BULL. AUSTRAL. MATH. SOC. 47H05
VOL. 18 (1978), 77-82.

Monotone operators and dentability

S.P. Fitzpatrick

P.S. Kenderov has shown that every monotone operator on an Asplund Banach space is continuous on a dense G_δ subset of the interior of its domain. We prove a general result which yields as special cases both Kenderov's Theorem and a theorem of Collier on the Fréchet differentiability of weak* lower semicontinuous convex functions.

Let E be a real Banach space with dual E^*. A multivalued mapping $T : E \to E^*$ is called a *monotone operator* on E if $\langle x^*-y^*, x-y\rangle \geq 0$ whenever $x^* \in Tx$ and $y^* \in Ty$. It is called *maximal monotone* if, in addition, its graph

$$G(T) = \{(x, x^*) : x \in E, x^* \in Tx\}$$

is not properly contained in the graph of any other monotone operator on E.

We say that a monotone operator T on E is *locally bounded* at $x \in E$ if there is a neighborhood U of x such that $T(U) = \cup\{Ty : y \in U\}$ is a bounded subset of E^*. We define the *domain* of T to be $D(T) = \{x \in E : Tx \neq \emptyset\}$, and we say that T is *continuous* at a point $x \in D(T)$ if, whenever $x_n \to x$, $x_n^* \in Tx_n$, and $x^* \in Tx$, we have $\|x_n^*-x^*\| \to 0$. This is the same as being single-valued and norm-to-norm upper semicontinuous at x, where T is said to be *upper semi-continuous* at $x \in E$ if given any neighborhood V of 0 in E^*, there is a neighborhood

Received 1 November 1977. The author would like to thank R.R. Phelps for his encouragement, R.T. Rockafellar for suggesting the definition of T_F, and P.S. Kendorov for his correspondence, all of which were very helpful. This work was supported by a CSIRO Postgraduate Studentship.

https://doi.org/10.1017/S0004972700007826 Published online by Cambridge University Press

U of x such that $T(U) \subset Tx + V$. (We will always be using the norm topology in E and the norm or the weak* topology in E^* .)

Let F be a norm closed subspace of E^* . Then an *F-slice* of a nonempty subset C of E is a set of the form

$$S(f, \alpha, C) = \{x \in C : \langle f, x \rangle > M(f, C) - \alpha\} ,$$

where $M(f, C) = \sup\{\langle f, x \rangle : x \in C\}$, $\alpha > 0$, and $f \in F$. We say C is *F-dentable* if for every $\varepsilon > 0$ there is an F-slice of C of diameter less than ε . There are only two choices for F of interest to us; when F is all of E^* (in which case an F-slice is simply called a "slice"), or where E is itself a dual space and F is its predual; that is, when $E = F^*$, so $F \subset F^{**} = E^*$ (and F-slices are called "weak* slices"). In these two cases, we use the terms "dentability" and "weak* dentability". A space E has the *Radon-Nikodym property* if every bounded subset of E is dentable.

Let $f : E \to (-\infty, \infty]$ be a lower semicontinuous convex function. The *subdifferential* ∂f of f is defined by setting, for $x \in E$,

$$\partial f(x) = \{x^* \in E^* : \langle x^*, y-x \rangle \le f(y)-f(x) \text{ for all } y \text{ in } E\} .$$

It is easy to see that ∂f is a monotone operator. Minty [9] showed that the subdifferential of a continuous convex function is maximal monotone, and Rockafellar [14] showed the same for arbitrary proper lower semicontinuous functions (see also Taylor [15]). Note that if f is continuous on an open convex set $C \subset E$, then by the separation theorem $C \subset \text{int } D(\partial f)$.

We call E an *Asplund space* if every lower semicontinuous convex function on E is Fréchet differentiable on a dense G_δ subset of the set of points where it is continuous. Asplund [1] showed that if E is an Asplund space, then every bounded subset of E^* is weak* dentable, and Namioka and Phelps [10] proved the converse. The subdifferential ∂f of a convex function f is continuous at a point x of its domain if and only if f is Fréchet differentiable at x (see Asplund and Rockafellar [2]), so the following result of Kenderov [7] generalizes that of Namioka and Phelps.

THEOREM 1 (Kenderov). *If E is an Asplund space, then every mono-*

https://doi.org/10.1017/S0004972700007826 Published online by Cambridge University Press

tone operator T *on* E *is continuous on a dense* G_δ *subset of* $\text{int}\, D(T)$.

Special cases of this result were obtained earlier by Robert [12], Fitzpatrick [5], and Kenderov and Robert [8]. We will prove the following result, which yields Theorem 1 when $F = E^*$ and $C = \text{int}\, D(T)$.

THEOREM 2. *Let* F *be a closed subspace of* E^* *such that every bounded subset of* F *is* E*-dentable. Let* T *be a monotone operator on* E *and* C *an open subset of* $D(T)$. *If* $Tx \cap F \neq \emptyset$ *for* x *in a dense subset of* C , *then* T *is continuous on a dense* G_δ *subset of* C .

Note that if $f : E^* \to (-\infty, \infty]$ is a weak* lower semicontinuous convex function then $\partial f(x) \cap E$ is nonempty for a dense set of x in C where C equals the domain of (norm) continuity of f . (This follows from Phelps [11], or Brøndsted and Rockafellar [3].) Applying Theorem 2 to $T = \partial f$ with E considered as a subspace of the dual of E^* , we get the following result.

COROLLARY 3 (Collier [4]). *Let* E *have the Radon-Nikodym property and let* f *be a weak* lower semicontinuous convex function on* E^* . *Then* f *is Fréchet differentiable on a dense* G_δ *subset of its domain of continuity.*

To prove Theorem 2 we need some preliminary results about maximal monotone operators.

PROPOSITION 4 (Rockafellar [13]). *Let* T *be a maximal monotone operator on* E *with* $\text{int}(\text{co}\, D(T)) \neq \emptyset$. *Then* $\text{int}\, D(T)$ *is convex,* $\overline{D(T)} = \overline{\text{int}\, D(T)}$, *and* T *is locally bounded at each point of* $\text{int}\, D(T)$.

The next result follows readily from local boundedness.

PROPOSITION 5 (Kenderov [6]). *If* T *is a maximal monotone operator then* T *is norm-to-weak* upper semicontinuous at each point of* $\text{int}\, D(T)$.

Now with $F \subset E^*$, T , and C as in Theorem 2, we can assume without loss of generality that T is maximal monotone. We write $\overline{\text{co}}\, A$ for the weak* closed convex hull of a subset A of E^* . Define T_F by

$$T_F(x) = \bigcap_{\varepsilon>0} \overline{\text{co}}\big(T[B(x, \varepsilon)] \cap F\big) \subset E^* \quad (x \in C) ,$$

where $B(x, \varepsilon)$ denotes the closed ball $\{y \in E : \|y-x\| \leq \varepsilon\}$. It is clear

https://doi.org/10.1017/S0004972700007826 Published online by Cambridge University Press

from Proposition 4 and our assumptions on T that $T_F(x)$ is a nonempty weak* compact convex subset of E^* for all $x \in C$.

LEMMA 6. *The set valued map* T_F *is monotone and* $T_F x = Tx$ *for each* x *in the open subset* C *of* $D(T)$.

Proof. Let $x \in C$ and suppose $x^* \in T_F x \backslash Tx$. By maximality of T there is $y \in E$ and $y^* \in Ty$ such that $\langle x^*-y^*, x-y \rangle = \delta < 0$. By definition of $T_F x$, for each $n \geq 1$ we can find $x_n \in B(x, n^{-1})$ and $x_n^* \in Tx_n \cap F$ such that $\langle x_n^*-y^*, x-y \rangle < \delta/2$. By local boundedness of T at x , for large n we have $\langle x_n^*-y^*, x_n-y \rangle < \delta/3 < 0$, which contradicts the monotonicity of T . So $T_F x \subset Tx$; hence T_F is monotone.

Now suppose $x \in C$ and $x^* \in Tx \backslash T_F x$. By the separation theorem, there is $z \in E$, $\|z\| = 1$, such that $\langle x^*, z \rangle > M(z, T_F x)$. So there is $\varepsilon > 0$ such that $B(x, \varepsilon) \subset C$ and

$$\langle x^*, z \rangle > M\left(z, \overline{\mathrm{co}}\left(T[B(x, \varepsilon)] \cap F\right)\right) .$$

Now if $w^* \in T_F\left(x+(\varepsilon/2)z\right)$, then monotonicity of T yields $0 \leq \langle x^*-z^*, \left(x+(\varepsilon/2)z\right)-x \rangle = \langle w^*-z^*, (\varepsilon/2)z \rangle$. Since $B\left(x+(\varepsilon/2)z, \varepsilon/2\right) \subset B(x, \varepsilon)$, any such w^* is in $\overline{\mathrm{co}}\left(T[B(x, \varepsilon)] \cap F\right)$, which contradicts

$$\langle w^*, z \rangle \geq \langle x^*, z \rangle > M\left(z, \overline{\mathrm{co}}\left(T[B(x, \varepsilon)] \cap F\right)\right) .$$

Hence $T_F x = Tx$ for all $x \in C$.

Now we use a modification of the main idea of Kenderov's proof [7] to complete the proof of Theorem 2.

Let $V_n = \bigcup_{y^* \in E^*} \mathrm{int}\left\{x \in C : Tx \subset B\left(y^*, n^{-1}\right)\right\}$ and let $G = \bigcap_n V_n$. Clearly G is the set of points of C where T is continuous, and V_n is open for each n ; so we only need to show that each V_n is dense in G .

Suppose $x \in C$ and $\varepsilon > 0$. By Proposition 4, there is an open convex neighborhood U of x , $U \subset B(x, \varepsilon) \subset C$, such that $T(U)$ is

https://doi.org/10.1017/S0004972700007826 Published online by Cambridge University Press

bounded. Let $A = T(U) \cap F$, which is by assumption nonempty and bounded. It follows that there is a slice $S = S(z, \alpha, A)$ of A with diameter less than $(2n)^{-1}$ and $z \in E$. Let $v^* \in S$, $v^* \in Tv$ with $v \in U$. For sufficiently small $\beta > 0$, the point $w = v + \beta z$ is in U. If $w^* \in Tw$, we have

$$0 \leq \langle w^*-v^*, w-v\rangle = \beta\langle w^*-v^*, z\rangle ,$$

so that $\langle w^*, z\rangle \geq \langle v^*, z\rangle > M(z, A) - \alpha$. By Proposition 5, there is an open neighborhood W of w, $W \subset U$, such that

$$T(W) \subset Tw + \{y^* \in E^* : |\langle y^*, z\rangle| < \langle v^*, z\rangle - M(z, A) + \alpha\} ;$$

so $T(W) \cap F \subset S(z, \alpha, A)$. Since S is contained in some closed ball $B(y^*, n^{-1})$ with $y^* \in F$, the set $T_F(W)$ is contained in $B(y^*, n^{-1})$ (since the ball is weak* closed). By Lemma 6, $T(W) \subset B(y^*, n^{-1})$, so $w \in V_n$. Hence V_n is dense, which completes the proof.

References

[1] Edgar Asplund, "Fréchet differentiability of convex functions", *Acta Math.* 121 (1968), 31-47.

[2] E. Asplund and R.T. Rockafellar, "Gradients of convex functions", *Trans. Amer. Math. Soc.* 139 (1969), 443-467.

[3] A. Brøndsted and R.T. Rockafellar, "On the subdifferentiability of convex functions", *Proc. Amer. Math. Soc.* 16 (1965), 605-611.

[4] James B. Collier, "The dual of a space with the Radon-Nikodym property", *Pacific J. Math.* 64 (1976), 103-106.

[5] S.P. Fitzpatrick, "Continuity of nonlinear monotone operators", *Proc. Amer. Math. Soc.* 62 (1977), 111-116.

[6] P.S. Kenderov, "The set-valued monotone mappings are almost everywhere single-valued", *C.R. Acad. Bulgare Sci.* 27 (1974), 1173-1175.

[7] P.S. Kenderov, "Monotone operators in Asplund spaces", *C.R. Acad. Bulgare Sci.* (to appear).

[8] Petar Kenderov et Raoul Robert, "Nouveaux résultats génériques sur les opérateurs monotones dans les espaces de Banach", *C.R. Acad. Sci. Paris Sér. A* 282 (1976), 845-847.

[9] George J. Minty, "On the monotonicity of the gradient of a convex function", *Pacific J. Math.* 14 (1964), 243-247.

[10] I. Namioka and R.R. Phelps, "Banach spaces which are Asplund spaces", *Duke Math. J.* 42 (1975), 735-750.

[11] R.R. Phelps, "Weak* support points of convex sets in E^* ", *Israel J. Math.* 2 (1964), 177-182.

[12] Raoul Robert, "Points de continuité des multi-applications semi-continues supérieurement", *C.R. Acad. Sci. Paris Sér. A* 278 (1974), 413-415.

[13] R.T. Rockafellar, "Local boundedness of nonlinear, monotone operators", *Michigan Math. J.* 16 (1969), 397-407.

[14] R.T. Rockafellar, "On the maximal monotonicity of subdifferential mappings", *Pacific J. Math.* 33 (1970), 209-216.

[15] Peter D. Taylor, "Subgradients of a convex function obtained from a directional derivative", *Pacific J. Math.* 44 (1973), 739-747.

Department of Mathematics,
University of Washington,
Seattle,
Washington,
USA.