WEYL'S THEOREM HOLDS FOR p-HYPONORMAL OPERATORS*

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1. Introduction. Let $\mathscr{H}$ be a complex Hilbert space and $B(\mathscr{H})$ the algebra of all bounded linear operators on $\mathscr{H}$. Let $\mathscr{K}(\mathscr{H})$ be the algebra of all compact operators of $B(\mathscr{H})$. For an operator $T \in B(\mathscr{H})$, let $\sigma(T)$, $\sigma_p(T)$, $\sigma_\pi(T)$ and $\pi_{oo}(T)$ denote the spectrum, the point spectrum, the approximate point spectrum and the set of all isolated eigenvalues of finite multiplicity of T, respectively. We denote the kernel and the range of an operator T by $\ker(T)$ and $R(T)$, respectively. For a subset $\mathscr{Y}$ of $\mathscr{H}$, the norm closure of $\mathscr{Y}$ is denoted by $\bar{\mathscr{Y}}$. The Weyl spectrum $\omega(T)$ of $T \in B(\mathscr{H})$ is defined as the set

$$\omega(T) = \bigcap_{K \in \mathscr{K}(\mathscr{H})} \sigma(T + K).$$

We say that Weyl's theorem holds for T if the following equality holds;

$$\omega(T) = \sigma(T) - \pi_{oo}(T).$$

An operator $T \in B(\mathscr{H})$ is said to be *p-hyponormal* if $(T^*T)^p \geq (TT^*)^p$. Especially, when $p = 1$ and $p = \frac{1}{2}$, T is called *hyponormal* and *semi-hyponormal*, respectively. It is well known that a p-hyponormal operator is q-hyponormal for $q \leq p$ by Löwner's Theorem. In [**8**], Coburn showed that Weyl's theorem holds for hyponormal operators. In this paper, we shall prove the following results.

THEOREM 0. *Let T be a p-hyponormal operator on $\mathscr{H}$ where $0 < p < 1$. Then Weyl's theorem holds for T.*

2. Proof of Theorem 0. Throughout this section, let p satisfy $0 < p < 1$. First in [**2**] Baxley proved the following result.

THEOREM A (Lemma 3 of [**2**]). *Let $T \in B(\mathscr{H})$. Suppose that T satisfies the following condition C-1.*

C-1. *If $\{\lambda_n\}$ is a infinite sequence of distinct points of the set of eigenvalues of finite multiplicity of T and $\{x_n\}$ is any sequence of corresponding normalized eigenvectors, then the sequence $\{x_n\}$ does not converge.*

Then

$$\sigma(T) - \pi_{oo}(T) \subset \omega(T).$$

Chō and Huruya proved the following result.

THEOREM B (Corollary 5 of [**5**]). *Let T be p-hyponormal. Let $\alpha, \beta \in \sigma_p(T)$ where $\alpha \neq \beta$. If x and y are eigenvectors of α and β, respectively, then $(x, y) = 0$.*

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By Theorem B, it follows that if T is p-hyponormal, then T satisfies C-1. Hence it is clear that if T is p-hyponormal, then

$$\sigma(T) - \pi_{oo}(T) \subset \omega(T).$$

For the proof of the converse inclusion relation, we shall prove the following result.

THEOREM 1. *Let T be p-hyponormal. If λ is an isolated point of $\sigma(T)$, then $\lambda \in \sigma_p(T)$.*

Since the theorem holds for $\lambda \neq 0$, by Theorem 1 of [**7**], we need only prove the case $\lambda = 0$.

For this proof, we need the Aluthge transform (cf. [**1**]). Let $U|T|$ be the polar decomposition of $T \in B(\mathscr{H})$. Then Aluthge introduced the transform

$$T \to \tilde{T} = |T|^{1/2} U |T|^{1/2},$$

and proved the following result.

THEOREM C (Theorems 1 and 2 of [**1**]). *Let T be p-hyponormal. Then $\tilde{T} = |T|^{1/2} U |T|^{1/2}$ is $(p + \frac{1}{2})$-hyponormal.*

Though the operator U in Aluthge's paper is unitary, it is easy to check that Theorem C holds for any p-hyponormal operator.

We need some further results.

LEMMA 1. *Let $T = U|T|$ be p-hyponormal. Then $\sigma(T) = \sigma(\tilde{T})$, where $\tilde{T} = |T|^{1/2} U |T|^{1/2}$.*

Proof. To see this write $T = (U|T|^{1/2})|T|^{1/2}$ and consider separately $\lambda = 0$ and $\lambda \neq 0$.

LEMMA 2. *Let T be semi-hyponormal. If 0 is an isolated point of $\sigma(T)$, then $0 \in \sigma_p(T)$.*

Proof. Let $T = U|T|$ be the polar decomposition of T and $\tilde{T} = |T|^{1/2} U |T|^{1/2}$. Since 0 belongs to the boundary of $\sigma(T)$, by Lemma 1 it follows $0 \in \sigma(\tilde{T}) = \sigma(T)$. Therefore, 0 is an isolated point of $\sigma(\tilde{T})$. Since, by Theorem C, $\tilde{T}$ is hyponormal, from a Stampfli result (Theorem 2 of [**10**]) it follows that 0 is an eigenvalue of $\tilde{T}$. Hence there exists a nonzero $x_0 \in \mathscr{H}$ such that $\tilde{T}x_0 = 0$. Since $|T|^{1/2} U |t|^{1/2} x_0 = 0$, we have $U|T|^{1/2} x_0 \in \ker(|T|^{1/2})$. Since, by Lemma 1 of [**5**], $\ker(T) \subset \ker(T^*)$, It follows that

$$T^*(U|T|^{1/2} x_0) = |T|^{3/2} x_0 = 0.$$

Hence $|T| x_0 = 0$. Therefore we have $0 \in \sigma_p(T)$.

Proof of Theorem 1 for $\lambda = 0$ and $0 < p < \frac{1}{2}$. Let $T = U|T|$ be the polar decomposition of T and $\tilde{T} = |\tilde{T}|^{1/2} U |T|^{1/2}$. By Lemma 1, it follows that $0 \in \sigma(\tilde{T})$ and 0 is an isolated point of $\sigma(\tilde{T})$. Since, by Theorem C, $\tilde{T}$ is semi-hyponormal, by Lemma 2 it follows that $0 \in \sigma_p(\tilde{T})$. Hence also it follows that $0 \in \sigma_p(T)$ on the analogy of the proof of Lemma 2.

Proof of the inclusion relation. $\omega(T) \subset \sigma(T) - \pi_{oo}(T)$.

Let $\lambda \in \pi_{oo}(T)$. By Theorem 4 of [**5**] or Theorem 2 of [**9**], we have

$$\ker(T - \lambda) \subset \ker((T - \lambda)^*) = (R(T - \lambda))^{\perp}.$$

Hence we have the following decomposition of $T - \lambda$:

$$T-\lambda=\begin{pmatrix}0 & 0\\ 0 & S\end{pmatrix} \quad \text{on} \quad \ker(T-\lambda)\oplus\overline{R((T-\lambda)^*)}.$$

Since

$$T=\begin{pmatrix}\lambda & 0\\ 0 & S+\lambda\end{pmatrix},$$

$S+\lambda$ is a p-hyponormal operator on $\overline{R((T-\lambda)^*)}$. If $\lambda\in\sigma(S+\lambda)$, by Theorem 1 we have $\lambda\in\sigma_p(S+\lambda)$ because λ is an isolated point of $\sigma(S+\lambda)$. This is a contradiction. Hence $\lambda\notin\sigma(S+\lambda)$. Therefore $0\notin\sigma(S)$. Let

$$K=\begin{pmatrix}I & 0\\ 0 & 0\end{pmatrix}.$$

Then $K\in\mathcal{K}(\mathcal{H})$ and

$$T+K-\lambda=\begin{pmatrix}I & 0\\ 0 & S\end{pmatrix}$$

is an invertible operator. Therefore $\lambda\notin\omega(T)$. Hence we have

$$\omega(T)\subset\sigma(T)-\pi_{oo}(T)$$

and the proof of the theorem is complete.

3. Application.

Corollary 1. *Let T be p-hyponormal. If $\pi_{oo}(T)=\emptyset$, then for every $K\in\mathcal{K}(\mathcal{H})$*

$$\|T\|\le\|T+K\|.$$

Proof. By Corollary 10 of [5], we have that $r(T)=\|T\|$, where $r(T)$ is the spectral radius of T. Hence from Theorem 1 it follows that $\|T\|\le\|T+K\|$ for every $K\in\mathcal{K}(\mathcal{H})$.

Corollary 2. *Let T be p-hyponormal. Then there exist orthogonal reducing subspaces $\mathcal{M}$ and $\mathcal{N}$ for T such that $\mathcal{H}=\mathcal{M}\oplus\mathcal{N}$, $T|_{\mathcal{M}}$ is a normal operator on $\mathcal{M}$ and*

$$\omega(T|_{\mathcal{N}})=\sigma(T|_{\mathcal{N}}).$$

Proof. For $\lambda\in\sigma_p(T)$, let

$$\mathcal{M}_\lambda=\{x \mid Tx=\lambda x\}.$$

Then, by Theorem 4 of [5], $\mathcal{M}_\lambda$ is a reducing subspace for T. Let

$$\mathcal{M}=\bigoplus_{\lambda\in\sigma_p(T)}\mathcal{M}_\lambda \quad \text{and} \quad \mathcal{N}=\mathcal{M}^\perp.$$

Then $\mathcal{M}$ reduces T and $T|_{\mathcal{M}}$ is normal. Let $S=T|_{\mathcal{N}}$. Then S is a p-hyponormal operator on $\mathcal{N}$. By Theorem 0, Weyl's theorem holds for S. Since $\pi_{oo}(S)=\emptyset$, it follows that $\omega(S)=\sigma(S)$.

COROLLARY 3. *Let T be p-hyponormal. Then*

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \iint_{\omega(T)} r^{2p-1}\, dr\, d\theta.$$

Proof. Let μ be planar Lebesgue measure. Then we have $\mu(\pi_{oo}(T)) = 0$. Hence the result follows from Theorem 5 of [**6**].

COROLLARY 4. *Let T be p-hyponormal. Then, for every* $K \in \mathcal{K}(\mathcal{H})$,

$$\|(T^*T)^p - (TT^*)^p\| \leq \frac{p}{\pi} \iint_{\sigma(T+K)} r^{2p-1}\, dr\, d\theta.$$

Proof. Since $\omega(T) \subset \sigma(T+K)$ for every $K \in \mathcal{K}(\mathcal{H})$, the result follows from Corollary 3.

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