqslant \int_\gamma \frac{|dz|}{|z|\log(1/|z|)} = \frac{\pi}{\log(1/x)}.$$

Since $\log(1/x) \to +\infty$ as $x \to 0^+$, we observe that $h_G(x, -x) \to 0$ as $x \to 0^+$, which contradicts the above.

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REFERENCES

[1] L. V. AHLFORS. *Conformal invariants: Topics in Geometric Function Theory* (McGraw-Hill, New York, 1973).

[2] G. D. ANDERSON, M. K. VAMANAMURTHY and M. K. VUORINEN. *Conformal Invariants, Inequalities and Quasiconformal Maps* (Wiley-Interscience, 1997).

[3] F. G. AVKHADIEV and K. J. WIRTHS. *Schwarz–Pick type inequalities. Frontiers in Mathematics* (Birkhäuser Verlag, Basel 2009).

[4] A. F. BEARDON. *The geometry of discrete groups.* Graduate Texts in Math. **91** (Springer-Verlag, New York, 1983).

[5] A. F. BEARDON. The Apollonian metric of a domain in $\mathbb{R}^n$. *Quasiconformal mappings and analysis* (Ann Arbor, MI, 1995), (Springer, New York, 1998), 91–108.

[6] A. F. BEARDON and D. MINDA. *The Schwarz–Pick Lemma and the hyperbolic metric* (Bookms 2018).

[7] A. F. BEARDON and CH. POMMERENKE. The Poincaré metric of plane domains. *J. London Math. Soc.* **18** (1978), 475–483.

[8] D. BETSAKOS and S. POULIASIS. Isometries for the modulus metric are quasiconformal mappings. *Trans. Amer. Math. Soc.* **372** (2019), 2735–2752.

[9] M. BRIDGEMAN and R. D. CANARY. Uniformly perfect domains and convex hulls: improved bounds in a generalization of a theorem of Sullivan. *Pure Appl. Math. Q.* **9** (2013), 49–71.

[10] M. R. BRIDSON and A. HAEFLIGER. *Metric spaces of non-positive curvature*. Fundamental Principles of Mathematical Sciences, **319** (Springer-Verlag, Berlin, 1999).

[11] S.M. BUCKLEY and D. A. HERRON. Quasihyperbolic geodesics are hyperbolic quasi-geodesics. *J. Eur. Math. Soc.* **22** (2020), 1917–1970.

[12] J. FERRAND. A characterisation of quasiconformal mappings by the behaviour of a function of three points. *Complex analysis*. *Lecture Notes in Math.* **1351** (Springer, Berlin, 1988), 110–123.

[13] J. FERRAND. Conformal capacities and conformally invariant functions on Riemannian manifolds. *Geom. Dedicata* **61** (1996), 103–120.

[14] J. FERRAND. Conformal capacities and extremal metrics. *Pacific J. Math.* **180** (1997), 41–49.

[15] J. FERRAND, G. J. MARTIN and M. K. VUORINEN. Lipschitz conditions in conformally invariant metrics. *J. Analyse Math.* **56** (1991), 187–210.

[16] J. B. GARNETT and D. E. MARSHALL. *Harmonic Measure* (Cambridge University Press, 2005).

[17] H. GAUSSIER. Metric properties of domains in $\mathbb{C}^n$. *In Geometric Function Theory in Higher Dimension*, Springer INdAM Ser. **26** (Springer, Cham, 2017), 143–155.

[18] F. W. GEHRING and K. HAG. *The ubiquitous quasidisk. With contributions by* Ole Jacob Broch. Mathematical Surveys and Monographs, **184** (American Mathematical Society, Providence, RI, 2012).

[19] F. W. GEHRING, G. J. MARTIN and B. PALKA. *An introduction to the theory of higher-dimensional quasiconformal mappings*. Math. Surveys Monogr. **216** (American Mathematical Society, Providence, RI, 2017).

[20] F. W. GEHRING and B. G. OSGOOD. Uniform domains and the quasihyperbolic metric. *J. Analyse Math.* **36** (1979), 50–74.

[21] F. W. GEHRING and B. P. PALKA. Quasiconformally homogeneous domains. *J. Analyse Math.* **30** (1976), 172–199.

[22] A. GOLBERG, T. SUGAWA and M. VUORINEN. Teichmüller's lemma in higher dimensions and its applications. *Comput. Methods Funct. Theory*, **20** (2020), 539–558.

[23] P. HARIRI, R. KLÉN and M. VUORINEN. *Conformally invariant metrics and quasiconformal mappings*. Springer Monogr. Math. (Springer, 2020).

[24] D. A. HERRON. Uniform domains and hyperbolic distance. *J. Anal. Math.* **143** (2021), 349–400.

[25] D. A. HERRON, Z. IBRAGIMOV and D. MINDA. Geodesics and curvature of Möbius invariant metrics. *Rocky Mountain J. Math.* **38** (2008), 891–921.

[26] D. A. HERRON and P. K. JULIAN. Ferrand's Möbius invariant metric. *J. Anal.* **21** (2013), 101–121.

[27] D. A. HERRON, W. MA and D. MINDA. Möbius invariant metrics bilipschitz equivalent to the hyperbolic metric. *Conform. Geom. Dyn.* **12** (2008), 67–96.

[28] M. JARNICKI and P. PFLUG. *Invariant Distances and Metrics in Complex Analysis*. Second extended edition. De Gruyter Expositions in Mathematics, **9** (Walter de Gruyter GmbH & Co. KG, Berlin, 2013).

[29] P. JÄRVI and M. VUORINEN. Uniformly perfect sets and quasiregular mappings. *J. London Math. Soc.* **54** (1996), 515–529.
[30] L. KEEN and N. LAKIC. *Hyperbolic Geometry from a Local Viewpoint* (Cambridge University Press, Cambridge, 2007).
[31] J. LELONG–FERRAND. Invariants conformes globaux sur les variétés riemanniennes (French). *J. Differential Geometry* **8** (1973), 487–510.
[32] J. L. LEWIS. Uniformly fat sets. *Trans. Amer. Math. Soc.* **308** (1988), 177–196.
[33] W. MA and D. MINDA. The hyperbolic metric and uniformly perfect regions. *J. Anal.* **20** (2012), 59–70.
[34] A. MARDEN and V. MARKOVIC. Characterisation of plane regions that support quasiconformal mappings to their domes. *Bull. Lond. Math. Soc.* **39** (2007), no. 6, 962–972.
[35] O. MARTIO. Equicontinuity theorem with an application to variational integrals. *Duke Math. J.* **42** (1975), no. 3, 569–581.
[36] O. MARTIO and J. SARVAS. Density conditions in the $n$-capacity. *Indiana Univ. Math. J.* **26** (1977), no. 3, 761–776.
[37] O. MARTIO and J. SARVAS. Injectivity theorems in plane and space. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** (1978/79), 383–401.
[38] P. PANSU. Quasiconformal mappings and manifolds of negative curvature. *Curvature and topology of Riemannian manifolds*. *Lecture Notes in Math.* **1201** (Springer, Berlin, 1986), 212–229.
[39] P. PANSU. Jacqueline Ferrand and her oeuvre. *Notices Amer. Math. Soc.* **65** (2018), no. 2, 201–205.
[40] A. Papadopoulos. *Metric spaces, convexity and non-positive curvature*. *IRMA Lectures in Mathematics and Theoretical Physics* **6** (European Mathematical Society, Zürich, 2014).
[41] P. PFLUG. Invariant metrics and completeness. *J. Korean Math. Soc.* **37** (2000), 269–284.
[42] CH. POMMERENKE. Uniformly perfect sets and the Poincaré metric. *Arch. Math.* **32** (1979), 192–199.
[43] CH. POMMERENKE. On uniformly perfect sets and Fuchsian groups. *Analysis* **4** (1984), 299–321.
[44] S. POULIASIS and A. YU. Solynin. Infinitesimally small spheres and conformally invariant metrics. *J. Anal. Math.* **143** (2021), no. 1, 179–205.
[45] YU. G. RESHETNYAK. *Space mappings with bounded distortion*. Trans. Math. Monogr. **73** (American Mathematical Society, Providence, RI, 1989).
[46] S. RICKMAN. *Quasiregular mappings*. *Ergeb. Math. Grenzgeb.* (3) **26** (Springer-Verlag, Berlin, 1993).
[47] P. SEITTENRANTA. Möbius-invariant metrics. *Math. Proc. Camb. Phil. Soc.* **125** (1999), 511–533.
[48] T. SUGAWA. Uniformly perfect sets: analytic and geometric aspects (Japanese). *Sugaku* **53** (2001), 387–402. (English trans: *Sugaku Expo.* 16 (2003), 225–242)
[49] T. SUGAWA, M. VUORINEN and T. ZHANG. On the hyperbolic distance of $n$-times punctured spheres. *J. Analyse Math.* **141** (2020), no. 2, 663–687.
[50] T. SUGAWA and T. ZHANG. Construction of nearly hyperbolic distance on punctured spheres. *Bull. Math. Sci.* **8** (2018), 307–323.
[51] P. TUKIA and J. VÄISÄLÄ. Quasisymmetric embeddings of metric spaces. *Ann. Acad. Sci. Fenn. Ser. A I Math.* **5** (1980), no. 1, 97–114.
[52] J. VÄISÄLÄ. *Lectures on n-dimensional quasiconformal mappings*. *Lecture Notes in Math.* **229** (Springer-Verlag, Berlin-New York, 1971).
[53] M. VUORINEN. Conformal invariants and quasiregular mappings. *J. Analyse Math.* **45** (1985), 69–115.
[54] M. VUORINEN. On quasiregular mappings and domains with a complete conformal metric. *Math. Z.* **194** (1987), no. 4, 459–470.
[55] M. VUORINEN. *Conformal geometry and quasiregular mappings*. *Lecture Notes in Math.* **1319** (Springer-Verlag, 1988).
[56] X. ZHANG. Isometries for the modulus metric in higher dimensions are conformal mappings. *Sci. China Math.* **64** (2021), no. 9, 1951–1958.