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ON SMALL SUBSPACE LATTICES IN HILBERT SPACE

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Abstract

We study the reflexivity and transitivity of a double triangle lattice of subspaces in a Hilbert space. We show that the double triangle lattice is neither reflexive nor transitive when some invertibility condition is satisfied (by the restriction of a projection under another). In this case, we show that the reflexive lattice determined by the double triangle lattice contains infinitely many projections, which partially answers a problem of Halmos on small lattices of subspaces in Hilbert spaces.

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1. Introduction

In [7], Halmos posed ten problems in operator theory, two of which were concerned with small subspace lattices in Hilbert spaces. The tenth problem asks whether every nontrivial strongly closed transitive atomic lattice is either self-conjugate or medial, that is, the intersection and union of any two nontrivial projections are 0 and I, respectively. Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. A subalgebra $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ is called *transitive* if the invariant subspace lattice $\mathrm{Lat}(\mathcal{A}) = \{P : (I - P)TP = 0, \forall T \in \mathcal{A}\}$ of $\mathcal{A}$ is $\{0, I\}$. A subspace lattice $\mathcal{L}$ is called *transitive* if the algebra $\mathrm{Alg}(\mathcal{L}) = \{T \in \mathcal{B}(\mathcal{H}) : (I - P)TP = 0, \forall P \in \mathcal{L}\}$ associated to $\mathcal{L}$ is equal to $\mathbb{C}I$ (the algebra consisting only scalar multiples of the identity operator I). In this paper, we shall assume that all subspaces of a Hilbert space are closed and they are represented by the orthogonal projections onto them; and that every subspace lattice contains 0 and I.

It is easy to check that any subspace lattice with only two nontrivial elements is not transitive. Halmos [7] gave an example of a transitive lattice of subspaces with only

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five nontrivial elements; and Harrison *et al.* [10] constructed an example with only four nontrivial projections. The existence of a transitive subspace lattice with only three nontrivial projections remains unknown. Partial results have been obtained by Hadwin *et al.* [6], Harrison [9] and Ge and Yuan [4, 5].

The ninth problem in [7] asks whether every complete Boolean algebra given by subspaces of a Hilbert space is reflexive. A subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is called *reflexive* if $\mathrm{Alg}(\mathrm{Lat}(\mathcal{A})) = \mathcal{A}$ and a subspace lattice $\mathcal{L}$ is called *reflexive* when $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) = \mathcal{L}$. The reflexive algebras are in the central role of the nonselfadjoint operator algebras which are closely related to operator theory and invariant subspaces of operators. Parallel to the theory of selfadjoint operator algebras (C^*-algebras and von Neumann algebras), many important results in nonselfadjoint algebras have been obtained in the past 50 years mathematicians such as Arveson [1], Larson [14], Davidson [2] and Lance [13].

It is not easy to determine the reflexivity of a given subspace lattice or a given subalgebra. Halmos [8] has shown that any atomic complete Boolean algebra is reflexive. Furthermore, Harrison [9] has shown that a finite distributive lattice is always reflexive. Thus, if a lattice has two or fewer nontrivial elements, it is reflexive. There are just two nonisomorphism types of nondistributive subspace lattices with only three nontrivial elements: the pentagon and the double triangle. These two classes are the most interesting small invariant subspace lattices to study.

Recently, Ge and Yuan [4, 5] studied the maximal triangularity of certain reflexive lattices. Those lattices assume some nice topological structures. For example, they show that the reflexive lattices generated by many double triangle lattices are homeomorphic to the classical two-dimensional sphere. This study initiated a new class of operator algebras which they call ‘Kadison–Singer algebras’ or ‘KS algebras’ for short (correspondingly ‘Kadison–Singer lattices’ or ‘KS lattices’). Several people followed their study and obtain many interesting reflexive algebras and lattices (see, for example [3, 11, 18]). Although some of the techniques developed in [4, 5] can be applied to study double triangle lattices in infinite factors, the reflexive lattice determined by a double triangle lattice in $\mathcal{B}(\mathcal{H})$ is in general unknown. The existence of a transitive double triangle lattice would imply such a reflexive lattice is trivial.

In this paper, we study the double triangle lattice of projections in $\mathcal{B}(\mathcal{H})$. When $\mathcal{B}(\mathcal{H})$ is replaced by a finite von Neumann algebra, two or any finite number of unbounded operators affiliated with the algebra have a common dense domain. This allows the construction of many bounded operators which leave the subspaces in the lattice invariant. For $\mathcal{B}(\mathcal{H})$, we believe some analogous results hold. Suppose that A and B are (unbounded) selfadjoint invertible operators (with unbounded inverses). Then it is easy to show that the algebra $\mathcal{A}_A = \{X \in \mathcal{B}(\mathcal{H}) : A^{-1}XA \text{ is bounded}\}$ is a (weak-operator) dense subalgebra of $\mathcal{B}(\mathcal{H})$. Define $\mathcal{A}_B$ similarly. We conjecture that $\mathcal{A}_A \cap \mathcal{A}_B$ is weak-operator dense in $\mathcal{B}(\mathcal{H})$ when A and B are affiliated with some nonatomic subalgebras of $\mathcal{B}(\mathcal{H})$ respectively.

The paper is organized as follows. In Section 2 we prove some basic results concerning double triangle lattices in $\mathcal{B}(\mathcal{H})$. Section 2 also contains a main result

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which describes the algebra of operators that leave all subspaces in a double triangle lattice invariant. In Section 3 we show that the algebra is not trivial if some invertibility is satisfied by the projections in a double triangle lattice in $\mathcal{B}(\mathcal{H})$. Moreover, we show that the reflexive lattice determined by the double triangle lattice is infinite. Furthermore, we prove the conjecture under the assumption that A and B are affiliated with a finite von Neumann algebra. In the final section, by using some of our techniques (different from [15]), we can again reduce Halmos' transitivity problem for small lattices to the case of double triangle lattices in $\mathcal{B}(\mathcal{H})$.

2. Preliminary results on double triangle lattices

We assume that $\mathcal{H}$ is a separable Hilbert space over the field of complex numbers. We use $\xi \otimes \eta$ to denote the rank-one operator $\xi \otimes \eta(\zeta) = \langle \zeta, \xi \rangle \eta$ where ξ, η and ζ are vectors in $\mathcal{H}$. Let P, Q, R be nontrivial (orthogonal) projections acting on $\mathcal{H}$ such that $\mathcal{L} = \{0, I, P, Q, R\}$ forms a double triangle lattice, that is, $P \wedge Q = P \wedge R = Q \wedge R = 0$ and $P \vee Q = P \vee R = Q \vee R = I$.

Proposition 2.1. *Suppose that $\mathcal{H}$ is infinite dimensional and $\mathcal{L} = \{0, I, P, Q, R\}$ a double triangle lattice. Then the ranges of any nontrivial projection in $\mathcal{L}$ and its orthogonal complement must be infinite dimensional.*

Proof. Suppose that $\dim(P(\mathcal{H})) = n < \infty$. As $P \wedge Q = 0$ and $P \vee Q = I$, then according to Kaplansky formula (see, for example, [12]), we have $P \vee Q - Q \sim P - P \wedge Q$ which implies that $\dim((I - Q)(\mathcal{H})) \leqslant n < \infty$, where $\sim$ denotes the usual Murray–von Neumann equivalence of projections. Similarly the dimension of $(I - R)(\mathcal{H})$ is also finite. Thus $(I - Q) \vee (I - R)$ is a finite-rank projection and $Q \wedge R \neq 0$. □

Throughout the rest of this paper, we assume that $\mathcal{H}$ is infinite dimensional. As $P(\mathcal{H})$ and $(I - P)(\mathcal{H})$ both are infinite dimensional, we can assume that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$ and $P = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$. If Q is such that $P \vee Q = I$ and $P \wedge Q = 0$, then according to the structure of two projections, we have the following result.

Lemma 2.2. *With the above notation, if $P \wedge Q = 0$ and $P \vee Q = I$, then Q must have the operator matrix form*

$$Q = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^* \sqrt{H(I-H)} & I - V^* H V \end{pmatrix}$$

where H is a positive contraction and V a partial isometry whose final projection agrees with the range projection of the operator $\sqrt{H(I-H)}$.

Proof. Suppose that $Q = \left(\begin{smallmatrix} H_1 & H_2 V \\ V^* H_2 & H_3 \end{smallmatrix}\right)$ where H_1 and H_3 are positive contractions and $H_2 V$ is the polar decomposition of the (2, 1)th entry in the operator matrix.

As $P \wedge Q = 0$, we have $\operatorname{Ker}(I - H_1) = 0$. Otherwise, we may assume that $0 \neq x \in \operatorname{Ker}(I - H_1)$. Then $Q\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right)$. Thus $P \wedge Q \neq 0$, which contradicts our assumption. Similarly, as $P \vee Q = I$, we have $\operatorname{Ker}(H_3) = 0$, since $0 \neq x \in \operatorname{Ker}(H_3)$ would imply that $(P \vee Q)\left(\begin{smallmatrix} 0 \\ x \end{smallmatrix}\right) = 0$, which is a contradiction.

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Furthermore, since $Q^2 = Q$,

$$\begin{aligned} H_1 &= H_1^2 + H_2^2 \\ H_2 V &= H_1 H_2 V + H_2 V H_3 \\ H_3 &= V^* H_2^2 V + H_3^2. \end{aligned}$$

Then $H_2 = \sqrt{H_1(I - H_1)}$ and

$$(I - H_1)\sqrt{H_1(I - H_1)}V = \sqrt{H_1(I - H_1)}V H_3.$$

As the kernel of the operator $\sqrt{I - H}$ is trivial, we find that $(I - H_1)H_1 V = H_1 V H_3$. By the last equation of the above system of equations and since $\mathrm{Ker}(H_3) = 0$, we obtain that $H_3 = I - V^* H_1 V$. □

REMARK 2.3. If the projections P, Q are in a finite factor, then the partial isometry V can be extended to a unitary. In this case, we have $Q = \left(\begin{smallmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & V^*(I-H)V \end{smallmatrix}\right)$. Note that in the proof of the above lemma, we have $P \wedge Q = 0$ and $P \vee Q = I$ if and only if Q has the operator matrix form in Lemma 2.2 and $\mathrm{Ker}(I - H) = 0$.

In this paper, we shall use $\mathrm{Ran}(A)$ to denote the range projection of the operator A and $\mathrm{Range}(A)$ to denote the actual range of A when the underlying space is given. When an operator S is unbounded, we denote the domain of S by $\mathcal{D}(S)$.

REMARK 2.4. Let us recall some properties of the operators H and V. Suppose that $E = VV^*$ and $F = V^*V$. Then

$$\begin{aligned} \mathrm{Ran}(\sqrt{H}) &= \mathrm{Ran}(\sqrt{H(I-H)}) = \mathrm{Ran}(\sqrt{H(I-H)^{-1}}) \\ &= \mathrm{Ran}(\sqrt{H}V) = \mathrm{Ran}(\sqrt{H(I-H)}V) \\ &= \mathrm{Ran}(\sqrt{H(I-H)^{-1}}V) = \mathrm{Ran}(\sqrt{I-H}V) = E. \end{aligned}$$

Furthermore, the restriction of H to $E(\mathcal{H}_0)$ is injective. Let

$$W = \begin{pmatrix} \sqrt{I-H} & \sqrt{H}V \\ -V^*\sqrt{H} & V^*\sqrt{I-H}V + (I-F) \end{pmatrix}.$$

Then $W^*W = WW^* = I$, that is, W is a unitary. Moreover, it is easy to check that $Q = W\left(\begin{smallmatrix} 0 & 0 \\ 0 & F \end{smallmatrix}\right)W^*$.

If $P \wedge Q = 0$, we have, for any $0 \neq \xi \in Q\mathcal{H} \ominus Q(I - P)\mathcal{H}$,

$$0 = \langle \xi, Q(I-P)\beta \rangle = \langle \xi, (I-P)\beta \rangle, \quad \forall \beta \in \mathcal{H},$$

which contradicts the fact that $Q \wedge P = 0$. Therefore we have the following lemma.

LEMMA 2.5. *If* $P \wedge Q = 0$, *then* $\mathrm{Ran}(Q) = \mathrm{Ran}(Q(I - P))$. *If* $P \vee Q = I$, *then* $\mathrm{Ran}(I - Q) = \mathrm{Ran}((I - Q)P)$.

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Now we can assume that in the double triangle lattice $\mathcal{L}$, the projections P, Q, R have the matrix forms

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & I - V^*HV \end{pmatrix}$$
$$R = \begin{pmatrix} K & \sqrt{K(I-K)}W \\ W^*\sqrt{K(I-K)} & I - W^*KW \end{pmatrix}$$

where H, K are positive contractions and V, W are partial isometries.

Lemma 2.6. *With the notation and assumptions given above, we have that $Q \wedge R = 0$ if and only if* $\mathrm{Ker}(\sqrt{H(I-H)^{-1}}V - \sqrt{K(I-K)^{-1}}W) = 0$*; and that $Q \vee R = I$ if and only if*

$$\mathrm{Ker}(V^*\sqrt{H(I-H)^{-1}} - W^*\sqrt{K(I-K)^{-1}}) = 0.$$

Proof. It is easy to see that $\binom{\xi_1}{\xi_2} \in Q(\mathcal{H})$ if and only if

$$\begin{pmatrix} I - H & -\sqrt{H(I-H)}V \\ -V^*\sqrt{H(I-H)} & V^*HV \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

Since $(I - Q) \wedge (I - P) = 0$, the above is equivalent to

$$\sqrt{I - H}\xi_1 - \sqrt{H}V\xi_2 = 0,$$

which implies that $\xi_1 = \sqrt{H(I-H)^{-1}}V\xi_2$.

Thus

$$\mathrm{Range}(Q) = \left\{\begin{pmatrix} \sqrt{H(I-H)^{-1}}V\xi \\ \xi \end{pmatrix} \,\middle|\, \xi \in \mathcal{D}(\sqrt{H(I-H)^{-1}}V)\right\},$$
$$\mathrm{Range}(R) = \left\{\begin{pmatrix} \sqrt{K(I-K)^{-1}}W\xi \\ \xi \end{pmatrix} \,\middle|\, \xi \in \mathcal{D}(\sqrt{K(I-K)^{-1}}W)\right\}.$$

Hence $Q \wedge R = 0$ if and only if

$$\mathrm{Ker}(\sqrt{H(I-H)^{-1}}V - \sqrt{K(I-K)^{-1}}W) = 0.$$

Note that

$$I - Q = \begin{pmatrix} I - H & -\sqrt{H(I-H)}V \\ -V^*\sqrt{H(I-H)} & V^*HV \end{pmatrix}.$$

Also, we find that $\binom{\xi_1}{\xi_2} \in (I - Q)\mathcal{H}$ if and only if

$$\begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & I - V^*HV \end{pmatrix}\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0.$$

Since $Q \wedge P = 0$, the above is equivalent to

$$V^*\sqrt{H(I-H)}\xi_1 + (I - V^*HV)\xi_2 = 0.$$

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If $V^*V = F$, then $I - V^*HV = (I - F) + V^*(I - H)V$. The above is equivalent to

$$V^*\sqrt{H(I-H)}\xi_1 + (I-F)\xi_2 + V^*(I-H)V\xi_2 = 0.$$

This implies that $(I - F)\xi_2 = 0$ and

$$V^*\sqrt{H(I-H)}\xi_1 + V^*(I-H)V\xi_2 = 0.$$

These two equations imply that $\xi_2 = -V^*\sqrt{H(I-H)^{-1}}\xi_1$. Thus

$$\mathrm{Range}(I-Q) = \left\{ \begin{pmatrix} \xi \\ -V^*\sqrt{H(I-H)^{-1}}\xi \end{pmatrix} \,\middle|\, \xi \in \mathcal{D}(\sqrt{H(I-H)^{-1}}) \right\},$$

$$\mathrm{Range}(I-R) = \left\{ \begin{pmatrix} \xi \\ -W^*\sqrt{K(I-K)^{-1}}\xi \end{pmatrix} \,\middle|\, \xi \in \mathcal{D}(\sqrt{K(I-K)^{-1}}W) \right\}.$$

Therefore we have $Q \vee R = I$ if and only if $(I - Q) \wedge (I - R) = 0$ which in turn is equivalent to $\mathrm{Ker}(V^*\sqrt{H(I-H)^{-1}} - W^*\sqrt{K(I-K)^{-1}}) = 0$. □

The following theorem gives a description of all elements in $\mathrm{Alg}(\mathcal{L})$ in terms of operator equations and will be used in later sections. Similar descriptions were given in [4, 5].

Theorem 2.7. *Suppose that the projections P, Q, R are the above-mentioned operator matrix forms and $\mathcal{L} = \{0, I, P, Q, R\}$ is a double triangle lattice. If $T \in \mathrm{Alg}(\mathcal{L})$, then there exist operators $T_1, T_2, T_3 \in \mathcal{B}(\mathcal{H}_0)$ which satisfy*

$$\begin{aligned} T_1\sqrt{H(I-H)^{-1}}V + T_2 &= \sqrt{H(I-H)^{-1}}VT_3, \\ T_1\sqrt{K(I-K)^{-1}}W + T_2 &= \sqrt{K(I-K)^{-1}}WT_3, \end{aligned}$$

such that $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$.

Conversely, if there are operators T_1, T_2, T_3 satisfying the above system of equations, then the operator T determined by the above operator matrix is in the algebra $\mathrm{Alg}(\mathcal{L})$.

Proof. $P \in \mathrm{Alg}(\mathcal{L})$, thus T must have the operator matrix form $\begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. Furthermore, $(I - Q)TQ = 0$ if and only if

$$\begin{aligned} \sqrt{I-H}(\sqrt{I-H}T_1\sqrt{H} + \sqrt{I-H}T_2V^*\sqrt{I-H} - \sqrt{H}VT_3V^*\sqrt{I-H})\sqrt{H} &= 0, \\ \sqrt{I-H}(\sqrt{I-H}T_1\sqrt{H(I-H)}V + \sqrt{I-H}T_2(I-V^*HV) - \sqrt{H}VT_3(I-V^*HV)) &= 0, \\ V^*\sqrt{H}(\sqrt{I-H}T_1\sqrt{H} + \sqrt{I-H}T_2V^*\sqrt{I-H} - \sqrt{H}VT_3V^*\sqrt{I-H})\sqrt{H} &= 0, \\ V^*\sqrt{H}(\sqrt{I-H}T_1\sqrt{H(I-H)}V + \sqrt{I-H}T_2(I-V^*HV) - \sqrt{H}VT_3(I-V^*HV)) &= 0. \end{aligned}$$

Suppose that $VV^* = E$ and $V^*V = F$. As $\mathrm{Ker}(I - H) = 0$, the second equation in the above system of equations implies that

$$\sqrt{I-H}T_1\sqrt{H(I-H)}V + \sqrt{I-H}T_2(I-V^*HV) - \sqrt{H}VT_3(I-V^*HV) = 0. \tag{2.1}$$

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By right-multiplying both side of Equation (2.1) $I - F$,

$$\sqrt{I-H}T_2(I-F) - \sqrt{H}VT_3(I-F) = 0. \tag{2.2}$$

Subtract (2.2) from (2.1) and

$$\sqrt{I-H}T_1\sqrt{H(I-H)}V + \sqrt{I-H}T_2V^*(I-H)V - \sqrt{H}VT_3V^*(I-H)V = 0. \tag{2.3}$$

As $\mathrm{Ran}(V^*(I-H)V) = F$ and $\mathrm{Ker}(V^*(I-H)V|_F) = 0$, Equation (2.3) implies that

$$(\sqrt{I-H}T_1\sqrt{H(I-H)^{-1}}V + \sqrt{I-H}T_2 - \sqrt{H}VT_3)F = 0. \tag{2.4}$$

Note that $V(I-F) = 0$. By combining Equations (2.2) and (2.4), we see that $(I-Q)TQ = 0$ implies that

$$\sqrt{I-H}T_1\sqrt{H(I-H)^{-1}}V + \sqrt{I-H}T_2 - \sqrt{H}VT_3 = 0.$$

By left-multiplying both sided of the above equation by the unbounded operator $\sqrt{(I-H)^{-1}}$, we obtain that $(I-Q)TQ = 0$ implies that

$$T_1\sqrt{H(I-H)^{-1}}V + T_2 = \sqrt{H(I-H)^{-1}}VT_3.$$

Similarly, $(I-R)TR = 0$ implies that

$$T_1\sqrt{K(I-K)^{-1}}W + T_2 = \sqrt{K(I-K)^{-1}}WT_3.$$

Conversely, when $T_1\sqrt{H(I-H)^{-1}}V + T_2 = \sqrt{H(I-H)^{-1}}VT_3$, by right-multiplying by $V^*\sqrt{I-H}$ and $I-F$ respectively,

$$T_1\sqrt{H} + T_2V^*\sqrt{I-H} = \sqrt{H(I-H)^{-1}}VT_3V^*\sqrt{I-H}$$

which implies that

$$\sqrt{I-H}T_1\sqrt{H} + \sqrt{I-H}T_2V^*\sqrt{I-H} = \sqrt{H}VT_3V^*\sqrt{I-H}$$

and $T_2(I-F) = \sqrt{H(I-H)^{-1}}VT_3(I-F)$. Therefore

$$\sqrt{I-H}T_1\sqrt{H(I-H)}V + \sqrt{I-H}T_2(I-V^*HV) - \sqrt{H}VT_3(I-V^*HV) = 0$$

which implies $(I-Q)TQ = 0$.

Similarly,

$$T_1\sqrt{K(I-K)^{-1}}W + T_2 = \sqrt{K(I-K)^{-1}}WT_3$$

implies that $(I-R)TR = 0$. □

In [5], the authors showed that if

$$\mathcal{D}(\sqrt{H(I-H)^{-1}}) \cap \mathcal{D}(\sqrt{K(I-K)^{-1}}W) \cap \mathcal{D}(W^*\sqrt{K(I-K)^{-1}}) \neq 0,$$

then the algebra $\mathrm{Alg}(\mathcal{L})$ is nontrivial. We can now obtain a similar result under a weaker assumption.

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Lemma 2.8. *With the notation given above and with the assumptions of Theorem 2.7, if we assume further that*

$$\mathcal{D}(\sqrt{H(I-H)^{-1}}V) \cap \mathcal{D}(\sqrt{K(I-K)^{-1}}W) \neq 0$$

and

$$\mathcal{D}(V^*\sqrt{H(I-H)^{-1}}) \cap \mathcal{D}(W^*\sqrt{K(I-K)^{-1}}) \neq 0,$$

then we have $\mathrm{Alg}(\mathcal{L}) \neq \mathbb{C}I$.

Proof. Suppose that

$$\xi \in \mathcal{D}(V^*\sqrt{H(I-H)^{-1}}) \cap \mathcal{D}(W^*\sqrt{K(I-K)^{-1}})$$

and

$$\eta \in \mathcal{D}(\sqrt{H(I-H)^{-1}}V) \cap \mathcal{D}(\sqrt{K(I-K)^{-1}}W).$$

We define two rank-one operators

$$T_1 = \xi \otimes (\sqrt{H(I-H)^{-1}}V - \sqrt{K(I-K)^{-1}}W)\eta$$

and

$$T_3 = (V^*\sqrt{H(I-H)^{-1}} - W^*\sqrt{K(I-K)^{-1}})\xi \otimes \eta.$$

Furthermore, we define the operator T_2 by

$$\begin{aligned} T_2 x = {}& \sqrt{H(I-H)^{-1}}VT_3 x \\ & - \langle x, V^*\sqrt{H(I-H)^{-1}}\xi\rangle(\sqrt{H(I-H)^{-1}}V - \sqrt{K(I-K)^{-1}}W)\eta \end{aligned}$$

for any $x \in \mathcal{H}$. It is easy to check that T_2 is a bounded operator.

By a simple calculation, it follows that

$$\begin{aligned} \sqrt{I-H}T_1\sqrt{H} + \sqrt{I-H}T_2V^*\sqrt{I-H} &= \sqrt{H}VT_3V^*\sqrt{I-H}, \\ \sqrt{I-K}T_1\sqrt{K} + \sqrt{I-K}T_2W^*\sqrt{I-K} &= \sqrt{K}WT_3W^*\sqrt{I-K}. \end{aligned}$$

The above equations imply that

$$\begin{aligned} \sqrt{I-H}T_1\sqrt{H(I-H)^{-1}}VF + \sqrt{I-H}T_2F &= \sqrt{H}VT_3F, \\ \sqrt{I-K}T_1\sqrt{K(I-K)^{-1}}WF' + \sqrt{I-K}T_2F' &= \sqrt{K}WT_3F', \end{aligned}$$

where $F = V^*V$ and $F' = W^*W$.

Furthermore,

$$\begin{aligned} \sqrt{I-H}T_2(I-F) &= \sqrt{H}VT_3(I-F), \\ \sqrt{I-K}T_2(I-F') &= \sqrt{K}WT_3(I-F'). \end{aligned}$$

Now we can define a bounded operator $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$. It follows that $T \in \mathrm{Alg}(\mathcal{L})$. □

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3. Double triangle lattices in $\mathcal{B}(\mathcal{H})$

In [6], the authors studied certain triangle lattices determined by two bounded operators. Here is their construction. Suppose that $\mathcal{K}$ is a Hilbert space and A, B are bounded linear operators acting on $\mathcal{K}$ whose ranges are dense in $\mathcal{K}$ and $\mathrm{Range}(A) \cap \mathrm{Range}(B) = 0$. Let $\mathcal{H} = \mathcal{K} \oplus \mathcal{K}$. They constructed a double triangle lattice $\{0, \mathcal{K} \oplus 0, \mathcal{K} \oplus \mathcal{K}, \mathcal{G}(A), \mathcal{G}(B)\}$ where $\mathcal{G}(A)$ denotes the graph of A. The authors then proved that if a bounded linear operator on $\mathcal{K}$ preserving the ranges of A, B invariant must be a scalar multiple of the identity operator, then the double triangle lattice $\{0, \mathcal{K} \oplus 0, \mathcal{K} \oplus \mathcal{K}, \mathcal{G}(A), \mathcal{G}(B)\}$ is transitive.

Here we can represent the projections in the above lattice in an operator matrix form. In fact, let $H = (I + AA^*)^{-1}$ and $K = (I + BB^*)^{-1}$, and V, W be the unitaries in the polar decompositions of A and B respectively. Let $P(\mathcal{H}) = \mathcal{K} \oplus 0$ and the projections Q and R be the operator matrices represented by H, V and K, W respectively as in Section 2. Then the above double triangle lattice is the same as the lattice $\{0, I, P, Q, R\}$.

In the above-mentioned case, the operators H and K are invertible and V and W are unitaries. As is shown in the previous section, in general, H and K are not invertible and V and W are only partial isometries. In this section, we will show that if one of the operators $I - H$ and $I - K$ is invertible, then the double triangle lattice $\mathcal{L} = \{0, I, P, Q, R\}$ is neither transitive nor reflexive.

Before proving our general result, we first study a special case. Suppose that $\mathcal{H}_0$ is a separable infinite-dimensional Hilbert space and $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0$. Let $P, Q, R \in \mathcal{B}(\mathcal{H})$ be the projections having the following matrix forms with respect to our space decomposition of $\mathcal{H}$:

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix},$$
$$R = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & I - V^*HV \end{pmatrix},$$

where $0 \leqslant H \leqslant I$ and V is a partial isometry. Let $\mathcal{L} = \{0, I, P, Q, R\}$ and suppose that $\mathcal{L}$ is a double triangle lattice.

We define $S = \sqrt{H(I-H)^{-1}}V - I$. Then S is a densely defined closed, in general, unbounded operator. From $Q \wedge R = 0$, $Q \vee R = I$ and Lemma 2.6, we have $\mathrm{Ker}(S^*) = \mathrm{Ker}(V^*\sqrt{H(I-H)^{-1}} - I) = 0$ and $\mathrm{Ker}(S) = \mathrm{Ker}(\sqrt{H(I-H)^{-1}}V - I) = 0$. Hence the ranges of S and S^* are both dense in $\mathcal{H}_0$.

Suppose that $T \in \mathrm{Alg}(\mathcal{L})$ is any given element. Since $P \in \mathcal{L}$, T must be an upper triangle operator matrix. From $Q \in \mathcal{L}$, we have $T = \left(\begin{smallmatrix} T_1 & T_2 \\ 0 & T_3 \end{smallmatrix}\right)$ such that $T_2 = T_3 - T_1$. From Theorem 2.7, we further have $T_1 S - S T_3 = 0$. By using properties of unbounded operators, we state the following result.

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Lemma 3.1. *With the above notation, if $T \in \mathrm{Alg}(\mathcal{L})$, then there is an $A \in \mathcal{B}(\mathcal{H}_0)$ such that $S^{-1}AS$ is bounded and*

$$T = \begin{pmatrix} A & S^{-1}AS - A \\ 0 & S^{-1}AS \end{pmatrix}.$$

Conversely, if $A \in \mathcal{B}(\mathcal{H}_0)$ such that $S^{-1}AS$ is bounded, then the above operator T belongs to $\mathrm{Alg}(\mathcal{L})$.

Remark 3.2. It is easy to see that there are many bounded operators A such that $S^{-1}AS$ is bounded. In fact, for any $\xi \in \mathcal{D}(V^*\sqrt{H(I-H)^{-1}} - I)$ and $\eta \in \mathcal{D}(\sqrt{H(I-H)^{-1}}V - I)$, we define the rank-one operators $T_1 = \xi \otimes S\eta$ and $T_3 = S^*\xi \otimes \eta$ respectively. Then the operator $T = \begin{pmatrix} T_1 & T_3 - T_1 \\ 0 & T_3 \end{pmatrix} \in \mathrm{Alg}(\mathcal{L})$. Thus the algebra $\mathrm{Alg}(\mathcal{L})$ is infinite-dimensional. Furthermore, if $E = VV^*$ and $F = V^*V$, then for any $\xi \in (I-E)(\mathcal{H}_0)$ and $\eta \in (I - F)(\mathcal{H}_0)$, since $S\eta = -\eta$ and $S^*\xi = -\xi$, we have that the operator $\begin{pmatrix} \xi\otimes\eta & 0 \\ 0 & \xi\otimes\eta \end{pmatrix}$ is in the algebra $\mathrm{Alg}(\mathcal{L})$. Hence all the tensor products of the bounded operators from $(I-E)(\mathcal{H}_0)$ into $(I-F)(\mathcal{H}_0)$ with the identity operator I_2 in $\mathcal{M}_2(\mathbb{C})$ are in $\mathrm{Alg}(\mathcal{L})$.

Furthermore, since S is a densely defined closed operator, by an argument similar to that used in [11], there exists a net $\{F_\lambda : \lambda > 0\}$ of projections such that $S^{-1}F_\lambda$, $F_\lambda S$ are bounded and $F_\lambda \to I$ as $\lambda \to 0$ under the strong-operator topology. Thus the $(1, 1)$th entry of the element in the algebra $\mathrm{Alg}(\mathcal{L})$ is dense in $\mathcal{B}(\mathcal{H}_0)$ under the strong-operator topology.

Lemma 3.3. *With the notation given as above, for any $E' \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \setminus \mathcal{L}$, we have $P \wedge E' = 0$ and $P \vee E' = I$.*

Proof. Suppose that $P \wedge E' = \begin{pmatrix} E'_0 & 0 \\ 0 & 0 \end{pmatrix} \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$, where E'_0 is a projection. Then for any $\xi \in \mathcal{D}(S^*)$ and $\eta \in \mathcal{D}(S)$, by using of the operators T_1, T_3 and T defined in the above remark, we have $(I - E'_0)T_1E'_0 = 0$. This means that for any $x \in \mathcal{H}_0$, $\xi \in \mathcal{D}(S^*)$ and $\eta \in \mathcal{D}(S)$,

$$\langle E'_0x, \xi\rangle S\eta = \langle E'_0x, \xi\rangle E'_0S\eta.$$

As the domains of S^* are dense in $\mathcal{H}_0$ and $\mathrm{Ker}(S^*) = 0$, which implies that the image of S is also dense in $\mathcal{H}_0$, we obtain that $E'_0 = 0$ or $E'_0 = I$.

If $E'_0 = I$, then $E' = \begin{pmatrix} I & 0 \\ 0 & E'_1 \end{pmatrix}$. Thus for any $\xi \in \mathcal{D}(S^*)$ and $\eta \in \mathcal{D}(S)$ and $T_3 = S^*\xi \otimes \eta$, we also have $(I - E'_1)T_3E'_1 = 0$. Similarly, we also have $E'_1 = 0$ or $E'_1 = I$. But $E' \notin \mathcal{L}$, and it follows that $E' \wedge P = 0$.

We can similarly show that $P \vee E' = I$. □

The following result is similar to that in [5, 11]. But the arguments used are different. We can also determine the elements of the lattice $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$. But its topological structure is undetermined.

Theorem 3.4. *For any projection $E' \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \setminus \{0, I, P\}$, there are a positive contraction K and a partial isometry U with range projection K in $\mathcal{B}(\mathcal{H}_0)$ such that*

$$E' = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & I - U^*KU \end{pmatrix},$$

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where $\sqrt{K(I-K)^{-1}}$ *(or* K*) and* U *are determined by the polar decomposition of* $aS+I$ *for some* $a \in \mathbb{C}$*.*

Conversely, for any given complex number $a \in \mathbb{C}$*, the polar decomposition of* $I + aS$ *uniquely determines* U *and* K*, which gives rise to a projection* E' *(in the above form) in* $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \setminus \{0, I, P\}$*.*

Proof. Suppose that $E' \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \setminus \{0, I, P\}$. According to the above lemma, we have $E' \wedge P = 0$ and $E' \vee P = I$. Thus according to Lemma 2.2,

$$E' = \begin{pmatrix} K & \sqrt{K(I-K)}U \\ U^*\sqrt{K(I-K)} & I - U^*KU \end{pmatrix}$$

where K is a positive contraction with $\mathrm{Ker}(I-K) = 0$ and U is a partial isometry with the range projection of K or $\sqrt{H(I-H)}$ as final projection.

For any $\xi \in \mathcal{D}(V^*\sqrt{H(I-H)^{-1}} - I)$ and $\eta \in \mathcal{D}(\sqrt{H(I-H)^{-1}}V - I)$, let $T_1 = \xi \otimes S\eta$ and $T_3 = S^*\xi \otimes \eta$ respectively. Then the operator $T = \begin{pmatrix} T_1 & T_3 - T_1 \\ 0 & T_1 \end{pmatrix} \in \mathrm{Alg}(\mathcal{L})$. According to the proof of Theorem 2.7, the equation $(I - E')TE' = 0$ implies that

$$\sqrt{I-K}T_1(\sqrt{K(I-K)}U - (I - U^*KU)) = (\sqrt{K}U - \sqrt{I-K})T_3(I - U^*KU).$$

Then for any $x \in \mathcal{H}_0$,

$$\begin{aligned} &\langle(\sqrt{K(I-K)}U - (I - U^*KU))x, \xi\rangle\sqrt{I-K}S\eta \\ &\quad = \langle(I - U^*KU)x, S^*\xi\rangle(\sqrt{K}U - \sqrt{I-K})\eta. \end{aligned} \tag{3.1}$$

If $(\sqrt{K(I-K)}U - (I - U^*KU))x = 0$ for any $x \in \mathcal{H}_0$, then U must be injective and $\mathrm{Ran}(\sqrt{K(I-K)}) = \mathcal{H}_0$ as $\mathrm{Ker}(I - U^*KU) = 0$. Thus we obtain that U is a unitary and $\sqrt{K(I-K)}U = U^*(I-K)U$, which implies $E' = Q$.

If $E' \neq Q$, then we can choose a vector x_0 such that $(\sqrt{K(I-K)}U - (I - U^*KU))x_0 \neq 0$. Moreover, as $\mathcal{D}(V^*\sqrt{H(I-H)^{-1}} - I)$ is dense, we also can pick a vector $\xi_0 \in \mathcal{D}(V^*\sqrt{H(I-H)^{-1}} - I)$ such that

$$\langle(\sqrt{K(I-K)}U - (I - U^*KU))x_0, \xi_0\rangle \neq 0.$$

Thus Equation (3.1) implies that there is a nonzero constant $a \in \mathbb{C}$ such that

$$a\sqrt{I-K}S\eta = (\sqrt{K}U - \sqrt{I-K})\eta$$

for any $\eta \in \mathcal{D}(\sqrt{H(I-H)^{-1}}V - I)$. Thus

$$\sqrt{K(I-K)^{-1}}U = I + aS.$$

Conversely, if K and U are defined by the above equation as the polar decomposition of the right-hand side, then it is easy to check that $E' \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$. □

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Note that $a = 0$ corresponds to the projection Q and $a = 1$ the projection R.

In the rest part of this section, we will discuss more general double triangle lattices in $\mathcal{B}(\mathcal{H})$. Let A be an invertible bounded operator and P a projection. Let $\overline{P}$ be the range projection of APA^{-1}. Note that

$$\overline{P}(\mathcal{H}) = \mathrm{Range}(APA^{-1}) = \mathrm{Range}(AP) = \{APx : x \in \mathcal{H}\}.$$

For the reader's convenience, we provide the proof of the following two well-known results on the similarity of lattices.

Lemma 3.5. *With the notation given above, suppose that P and Q are two projections. Then*

$$\overline{P} \wedge \overline{Q} = \overline{P \wedge Q}, \quad \overline{P} \vee \overline{Q} = \overline{P \vee Q}.$$

Proof. The first result follows from the following observations:

$$\begin{aligned} \xi \in (\overline{P} \wedge \overline{Q})(\mathcal{H}) &\Leftrightarrow APA^{-1}\xi = \xi = AQA^{-1}\xi \\ &\Leftrightarrow A^{-1}\xi \in P \wedge Q \Leftrightarrow \xi \in \overline{P \wedge Q}. \end{aligned}$$

It is easy to show that $\overline{P} \vee \overline{Q} \leq \overline{P \vee Q}$. Conversely, we have to show that if $\xi \in \overline{P \vee Q}$, then $\xi \in \overline{P} \vee \overline{Q}$. Assume that this is not true. Then there is a vector $\xi \in \overline{P \vee Q}(\mathcal{H})$ but $\xi \perp (\overline{P} \vee \overline{Q})(\mathcal{H})$. Thus there is a vector $\xi_0 \in (P \vee Q)(\mathcal{H})$ such that for any $x \in P(\mathcal{H})$ and any $y \in Q(\mathcal{H})$,

$$\langle A\xi_0, Ax\rangle = 0, \quad \langle A\xi_0, Ay\rangle = 0.$$

This implies that for any $x \in P(\mathcal{H})$ and any $y \in Q(\mathcal{H})$ we have $\langle A^*A\xi_0, x\rangle = 0$ and $\langle A^*A\xi_0, y\rangle = 0$. Thus $A^*A\xi_0 \perp (P + Q)(\mathcal{H})$. But $\mathrm{Ran}(P + Q) = P \vee Q$. Thus $A^*A\xi_0 \perp (P \vee Q)(\mathcal{H})$ and $A^*A\xi_0 = 0$. It follows that $\xi_0 = 0$. □

It follows that if $\{0, I, P, Q, R\}$ is a double triangle lattice, then so is $\{0, I, \overline{P}, \overline{Q}, \overline{R}\}$.

Remark 3.6. In general, we do not have $I - \overline{P} = \overline{I - P}$. For example, let $P = \left(\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right)$ and $A = \left(\begin{smallmatrix} I & -I \\ 0 & I \end{smallmatrix}\right)$, then $\overline{P} = \mathrm{Ran}(APA^{-1}) = P$. But

$$A(I - P)A^{-1} = \begin{pmatrix} 0 & -I \\ 0 & I \end{pmatrix} \quad \text{and} \quad \overline{I - P} = \mathrm{Ran}(A(I - P)A^{-1}) = \frac{1}{2}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix}.$$

Note also that $I - \overline{P} = \left(\begin{smallmatrix} 0 & 0 \\ 0 & I \end{smallmatrix}\right)$.

However we can prove the following result.

Lemma 3.7. *Suppose $\mathcal{L}$ is a lattice and A is an invertible operator. For any $P \in \mathcal{L}$, we define $\overline{P} = \mathrm{Ran}(APA^{-1})$ and $\overline{\mathcal{L}} = \{\overline{P} : P \in \mathcal{L}\}$. Then we have* $\mathrm{Alg}(\overline{\mathcal{L}}) = A(\mathrm{Alg}(\mathcal{L}))A^{-1}$ *and* $\mathrm{Lat}(\mathrm{Alg}(\overline{\mathcal{L}})) = \{\overline{P} : P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))\}$.

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PROOF. First, from

$$\begin{aligned} T \in \mathrm{Alg}(\mathcal{L}) &\Leftrightarrow \forall P \in \mathcal{L}, PTP = TP \\ &\Leftrightarrow \forall P \in \mathcal{L}, APA^{-1} \cdot ATA^{-1} \cdot AP = ATA^{-1} \cdot AP \\ &\Leftrightarrow ATA^{-1} \in \mathrm{Alg}(\overline{\mathcal{L}}), \end{aligned}$$

we have $A(\mathrm{Alg}(\mathcal{L}))A^{-1} = \mathrm{Alg}(\overline{\mathcal{L}})$.

Secondly, according to the previous result and

$$\begin{aligned} Q \in \mathrm{Lat}(\mathrm{Alg}(\overline{\mathcal{L}})) &\Leftrightarrow \forall T \in \mathrm{Alg}(\mathcal{L}), QATA^{-1}Q = ATA^{-1}Q \\ &\Leftrightarrow \forall T \in \mathrm{Alg}(\mathcal{L}), A^{-1}QATA^{-1}QA = TA^{-1}QA \\ &\Leftrightarrow \mathrm{Ran}(A^{-1}QA) = \mathrm{Ran}(A^{-1}Q) \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L})), \end{aligned}$$

we have $\mathrm{Lat}(\mathrm{Alg}(\overline{\mathcal{L}})) = \{\overline{P} : P \in \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))\}$. □

Now suppose that the projections $P, Q, R \in \mathcal{B}(\mathcal{H})$ have the matrix forms

$$\begin{aligned} P &= \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad Q = \begin{pmatrix} H & \sqrt{H(I-H)}V \\ V^*\sqrt{H(I-H)} & I - V^*HV \end{pmatrix}, \\ R &= \begin{pmatrix} K & \sqrt{K(I-K)}W \\ W^*\sqrt{K(I-K)} & W^*(I-K)W \end{pmatrix}, \end{aligned}$$

where H, K are positive contractions and V, W are partial isometries with the range projections of H and K as their final projections, respectively. Then we have the following conclusion.

THEOREM 3.8. *With the notation given above, let $\mathcal{L} = \{0, I, P, Q, R\}$. Suppose that $\mathcal{L}$ is a double triangle lattice and one of the operators $I - H$ and $I - K$ is invertible. Then $\mathcal{L}$ is neither reflexive nor transitive.*

PROOF. Without loss of generality, we assume that $I - H$ is invertible. Then $\sqrt{I-H}$ is also invertible.

Let $A = \begin{pmatrix} I & I - \sqrt{H(I-H)^{-1}}V \\ 0 & I \end{pmatrix}$. Then $A^{-1} = \begin{pmatrix} I & \sqrt{H(I-H)^{-1}}V - I \\ 0 & I \end{pmatrix}$. It is easy to check that $\overline{P} = \mathrm{Ran}(APA^{-1}) = P$. Furthermore, as

$$AQ = \begin{pmatrix} V^*\sqrt{H(I-H)} & I - V^*HV \\ V^*\sqrt{H(I-H)} & I - V^*HV \end{pmatrix}$$

and

$$\{V^*\sqrt{H(I-H)}x + (I - V^*HV)y : x, y \in \mathcal{H}_0\} = \mathcal{H}_0,$$

we have

$$(AQA^{-1})(\mathcal{H}) = \left\{ \begin{pmatrix} x \\ x \end{pmatrix} : x \in \mathcal{H}_0 \right\}$$

which implies that $\overline{Q} = \frac{1}{2}\begin{pmatrix} I & I \\ I & I \end{pmatrix}$.

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Let $\overline{\mathcal{L}} = \{0, I, \overline{P}, \overline{Q}, \overline{R}\}$. According to Lemma 3.5, $\overline{\mathcal{L}}$ is also a double triangle lattice. According to Remark 3.2, we know that $\mathrm{Alg}(\overline{\mathcal{L}})$ is infinite-dimensional. According to Theorem 3.4, $\overline{\mathcal{L}} \subsetneq \mathrm{Lat}(\mathrm{Alg}(\overline{\mathcal{L}}))$. Finally, according to Lemma 3.7, we know that $\mathrm{Alg}(\mathcal{L})$ is infinite-dimensional and $\mathcal{L} \subsetneq \mathrm{Lat}(\mathrm{Alg}(\mathcal{L}))$. □

When P, Q, R are projections in a finite factor $\mathcal{M}$, suppose that $\mathcal{L} = \{0, I, P, Q, R\}$ is a double triangle lattice. Ge and Yuan [5] and Hou and Yuan [11] have shown that for any three distinct projections E_i $(i = 1, 2, 3)$ in $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \setminus \{0, I\}$, $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) = \mathrm{Lat}(\mathrm{Alg}(\{E_1, E_2, E_3\}))$ and $\mathrm{Alg}(\mathcal{L}) = \mathrm{Alg}(\{E_1, E_2, E_3\})$. This implies that $\mathrm{Alg}(\mathcal{L})$ is a KS algebra in the sense of Ge and Yuan [4, 5]. Furthermore, they have shown that $\mathrm{Lat}(\mathrm{Alg}(\mathcal{L})) \setminus \{0, I\}$ is homeomorphic to the sphere $\mathbb{S}^2$ under the strong-operator topology.

In the proof of these results, there are two crucial points: first, is that the domains of the unbounded operators concerned contain a common dense subspace; and secondly, that there is an increasing net $\{E_\lambda : \lambda > 0\}$ of projections with strong-operator limit I as $\lambda \to 0$ such that the left or right multiplications of those unbounded operators with E_λ are bounded. Thus they can use the operations of bounded operators.

As for the first point, the unbounded operators concerned are affiliated with the finite factor and the domains of these operators contain a common dense subspace such that these unbounded operators form an algebra. However, this is not true in a type I_∞ von Neumann algebra or $\mathcal{B}(\mathcal{H})$. Von Neumann [16] asserts that for a given unbounded self-adjoint operator T there always exists a unitary U such that $\mathcal{D}(T) \cap \mathcal{D}(U^*TU) = 0$.

As for the second point, suppose that $\mathcal{M}$ is a factor of type II_1 with the unique trace τ and T, S are two closed densely defined unbounded operators affiliated with $\mathcal{M}$. Let $T = HU$ and $S = KV$ be the polar decompositions of T and S respectively. Note that H and K are positive unbounded operators affiliated with $\mathcal{M}$. Thus the spectral projections of H and K are contained in $\mathcal{M}$. For any $\varepsilon > 0$, we can choose the spectral projections E_ε of H and F_ε of K such that $E_\varepsilon H$ and $F_\varepsilon K$ are bounded and $\tau(E_\varepsilon) > 1 - (\varepsilon/2)$ and $\tau(F_\varepsilon) > 1 - (\varepsilon/2)$. We define $P_\varepsilon = E_\varepsilon \wedge F_\varepsilon$. Then $P_\varepsilon T$ and $P_\varepsilon S$ are bounded and

$$\tau(P_\varepsilon) = \tau(E_\varepsilon) + \tau(F_\varepsilon) - \tau(E_\varepsilon \vee F_\varepsilon) > 1 - \varepsilon.$$

Then the net $\{P_\varepsilon\}$ of projections gives what they need. Using this fact, it is not hard to deduce the following lemma, and we leave the proof as an exercise.

Lemma 3.9. *Suppose that A and B are closed densely defined invertible (perhaps with unbounded inverse) operators affiliated with the II_1 factor $\mathcal{M}$. Then $\mathcal{A}_A \cap \mathcal{A}_B$ is weak-operator dense in $\mathcal{B}(\mathcal{H})$.*

4. Transitivity of small lattices

Longstaff [15] has reduced the transitivity of lattices with three nontrivial projections to the case of double triangle lattices. By using completely different methods, we shall prove this result again in this section.

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Suppose that $\mathcal{H}$ is a separable Hilbert space and $P, Q, R \in \mathcal{B}(\mathcal{H})$ are three projections. Assume that $\mathcal{L} = \{0, I, P, Q, R\}$. Let $\mathcal{L}^{\perp} = \{0, I, I - P, I - Q, I - R\}$. We have the following simple facts.

Lemma 4.1. $\mathrm{Alg}(\mathcal{L}) = \mathbb{C}I$ *if and only if* $\mathrm{Alg}(\mathcal{L}^{\perp}) = \mathbb{C}I$.

This is an easy corollary of the observation that

$$T \in \mathrm{Alg}(\mathcal{L}) \Leftrightarrow T^* \in \mathrm{Alg}(\mathcal{L}^{\perp}).$$

Lemma 4.2. *Suppose that* $\mathrm{Alg}(\mathcal{L}) = \mathbb{C}I$ *and* $E = P \wedge Q \neq 0$. *Let* $R = \left(\begin{smallmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{smallmatrix}\right)$ *be the operator matrix form of R with respect to the orthogonal decomposition* $\mathcal{H} = E(\mathcal{H}) \oplus E^{\perp}(\mathcal{H})$. *Then it follows that* $\mathrm{Ker}(I - R_{11}) = 0$ *and* $\mathrm{Ker}(R_{11}) = 0$.

Proof. Assume that $\alpha \in \mathrm{Ker}(I - R_{11})$ is a nonzero vector. Let $\xi := \left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right)$ and $Tx := \langle x, \xi\rangle \xi$ for any $x \in \mathcal{H}$. Then T is a rank-one operator and $T \in \mathrm{Alg}(\mathcal{L})$, which is a contradiction.

If $0 \neq \alpha \in \mathrm{Ker}(R_{11})$, then we similarly define the vector ξ and the operator T as above. As $P\xi = \xi$ and $Q\xi = \xi$, we have

$$TP = PTP, \quad TQ = QTQ.$$

Furthermore, as

$$R\xi = \begin{pmatrix} 0 \\ R_{21}\alpha \end{pmatrix} = R^2\xi = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix} \begin{pmatrix} 0 \\ R_{21}\alpha \end{pmatrix} = \begin{pmatrix} R_{12}R_{21}\alpha \\ R_{22}R_{21}\alpha \end{pmatrix}$$

and $R_{12} = R_{21}^*$, we have $R_{21}\alpha = 0$. This implies that $TR = 0 = RTR$ and $T \in \mathrm{Alg}(\mathcal{L})$. □

Lemma 4.3. *With the notation given above, suppose that* $\mathcal{L}$ *is transitive and* $P \wedge Q = E \neq 0$. *Let* $R_{12} = HV$ *be the polar decomposition of* R_{12} *where* H *is positive and* V *a partial isometry. Then* $\mathrm{Ker}(H) = 0$ *and* $\mathrm{Ran}(V) = E$.

Proof. We just need to show that $\mathrm{Ker}(H) = 0$. As $R_{12}^* = R_{21}$ and $H^2 = R_{12}R_{21}$, we have that $\mathrm{Ker}(H) = 0$ if and only if $\mathrm{Ker}(R_{21}) = 0$. Suppose that $\alpha \in \mathrm{Ker}(R_{21})$ is a nonzero vector. Let $\xi = \left(\begin{smallmatrix} \alpha \\ 0 \end{smallmatrix}\right)$ and T be the rank-one operator $x \mapsto \langle x, \xi\rangle\xi$. Since $P\xi = Q\xi = \xi$, we have $PTP = TP$ and $QTQ = TQ$.

Furthermore, as

$$R\xi = \begin{pmatrix} R_{11}\alpha \\ 0 \end{pmatrix} \quad \text{and} \quad R^2\xi = \begin{pmatrix} R_{11}^2\alpha \\ R_{21}R_{11}\alpha \end{pmatrix},$$

we have $R_{11}\alpha = R_{11}^2\alpha$. But $\mathrm{Ker}(I - R_{11}) = 0$, thus we have $R_{11}\alpha = 0$ and $R\xi = 0$. Thus implies that $RTR = TR$ and $T \in \mathrm{Alg}(\mathcal{L})$, which is in contradiction to $\mathrm{Alg}(\mathcal{L}) = \mathbb{C}I$. □

By using the properties of two projections (see, for example, [12] or [17]), we have the following corollary.

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Corollary 4.4. *With the notation given above,*

$$R = \begin{pmatrix} R_{11} & \sqrt{R_{11}(I-R_{11})}V \\ V^*\sqrt{R_{11}(I-R_{11})} & V^*(I-R_{11})V+F \end{pmatrix}$$

where F is a projection such that $F \perp V^*V$.

Now we prove the main result of the section. Let $P, Q, R \in \mathcal{B}(\mathcal{H})$ be projections and $\mathcal{L} = \{0, I, P, Q, R\}$.

Theorem 4.5. *If $\mathrm{Alg}(\mathcal{L}) = \mathbb{C}I$, then the intersection and union of any two nontrivial distinct projections in $\mathcal{L}$ are zero and I respectively.*

Proof. By Lemma 4.1, we only need to show that $P \wedge Q = 0$.

Assume to the contrary that $P \wedge Q \neq 0$. According to the above lemmas, the projections P, Q, R must be the operator matrix forms

$$P = \begin{pmatrix} I & 0 \\ 0 & P_0 \end{pmatrix}, \quad Q = \begin{pmatrix} I & 0 \\ 0 & Q_0 \end{pmatrix},$$
$$R = \begin{pmatrix} R_{11} & \sqrt{R_{11}(I-R_{11})}V \\ V^*\sqrt{R_{11}(I-R_{11})} & V^*(I-R_{11})V+F \end{pmatrix},$$

where P_0 and Q_0 are projections.

As R_{11} is positive and $\mathrm{Ker}(R_{11}) = \mathrm{Ker}(I - R_{11}) = 0$, there exists a spectral projection E_0 of R_{11} such that both E_0R_{11} and $E_0(I - R_{11})$ are invertible on the subspace $E_0(\mathcal{H})$. Let $T = E_0$ and $A = \begin{pmatrix} T & -T\sqrt{R_{11}(T(I-R_{11}))^{-1}}V \\ 0 & 0 \end{pmatrix}$ where $(T(I - R_{11}))^{-1}$ is defined as the operator which is $(T(I - R_{11}))^{-1}$ on the range of E_0 and zero on the range of $I - E_0$.

It is a simple calculation to check that $(I - P)AP = 0$ and $(I - Q)AQ = 0$. Furthermore,

$$\begin{aligned}
(I-R)AR &= \begin{pmatrix} I-R_{11} & -\sqrt{R_{11}(I-R_{11})}V \\ -V^*\sqrt{R_{11}(I-R_{11})} & I-V^*(I-R_{11})V-F \end{pmatrix} AR \\
&= \begin{pmatrix} (I-R_{11})T & -(I-R_{11})T\sqrt{R_{11}(T(I-R_{11}))^{-1}}V \\ -V^*\sqrt{R_{11}(I-R_{11})}T & V^*\sqrt{R_{11}(I-R_{11})}T\sqrt{R_{11}(T(I-R_{11}))^{-1}}V \end{pmatrix} R \\
&= \begin{pmatrix} (I-R_{11})T & -T\sqrt{R_{11}(I-R_{11})}V \\ -V^*\sqrt{R_{11}(I-R_{11})}T & V^*TR_{11}V \end{pmatrix} \\
&\quad \times \begin{pmatrix} R_{11} & \sqrt{R_{11}(I-R_{11})}V \\ V^*\sqrt{R_{11}(I-R_{11})} & V^*(I-R_{11})V+F \end{pmatrix} = 0.
\end{aligned}$$

Hence $T \in \mathrm{Alg}(\mathcal{L}) \setminus \mathbb{C}I$, which is a contradiction. □

According to the above theorem, we can draw the following consequence.

Corollary 4.6. *Every pentagon lattice is not transitive.*

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