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GENERALIZED PRÜFER ANGLE AND VARIATIONAL METHODS FOR p-LAPLACIAN EIGENVALUE PROBLEMS ON THE HALF-LINE

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Abstract The nonlinear eigenvalue problem

$$-(|y'(x)|^{p-1}\operatorname{sgn} y'(x))' = (p-1)(\lambda - q(x))|y(x)|^{p-1}\operatorname{sgn} y(x)$$

for $0 \leqslant x < \infty$, fixed $p \in (1,\infty)$, and with $y'(0)/y(0)$ specified, is studied under conditions on q related to those of Brinck and Molčanov. Topics include Sturmian results, connections between problems on finite intervals and the half-line, and variational principles.

Keywords: Prüfer angle; p-Laplacian; nonlinear variational principles

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1. Introduction

We shall study the nonlinear eigenvalue problem

$$-(|y'(x)|^{p-1}\operatorname{sgn} y'(x))' = (p-1)(\lambda - q(x))|y(x)|^{p-1}\operatorname{sgn} y(x), \tag{1.1}$$

$0 \leqslant x < \infty$, for fixed $p \in (1,\infty)$, with λ a real parameter and with initial condition

$$\left(\frac{y'}{y}\right)(0) = \cot_p(\alpha), \quad \alpha \in [0,\pi_p). \tag{1.2}$$

The differential equation (1.1) is to be understood in the Carathéodory sense. Here $\cot_p = \sin_p' / \sin_p$, where $\sin_p$ is the generalized sine function with first positive zero at π_p. We remark that the notation $\sin_p$ has been used in different ways, and we refer the reader to [**2**, **8**, **11**] for properties of the functions used here. In particular,

$$|\sin_p \theta|^p + |\sin_p' \theta|^p = 1. \tag{1.3}$$

Our notation $\cot_p$ is non-standard, but we note that this function maps $[0,\pi_p)$ in a one-to-one fashion onto the extended reals. In particular, (1.2) is taken as the Dirichlet condition $y(0)=0$ if $\alpha=0$.

The potential function q is to be real valued and locally integrable, i.e. $q \in L_1(0,b)$ for any finite $b>0$, and we write

$$q = q^+ - q^-, \quad q^+ = \max(q,0).$$

We impose two further conditions on q, namely,

$$\text{there exists a constant } C>0 \text{ such that } \int_x^{x+1} q^- < C \text{ for all } x \geqslant 0 \tag{B0}$$

and

$$\text{for every } \varepsilon>0, \quad \lim_{x\to\infty} \int_x^{x+\varepsilon} q^+ = \infty. \tag{M1}$$

Condition (M1) (with q in place of q^+) was used by Molčanov for his seminal work [**12**] on spectral discreteness in the case $p=2$, when (1.1) is a Sturm–Liouville equation. Molčanov also assumed q to be bounded below. This was subsequently relaxed by Brinck [**5**] to (B0) but with q in place of q^-. Our combined conditions lie between those of Molčanov and Brinck. As an example, one could take

$$q(x) = \begin{cases} -x & \text{if } n \leqslant x < n+n^{-1},\ n=1,2,\dots, \\ x & \text{otherwise.} \end{cases}$$

Minor amendments would produce examples violating any prescribed functional lower or upper bounds.

Two methods of attack that have proved useful for equations of the form (1.1) on compact intervals use a generalized Prüfer angle and a generalized minimax principle, respectively. A generalized Prüfer method (involving the $\sin_p$ function used here) was described by Elbert [**8**], and there have been several subsequent investigations along these lines (see [**2**] for a review). On a half-line, modified Prüfer methods have been used in [**4**, **6**] and our results generalize theirs as follows. In [**4**], where $p=2$, q is assumed to be bounded below, while in the relevant part of [**6**] q has to satisfy certain smoothness and growth conditions. Although our overall strategy in §§ 2 and 3 is along the lines of these references, their methods would need considerable modification to deal with the assumptions here. Moreover, neither [**4**] nor [**6**] address our later topics, which include variational principles.

Historically, (Lyusternik–Schnirelman) variational methods, based on the Lagrange multiplier approach, precede those of Prüfer type, and we refer the reader to [**9**, **10**, **13**] for expositions and references. Despite the significant history of work in this area, there remain well-known open problems, and it is not obvious *a priori* whether a minimax principle will hold in our situation. The delicacy of the situation is already apparent on a compact interval, where all eigenvalues can indeed be characterized variationally as above for separated boundary conditions (cf. [**2**]), but this can fail for periodic boundary conditions [**3**].

As we shall see, the Prüfer angle method on a half-line leads to eigenfunctions that decay exponentially, and belong to L_r for any $r \geqslant 1$. In order to discuss our variational principles, however, we shall need to control eigenfunction derivatives as well, and accordingly we shall give the following definition.

Definition 1.1. An eigenvalue for the problem (1.1), (1.2) is a value of $\lambda \in \mathbb{R}$ for which there exists a $y \in W_p^1$ satisfying the differential equation (1.1) and the initial condition (1.2).

Here and below L_p and W_p^1 will refer to the half-line $(0, \infty)$.

Different definitions of 'eigenvalue' have appeared in the literature. For example, Brown and Reichel [**6**] merely require the solution y in Definition 1.1 to belong to L_p, while Drábek and Kufner [**7**] replace the system (1.1), (1.2) by its weak (variational) version, requiring solutions y to be in W_p^1 and, in addition, to satisfy

$$\lim_{x \to \infty} y(x) = 0. \tag{1.4}$$

They also define a strong solution with extra smoothness conditions guaranteeing (1.1), (1.2) as well as (1.4). In § 4 we shall show that, in our setting, the sets of eigenvalues coincide for all four of these definitions. Additional remarks are given after Theorem 4.1.

In § 2 we introduce the generalized Prüfer angle generated by (1.1), (1.2) and the first-order equation it satisfies. Section 3 contains a discussion of sets forming a partition of the real line and which are connected with limiting properties of the Prüfer angle. This sets the stage for § 4, where the eigenvalues are identified and oscillation and Sturmian results for the associated eigenfunctions are presented. Section 5 contains results relating problems on finite intervals to the half-line case, and variational approaches are also discussed. Our variational principle is established by a limiting argument, and we show by example that the Lagrange multiplier method cannot in general be used for potentials satisfying our conditions.

2. Generalized Prüfer angles

For a solution y of (1.1), (1.2) we shall introduce the generalized Prüfer (Elbert) angle θ via

$$y(\lambda, x) = \rho(\lambda, x) \sin_p \theta(\lambda, x), \qquad y'(\lambda, x) = \rho(\lambda, x) \sin_p' \theta(\lambda, x).$$

This leads to

$$\left(\frac{y'}{y}\right)(\lambda, x) = \cot_p \theta(\lambda, x),$$

and then we can use (1.1) and (1.3) to derive

$$\begin{aligned} \theta'(\lambda, x) &= |\sin_p' \theta(\lambda, x)|^p + (\lambda - q(x))|\sin_p \theta(\lambda, x)|^p \\ &= 1 - (q(x) - \lambda + 1)|\sin_p \theta(\lambda, x)|^p, \end{aligned} \tag{2.1}$$

$$\theta(\lambda, 0) = \alpha. \tag{2.2}$$

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Given α, λ, it can be seen from [**2**, § 2] that the solution y of (1.1), (1.2) can be recovered via the ρ, θ system, up to scalar multiples, from the unique solution of (2.1), (2.2).

We shall now consider the first-order equation

$$u'(x) = 1 - g(x)|\sin_p u(x)|^p, \tag{2.3}$$

where $g = q - \lambda + 1$ and q satisfies (B0) and (M1), which we take in the form

$$\text{given } A > 0,\ \varepsilon > 0, \quad \exists X_{A,\varepsilon} : x > X_{A,\varepsilon} \quad \Longrightarrow \quad \int_x^{x+\varepsilon} q^+ > A. \tag{2.4}$$

As for q, we write $g = g^+ - g^-$ and we note that g also satisfies (B0), (M1). The constant $C = C(\lambda)$ from (B0) is now λ-dependent, but we shall continue to write C.

The results of this section have analogues in [**4**], but the proofs are modified to allow the weaker assumption (B0).

Lemma 2.1. *Given $0 < \gamma < \delta < \pi_p$ and $\eta > 0$ there exists $X_{\gamma,\delta,\eta}$ such that for any solution of (2.3) the conditions*

$$x > X_{\gamma,\delta,\eta}, \quad u(x) \in (\gamma, \delta) \quad \text{and} \quad u(y) \leqslant \delta \text{ for all } y \in [x, x+\eta] \tag{2.5}$$

imply that there exists $\varepsilon \in (0, \eta)$ satisfying

$$u(x + \varepsilon) = \gamma. \tag{2.6}$$

Proof. Let $B = \min\{|\sin_p u|^p : u \in [\gamma, \delta]\}$. Then $0 < B \leqslant 1$. By virtue of (2.4) we select $X_{\gamma,\delta,\eta}$ so that

$$x > X_{\gamma,\delta,\eta} \quad \Longrightarrow \quad \int_x^{x+\eta} g^+ > \frac{\delta - \gamma + \eta + ([\eta] + 1)C}{B}. \tag{2.7}$$

Suppose then that $x > X_{\gamma,\delta,\eta}$ satisfies (2.5) but that no $\varepsilon \in (0, \eta)$ can be found to satisfy (2.6). Then $u(y) \in (\gamma, \delta]$ for all $y \in [x, x + \eta]$ and we have

$$\begin{aligned} u(x+\eta) &= u(x) + \int_x^{x+\eta} (1 - (g^+ - g^-)|\sin_p u|^p) \\ &\leqslant \delta + \eta - B \int_x^{x+\eta} g^+ + ([\eta] + 1)C \\ &< \gamma \end{aligned}$$

by (2.7). This is a contradiction. □

Lemma 2.2. *Given $0 < \gamma < \delta < \pi_p$ such that*

$$\delta - \gamma - Cm > 0, \tag{2.8}$$

where $m = \max\{|\sin_p u|^p : \gamma \leqslant u \leqslant \delta\}$, there exists a $Y_{\gamma,\delta}$ such that, for any solution of (2.3),

$$x > Y_{\gamma,\delta}, \quad u(x) \leqslant \gamma \quad \Longrightarrow \quad u(x + t) < \delta \text{ for all } t > 0.$$

Proof. Set

$$\eta = \frac{\delta - \gamma - Cm}{1 + Cm}$$

and take $Y_{\gamma,\delta} = X_{\gamma,\delta,\eta}$ from Lemma 2.1. Suppose that $x > Y_{\gamma,\delta}$ has $u(x) \leqslant \gamma$ and that $z > x$ has $u(z) = \delta$. We can assume that z is the minimum of all points $r > x$, where $u(r) = z$. Select $y \in [x, z]$ so that $u(y) = \gamma$ and $u(w) \in (\gamma, \delta]$ for all $w \in (y, z]$. Now

$$\begin{aligned}
\delta - \gamma &= u(z) - u(y) \\
&= \int_y^z (1 - g|\sin_p u|^p) \\
&\leqslant z - y + m \int_y^z g^- \\
&< z - y + m\{[z - y] + 1\}C \\
&\leqslant (z - y)(1 + Cm) + Cm,
\end{aligned}$$

so $z - y > \eta$. Thus, we can apply Lemma 2.1 over the interval $(z - \eta, z)$ to obtain a point $w \in (z - \eta, z)$, where $u(w) = \gamma$, giving a contradiction. □

The next result parallels Corollary 2.3 of [**4**] but extra care must be taken with the proof in the light of condition (2.8).

Lemma 2.3. *Let u be a solution of (2.3). If $\liminf_{x\to\infty} u(x) < \pi_p$, then*

$$\limsup_{x\to\infty} u(x) \leqslant 0.$$

Proof. By assumption there exists a small enough $\varepsilon > 0$ with $\pi_p - \varepsilon \in (\pi_p/2, \pi_p)$ and $x_n \to \infty$ with $u(x_n) \leqslant \pi_p - 2\varepsilon$ for each n such that (2.8) is satisfied with $\gamma = \pi_p - 2\varepsilon, \delta = \pi_p - \varepsilon$. Suppose $\limsup_{x\to\infty} u(x) > 0$, so there is small $\eta > 0$ and $y_n \to \infty$ satisfying

$$u(y_n) > 2\eta \tag{2.9}$$

for each n, such that (2.8) holds with $\gamma = \eta$, $\delta = 2\eta$. By Lemma 2.2 (with $\gamma = \pi_p - 2\varepsilon$ and $\delta = \pi_p - \varepsilon$) there exists N_1 such that $u(x) < \pi_p - \varepsilon$ for all $x > x_n$ with $n > N_1$. Furthermore, by Lemma 2.2 with $\gamma = \eta$, $\delta = \pi_p - \varepsilon$ and $\eta = 1$, say, there exist N_2 and $z_n > y_n$ such that $u(z_n) = \eta$ for all $n > N_2$. Now applying Lemma 2.2 again (with $\gamma = \eta$, $\delta = 2\eta$), we can find N_3 such that $u(x) < 2\eta$ for all $x > z_n$ and $n > N_3$, contradicting (2.9). □

Lemma 2.4. *With q as above, suppose that u_μ satisfies $u_\mu' = 1 - (q - \mu + 1)|\sin_p u_\mu|^p$ on $[0, \infty)$ with $u_\mu(a)$ continuous in $\mu \geqslant \lambda$ for some fixed $a \geqslant 0$ and $\lambda \in \mathbb{R}$. If $u_\lambda(x) \to 0$ as $x \to \infty$, then there exists $\nu > \lambda$ such that*

$$\lambda < \mu < \nu \quad \Longrightarrow \quad \limsup_{x\to\infty} u_\mu(x) < \pi_p.$$

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Proof. We first note that $q-\mu+1$ satisfies (B0), with the constant C being chosen independently of μ, provided that $\mu-\lambda$ is sufficiently small and, in a similar fashion, $q-\mu+1$ satisfies (M1) with the quantities $X_{\gamma,\delta,\eta}$, $Y_{\gamma,\delta}$ also being μ-independent for sufficiently small $\mu-\lambda$. The proof now follows that of [**4**, Lemma 2.4]. □

The following property of θ is central for much of the next section.

Lemma 2.5. *For a given $\lambda\in R$, there exists an integer $n=n(\lambda)\geqslant 0$ such that*

$$n\pi_p<\theta(\lambda,x)<(n+1)\pi_p \text{ for all } x \text{ sufficiently large.}$$

Proof. Suppose that $\theta(\lambda,x)\to\infty$ as $x\to\infty$. Let $0<\gamma<\delta<\pi_p$ be such that $\delta-\gamma-Cm>0$, where $m=\max\{|\sin_p u|^p:\gamma\leqslant u\leqslant\delta\}$. Note that this can be achieved by taking $\gamma=\delta/2$ close enough to 0. Set

$$x_n=\min\{x:\theta(\lambda,x)=n\pi_p+\gamma\},\quad n\geqslant 1.$$

Since

$$\begin{aligned}n\pi_p+\gamma-\alpha&=\theta(\lambda,x_n)-\theta(\lambda,0)\\&=\int_0^{x_n}(1-(q(t)-\lambda+1)|\sin_p\theta(\lambda,t)|^p)\,\mathrm{d}t\\&\leqslant x_n(2+|\lambda|)+C([x_n]+1),\end{aligned}$$

we see that $x_n\to\infty$. Now use Lemma 2.2 to find $Y_{\gamma,\delta}$ and fix N so that $x_N>Y_{\gamma,\delta}$. Note that $u(x)=\theta(\lambda,x)-N\pi_p$ satisfies the differential equation (2.3) and also that $u(x_N)=\gamma$. Lemma 2.2 then shows that $u(x)<\delta$ for all $x>x_N$, giving a contradiction. Since, from [**2**, Lemma 2.3],

$$\theta \text{ increases through integer multiples of } \pi_p, \tag{2.10}$$

the result is established. □

3. The sets Λ_n

From now on we shall adopt the following notation for $t\in\mathbb{R}$:

$$[t]^{p-1}=|t|^{p-1}\operatorname{sgn}t=|t|^{p-2}t,$$

so (1.1) will be written in the form

$$-([y']^{p-1})'=(p-1)(\lambda-q)[y]^{p-1}. \tag{3.1}$$

For each integer $n\geqslant 0$ we define

$$\begin{aligned}\Lambda_n&=\{\lambda\in\mathbb{R}:n\pi_p<\theta(\lambda,x)<(n+1)\pi_p \text{ for all } x \text{ sufficiently large}\},\\\Lambda_n^+&=\{\lambda\in\Lambda_n:\theta(\lambda,x)\to(n+1)\pi_p \text{ as } x\to\infty\},\\\Lambda_n^-&=\{\lambda\in\Lambda_n:\theta(\lambda,x)\to n\pi_p \text{ as } x\to\infty\}.\end{aligned}$$

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By Lemma 2.5, the sets Λ_n form a partition of $\mathbb{R}$ and, in particular, if $\lambda \in \Lambda_n \setminus \Lambda_n^+$, then by Lemma 2.3 and (2.10) we have $\lambda \in \Lambda_n^-$, so

$$\Lambda_n = \Lambda_n^- \cup \Lambda_n^+.$$

Lemma 3.1. *Suppose $\lambda \in \Lambda_n^+$ and that y satisfies (1.1), (1.2). Then y' is bounded on $[0,\infty)$.*

Proof. Since $\theta(\lambda, x) \to (n+1)\pi_p$ from below, we can assume without loss of generality that, for x sufficiently large,

$$y(x) > 0 \quad \text{and} \quad y'(x) < 0$$

and that

$$\frac{y'}{y}(x) < -1,$$

so $y(x) < A\mathrm{e}^{-x} < B$ for constants A, B and $x \geqslant X_1$ say. Assume that for a sequence $x_j \to \infty$ we have $y'(x_j) \to -\infty$. Then, for $x \geqslant X_1$ and $0 \leqslant t \leqslant 1$,

$$\begin{aligned}
([y']^{p-1})' = (p-1)(q-\lambda)[y]^{p-1} &\geqslant -(p-1)(q-\lambda)^- y^{p-1},\\
\int_{x-t}^{x} ([y']^{p-1})' \geqslant -(p-1)\int_{x-t}^{x} (q-\lambda)^- y^{p-1} &\geqslant -(p-1)B^{p-1}(C+|\lambda|),\\
[y']^{p-1}(x) - [y']^{p-1}(x-t) \geqslant -(p-1)B^{p-1}(C+|\lambda|),&
\end{aligned}$$

whence

$$|y'|^{p-1}(x) - |y'|^{p-1}(x-t) \leqslant (p-1)B^{p-1}(C+|\lambda|).$$

Now, choosing j large enough to ensure that

$$|y'(x_j)|^{p-1} > (p-1)(B^{p-1}(C+|\lambda|)) + (2B)^{p-1},$$

we see that

$$\begin{aligned}
|y'|(x_j - t) &\geqslant 2B,\\
y'(x_j - t) &\leqslant -2B,\\
\int_0^1 y'(x_j - t)\,\mathrm{d}t &\leqslant -2B,\\
y(x_j) - y(x_j - 1) &\leqslant -2B,
\end{aligned}$$

whence

$$y(x_j) < -2B + y(x_j - 1) \leqslant -B < 0.$$

This contradiction establishes the result. □

Lemma 3.2. *Suppose that y is a measurable function satisfying*

$$|y(x)| < A\mathrm{e}^{-x/p} \tag{3.2}$$

for sufficiently large x. Then $q^- y^p \in L_1(0,\infty)$.

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Proof. We take X_p large enough to ensure that (3.2) holds for $x > X_p$. Then, for $k > X_p$, by (B0) we have

$$\int_k^{k+1} q^- |y|^p < C\mathrm{e}^{-k},$$

so

$$\int_{[X_p]+1}^{\infty} q^- y^p = \sum_{k \geqslant [X_p]+1} \int_k^{k+1} q^- y^p \leqslant C \sum_0^{\infty} \mathrm{e}^{-k} < \infty.$$

$\square$

Theorem 3.3. *Each set Λ_n^+ contains at most one point.*

Proof. Suppose that λ and μ both belong to Λ_k^+ and that $\lambda < \mu$, so

$$\theta(\lambda, x) < \theta(\mu, x) < (k+1)\pi_p$$

for all x. Suppose that y and z are solutions of (1.1), (1.2) corresponding to λ and μ, respectively. We define x_0 by

$$\begin{aligned} \theta(\lambda, x_0) &= k\pi_p \quad \text{when } k \geqslant 1, \\ x_0 &= 0 \qquad\ \ \text{when } k = 0, \end{aligned}$$

and we take v to be the solution of the initial-value problem consisting of the differential equation (1.1) on $[x_0, \infty)$ with μ in place of λ and subject to the initial condition $v(x_0) = 0$ when $k \geqslant 1$ or $k = \alpha = 0$, and $(v'/v)(x_0) = \cot_p(\alpha)$ when $k = 0 \neq \alpha$. For $k = 0$, note that $v = z$ and, furthermore, y and v are of one sign, which we take to be positive on (x_0, ∞).

If we define an angle ϕ on $[x_0, \infty)$ via $v'/v = \cot_p \phi$, then

$$\theta(\lambda, x) - k\pi_p < \phi < \theta(\mu, x) - k\pi_p$$

so that $\phi \to \pi_p$ from below as $x \to \infty$. As in the proof of Lemma 3.1, we now have

$$v \to 0 \text{ exponentially and } v' \text{ remains bounded as } x \to \infty. \tag{3.3}$$

For small $\varepsilon > 0$ we use

$$w = \frac{y^p}{(v+\varepsilon)^{p-1}},$$

so

$$w' = \frac{py^{p-1}y'}{(v+\varepsilon)^{p-1}} - \frac{(p-1)y^p v'}{(v+\varepsilon)^p}.$$

Thus, by Lemma 3.1 and (3.3), $w, w' \to 0$ exponentially as $x \to \infty$ and, in particular, $w, w' \in L_1(0, \infty)$. Now [**1**, Theorem 1.1] shows that

$$R = R(y, v, \varepsilon) := |y'|^p - w'|v'|^{p-2}v' \geqslant 0 \text{ pointwise},$$

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and hence, for any $b > x_0$,

$$\begin{aligned}0 &\leqslant \int_{x_0}^{b} R = \int_{x_0}^{b} [y']^{p-1} y' - \int_{x_0}^{b} [v']^{p-1} w' \\ &= (p-1)\int_{x_0}^{b} (\lambda - q) y^p - (p-1)\int_{x_0}^{b} (\mu - q) y^p \biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1} + B|_{x_0}^{b} \\ &< (p-1)\int_{x_0}^{b} y^p \biggl(\lambda - \mu\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr) + (p-1)\int_{x_0}^{b} q^- y^p \biggl(1 - \biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr) + B|_{x_0}^{b},\end{aligned}$$

where

$$B = [y']^{p-1} y - [v']^{p-1} w.$$

Let $b \to \infty$ and note, again by Lemma 3.1 and (3.3), that $B(b) \to 0$. This gives

$$\begin{aligned}0 \leqslant (p-1)\int_{x_0}^{\infty} y^p \biggl(\lambda - \mu\biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr) &+ (p-1)\int_{x_0}^{\infty} q^- y^p \biggl(1 - \biggl(\frac{v}{v+\varepsilon}\biggr)^{p-1}\biggr) \\ &- [\cot_p(\alpha)]^{p-1} y(x_0)^p \biggl(1 - \biggl(\frac{v(x_0)}{v(x_0)+\varepsilon}\biggr)^{p-1}\biggr),\end{aligned}$$

where the last term is taken to be 0 unless $k = 0 \neq \alpha$. Now let $\varepsilon \to 0$ and, noting Lemma 3.2, use Lebesgue's dominated convergence theorem to obtain

$$0 \leqslant \int_{x_0}^{\infty} y^p (\lambda - \mu) < 0.$$

This contradiction establishes the result. □

Let us define

$$\lambda_n = \sup \Lambda_n^-$$

for $n = 0, 1, 2, \dots$. It follows from [**2**] that

$$\theta(\lambda, x) \to \infty \quad \text{as } \lambda \to \infty \text{ for any } x > 0$$

and thus each Λ_n^- is bounded above, so $\lambda_n < \infty$. The next result gives, in particular, the complementary inequality $\lambda_n > -\infty$ as well as a complete characterization of the sets Λ_n.

Theorem 3.4. *The λ_n, $n \geqslant 0$, are finite and increase strictly with n to ∞. Setting $\lambda_{-1} = -\infty$, we have, for all $n \geqslant 0$,*

$$\Lambda_n^- = (\lambda_{n-1}, \lambda_n), \qquad \Lambda_n^+ = \{\lambda_n\}, \qquad \Lambda_n = (\lambda_{n-1}, \lambda_n].$$

Proof. Since $u^{-1}|\sin_p u|^p \to 0$ as $u \to 0$ and $(u - \pi_p)^{-1}|\sin_p u|^p \to 0$ as $u \to \pi^p$, we can choose λ, δ to satisfy (2.8). Now [**3**, Lemma 2.4] shows that $\theta(\lambda, Y_{\lambda,\delta}) \to 0$ as $\lambda \to -\infty$, where $Y_{\lambda,\delta}$ is as in Lemma 2.2, so we can choose λ_* so that $\theta(\lambda_*, Y_{\lambda,\delta}) < \gamma$. From Lemma 2.2, $\theta(\lambda_*, x) < \delta$ for all $x > Y_{\lambda,\delta}$, so, by Lemma 2.3, $\lambda_* \in \Lambda_0^-$. In particular,

$\Lambda_0^- \neq \emptyset$, so λ_0 is finite. The remainder of the proof follows the lines of [**4**], so we shall be brief. Note that $\theta(\lambda, x)$ increases monotonically in λ for any x and thus, by (2.10), the sets Λ_n^-, Λ_n^+ and Λ_n are intervals. Now Lemmas 2.3 and 2.4 can be used to prove that Λ_n^- is open for each n (cf. [**4**, Lemma 3.3]) and in particular $\Lambda_0^- = (-\infty, \lambda_0)$. With the aid of Theorem 3.3 we then conclude that $\Lambda_0^+ = \{\lambda_0\}$ and $\Lambda_0^- = (-\infty, \lambda_0]$. Finally, the proof may be completed by induction on n (cf. [**4**, Theorem 4.2]). □

4. Eigenvalues and eigenfunctions

The first result of this section identifies the points λ_n as the eigenvalues of our problem via Definition 1.1 and also shows that, for precisely these points, the associated eigenfunctions have the properties needed for all four definitions of 'eigenvalue' discussed in § 1. Thus, the sets of eigenvalues are identical under all these definitions for potentials satisfying our conditions.

Theorem 4.1. *Under each definition discussed in § 1, λ is an eigenvalue if and only if $\lambda = \lambda_n$ for some n. Any eigenfunction associated with λ_n decays exponentially as $x \to \infty$ and belongs to W_p^1.*

Proof. If $\lambda \in \Lambda_n^+$ (respectively, Λ_n^-), then $\theta(\lambda, x)$ tends to a multiple of π_p from below (respectively, above). Thus, for any solution y of (1.1), (1.2) and for sufficiently large x,

$$\frac{y'}{y}(x) < -1 \quad \left(\text{respectively, } \frac{y'}{y}(x) > 1\right),$$

and so

$$y(x) < A\mathrm{e}^{-x} \quad (\text{respectively, } y(x) > A\mathrm{e}^{x})$$

for some constant A. Thus, if $\lambda \in \Lambda_n^-$, then $y \notin L_p$ and λ cannot be an eigenvalue for any of the definitions considered.

When $\lambda \in \Lambda_n^+$ we see from above that $y \in L_p$. It will suffice to show that $y' \in L_p$, since standard arguments (cf. [**7**]) show that a weak solution must satisfy (1.1), (1.2). From (1.1) multiplied throughout by y we have

$$-([y']^{p-1})'y + (p-1)q|y|^p = (p-1)\lambda|y|^p,$$

and, by integration over $[0, b]$,

$$\int_0^b |y'|^p + (p-1)\int_0^b q^+|y|^p = (p-1)\lambda\int_0^b |y|^p + (p-1)\int_0^b q^-|y|^p + [y']^{p-1}y|_0^b.$$

The terms on the right-hand side have finite limits as $b \to \infty$ by Lemmas 3.1 and 3.2 and hence the same is true for the terms on the left-hand side, which are non-negative. This gives $y' \in L_p$ and completes the proof. □

We should point out that, while the definitions of solution here and in [**7**] formally coincide, the two situations are not directly comparable. For example, we have a coefficient of 1 multiplying the leading term in (1.1) but this violates [**7**, condition (1.2)].

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Also, $q = 0$ in [**7**], and this violates our condition (M1), although it allows the Lagrange multiplier technique to be used for a variational principle. As Example 5.4 shows, this fails in general under our conditions.

Recall that, since the eigenvalues are characterized by the Prüfer system, it follows from § 2 that any eigenfunction corresponding to a given eigenvalue is unique up to scalar multiples. From this and Theorems 3.4 and 4.1 we immediately have the following result.

Corollary 4.2. *Any eigenfunction corresponding to λ_n has precisely n zeros in $(0, \infty)$.*

We turn now to further Sturmian properties of the eigenfunctions. Suppose that we have two potential functions $q, \tilde{q}$ and two initial angles α, $\tilde{\alpha}$ giving rise to eigenvalues, eigenfunctions and Prüfer angles λ_n, u_n, θ and $\tilde{\lambda}_n$, $\tilde{u}_n$, $\tilde{\theta}$, respectively.

Theorem 4.3.

(i) *If $q \leqslant \tilde{q}$ and $\alpha \geqslant \tilde{\alpha}$, then $\tilde{\lambda}_n \geqslant \lambda_n$ for $n \geqslant 0$, with $\tilde{\lambda}_n > \lambda_n$ if either $q < \tilde{q}$ or $\alpha > \tilde{\alpha}$.*

(ii) *If $q < \tilde{q}$ and $\alpha = \tilde{\alpha}$, then, for each n, between any two zeros of $\tilde{u}_n$ there is at least one zero of u_n.*

(iii) *For each $n \geqslant 0$, between any two zeros of u_n there is at least one zero of u_m, for any $m > n$.*

Proof. (i) With reference to (2.1), (2.2) we see that the conditions lead to $\theta(\lambda, x) \leqslant \tilde{\theta}(\lambda, x)$ for all λ and x with strict inequality if either $q < \tilde{q}$ or $\alpha > \tilde{\alpha}$. From this, the results of § 3 and Theorem 4.1, the claim follows readily.

(ii) Again from (2.1), (2.2) we have $\theta'(\lambda, x) > \tilde{\theta}'(\lambda, x)$ for all λ and x. Suppose that $\tilde{\theta}(\lambda, x_0) = k\pi_p$, $\tilde{\theta}(\lambda, x_1) = j\pi_p$ with $j \geqslant k + 1$. Then

$$\pi_p < (j - k)\pi_p = \tilde{\theta}(\lambda, x_1) - \tilde{\theta}(\lambda, x_0) < \theta(\lambda, x_1) - \theta(\lambda, x_0),$$

which establishes the result.

(iii) The argument follows similar lines to that in (ii). □

5. Finite-interval problems and variational results

For any choice of $b > 0$ we introduce the quantities λ_{nb} as the eigenvalues of the regular problem on $[0, b]$ consisting of (1.1), (1.2) subject to the terminal Dirichlet condition $y(b) = 0$. These eigenvalues can be characterized by Prüfer methods as discussed in [**2**], the essential relation being

$$\theta(\lambda_{nb}, b) = (n + 1)\pi_p. \tag{5.1}$$

Our first result concerns interlacing of the λ_{nb} with the λ_n discussed in the previous sections.

Lemma 5.1.

(i) *For all $b > 0$ and $n \geqslant 0$ we have*

$$\lambda_n < \lambda_{nb}.$$

(ii) *For all $n \geqslant 0$ there exists $b_n > 0$ so that*

$$\lambda_{n+1} > \lambda_{nb} \quad \text{for } b > b_n.$$

Proof. (i) Note by (2.1) that $\theta(\lambda, b)$ is strictly increasing in λ for any fixed b and that $\theta(\lambda_n, x) \to (n+1)\pi_p$ from below as $x \to \infty$ so that $\theta(\lambda_n, b) < (n+1)\pi_p$. The result now follows from (5.1).

(ii) Since $\theta(\lambda_{n+1}, x) \to (n+2)\pi_p$ as $x \to \infty$, we have $\theta(\lambda_{n+1}, b) > (n+1)\pi_p$ for b large enough. Now (5.1) and the monotonicity of θ in λ establish the claim. □

We turn next to the dependence of λ_{nb} on b and we give a specific sense in which half-line problems for (1.1), (1.2) can be approximated by problems on finite intervals.

Theorem 5.2. *For all $n \geqslant 0$, $\lambda_{nb} \downarrow \lambda_n$ as $b \to \infty$.*

Proof. The monotonicity of θ in λ and (5.1) show that λ_{nb} is decreasing in b. Since $\lambda_{nb} > \lambda_n$ by Lemma 5.1, there exists $\lambda_{n\infty}$ for which

$$\lambda_{nb} \downarrow \lambda_{n\infty} \geqslant \lambda_n. \tag{5.2}$$

Suppose that $\lambda_{n\infty} > \mu > \lambda_n$ if possible. Then there exists a b' for which

$$\theta(\mu, b') = (n+1)\pi_p = \lim_{x\to\infty} \theta(\lambda_n, x).$$

Thus, $\lambda_{n\infty} > \mu = \lambda_{nb'}$, contradicting (5.2). □

We now discuss various approaches to obtaining the eigenvalues variationally. First we require a number of definitions. We write

$$J(y) = \cot_p^*(\alpha)|y(0)|^p + \int_0^\infty (|y'|^p + (p-1)q|y|^p), \quad y \in U,$$

where

$$\cot_p^*(\alpha) = \begin{cases} \cot_p(\alpha) & \text{if } \alpha \neq 0, \\ 0 & \text{if } \alpha = 0, \end{cases}$$

and

$$U = \left\{ u \in W_p^1(0,\infty) : (p-1)\int |u|^p = 1,\ u(0) = 0 \text{ if } \alpha = 0 \right\}.$$

For $b > 0$ we shall also use

$$U_b = \{u \in U : \operatorname{supp} u \subset [0, b]\}.$$

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The following definitions will be used for the Lyusternik–Schnirelman characterization of the eigenvalues:

$$F_{nb} = \{A \subset U_b : A = -A,\ A \text{ is compact},\ \gamma(A) \geqslant n+1\},$$
$$F_{n\infty} = \bigcup_{b>0} F_{nb},$$

where the Krasnoselskij genus γ is given by

$$\gamma(A) = \inf\{m : \text{there is an odd continuous map from } A \text{ to } \mathbb{R}^m \setminus \{0\}\}.$$

For $A \in F_{n\infty}$, it will be convenient to write

$$s(A) = \sup_{u \in A} J(u).$$

We now have the following variational characterization of the eigenvalues λ_n.

Theorem 5.3. *For each $n \geqslant 0$,*

$$\lambda_n = \inf_{A \in F_{n\infty}} s(A).$$

Proof. We write

$$\mu_n = \inf_{A \in F_{n\infty}} s(A)$$

and note from [**2**] that

$$\lambda_{nb} = \inf_{A \in F_{nb}} s(A). \tag{5.3}$$

Thus, $\mu_n \leqslant \lambda_{nb}$ and so, by Theorem 5.2, $\mu_n \leqslant \lambda_n$.

Suppose that $\mu_n < \lambda_n$ and take $A' \in F_{n\infty}$ (whence $A' \in F_{nb'}$ for some b') to satisfy

$$\mu_n \leqslant s(A') < \lambda_n.$$

Then

$$\lambda_{nb'} = \inf_{A \in F_{nb'}} s(A) \leqslant s(A') < \lambda_n,$$

contradicting Lemma 5.1. This completes the proof. □

Let us define

$$F_n = \{A \subset U : A = -A,\ A \text{ is compact},\ \gamma(A) \geqslant n+1\}.$$

While clearly $F_{n\infty} \subset F_n$, the converse inclusion is false, so it is not clear whether one can replace $F_{n\infty}$ by F_n in Theorem 5.3 or even by

$$F_{nC} = \{A \in F_n : \text{ each function in } A \text{ has compact support}\}.$$

We note that a standard approach to deriving formulae of the type used in Theorem 5.3 is via the Lagrange multiplier method (cf. [**2**, **13**], where the problems are set on finite

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intervals). A similar approach has been used for half-line problems with more general coefficients (except that $q = 0$) in [**7**]. Such approaches require J to be C^1 on U (using the W_p^1 topology). The following example shows that J need not even be continuous let alone C^1 for q under our assumptions. In fact, we take $q(x) = \mathrm{e}^x$, which also satisfies the requirements of [**4**, **5**].

Example 5.4. Take $q(x) = \mathrm{e}^x$ and $\alpha = 0$. For $j = 3, 4, \dots$, set

$$u_j(x) = \begin{cases} c_j x & \text{for } 0 \leqslant x \leqslant 1, \\ c_j(2-x) & \text{for } 1 \leqslant x \leqslant 2, \\ e_j(x-j+1) & \text{for } j-1 \leqslant x \leqslant j, \\ e_j(j+1-x) & \text{for } j \leqslant x \leqslant j+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$u_\infty(x) = \begin{cases} cx & \text{for } 0 \leqslant x \leqslant 1, \\ c(2-x) & \text{for } 1 \leqslant x \leqslant 2, \\ 0 & \text{otherwise,} \end{cases}$$

where c_j, e_j and c will be chosen subsequently. We require

$$(p-1)\int |u_j|^p = (p-1)\int |u_\infty|^p = 1,$$

and, by straightforward calculation, this forces

$$c_j^p + e_j^p = \frac{p+1}{2(p-1)} = c^p.$$

Using W_p^1 norms, we also have

$$|u_j - u_\infty|^p = \frac{2(p+2)}{(p+1)}(|c_j - c|^p + |e_j|^p).$$

Now we take $e_j = 1/j$, so $c_j \to c$, $u_j \to u$ in W_p^1. Furthermore,

$$J(u_j) \geqslant \frac{(p-1)\mathrm{e}^{j-1}|e_j|^p}{(p+1)},$$

so $J(u_j) \to \infty$ but $J(u_\infty)$ is obviously finite. Hence, J is not a continuous functional on U.

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