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A NOTE ON ERDÖS–RENYI LAW OF LARGE NUMBERS

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ABSTRACT. In this note the Erdös–Renyi law of large numbers is extended to stationary Gaussian sequences.

1. **Introduction and main theorem.** A new law of large numbers for i.i.d. sequence of random variables was discovered by Erdös–Renyi (1970). The general problem of extending this theorem to stationary sequences, under various mixing conditions, appears to be quite difficult. In this note we deal with a stationary Gaussian sequence and show that such a sequence obeys Erdös–Renyi theorem under a mild condition on the correlation sequence. The same condition was used in Deo (1974) to prove Strassen's law of iterated logarithm for stationary, Gaussian sequences.

Let $\{\xi_n : 1 \le n < \infty\}$ be a stationary, Gaussian sequence with $E(\xi_1) = 1$, $E(\xi_1^2) = 1$ and $E(\xi_1 \xi_{n+1}) = r_n$, $n \ge 0$. Let $S_0 = 0$, $S_n = \sum_{j=1}^{n} \xi_j$ and for $1 \le k \le n$ let $\Theta(n, k) = \max_{0 \le j \le n-k} (S_{j+k} - S_j)/k$. We will assume that

$$\lim_{n\to\infty} n^{1+\beta} r_n = 0 \quad \text{for some} \quad \beta > 0. \tag{1}$$

Under (1) the series $\sum r_j$ converges absolutely. Write $\sigma^2 = 1 + 2\sum_{j=1}^{\infty} r_j$. We exclude the degenerate case $\sigma = 0$ and assume hereafter $\sigma > 0$. Let $[\cdot]$ be the usual largest integer function. The object of this note is to prove the following

THEOREM. *If* (1) *holds then, for each* $c > 0$,

$$\lim_{n\to\infty} \Theta(n, [c \log n]) = \sigma\sqrt{\frac{2}{c}} \tag{2}$$

with probability one.

Proof. We first show that

$$\varliminf_{n\to\infty} \Theta(n, [c \log n]) \ge \sigma\sqrt{\frac{2}{c}} \qquad \text{w.p. } 1. \tag{3}$$

Let $\varepsilon > 0$. The first step consists in proving that 3 positive numbers ν, δ

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(depending upon ε) such that

$$(4)\qquad P\{\Theta(n,[c\log n])<\sigma\left(\sqrt{\frac{2}{c}}-\varepsilon\right\}<n^{-\nu}+e^{-n^{\delta}},\quad \text{for all large } n.$$

Now break up the integers 1 through n into blocks of size $[c \log n]$ leaving a gap of size $[\log\log n]$ between two adjacent blocks. Let k_n denote the total number of such blocks of size $[c \log n]$ and let $J_1, J_2, \ldots, J_{k_n}$ denote these blocks. Thus J_1 consists of integers 1 through $[c \log n]$, J_2 consists of integers $[c \log n]+[\log\log n]$ through $2[c \log n]+[\log\log n]$ and so on. Note that k_n is approximately $n/(\log n+\log\log n)$ and as will be clear from computations below the last incomplete block if any, of size less than $[c \log n]$ can be safely ignored. Let $Y_{n,i}=\sum_{j\in J_i}\xi_j$, $1\le i\le k_n$. Now the probability in (4) is clearly dominated by

$$(5)\qquad P\left\{\max_{1\le i\le k_n}(\sigma^2[c\log n])^{-1/2}Y_{n,i}<\left(\sqrt{\frac{2}{c}}-\varepsilon\right)\sqrt{c\log n}\right\}.$$

Note that for each n, the variances of $Y_{n,i}$'s are equal; and, as $n\to\infty$, these are asymptotic to $\sigma^2 c \log n$. Hence if $0<\varepsilon'<\varepsilon$, the probability in (5) is, for large n, less than

$$(6)\qquad P\left\{\max_{1\le i\le K_n}(\mathrm{Var}\ Y_{n,i})^{-1/2}Y_{n,i}<\left(\sqrt{\frac{2}{c}}-\varepsilon'\right)\sqrt{c\log n}\right\}.$$

Let now $Z_1, Z_2, \ldots, Z_{k_n}$ be independent standard normal variables. We have,

$$(7)\qquad P\left\{\max_{1\le i\le k_n}Z_i<\left(\sqrt{\frac{2}{c}}-\varepsilon'\right)\sqrt{c\log n}\right\}=\prod_{i=1}^{k_n}P\left\{Z_i<\left(\sqrt{\frac{2}{c}}-\varepsilon'\right)\sqrt{c\log n}\right\}$$
$$=\prod_{i=1}^{k_n}\left[1-P\left\{Z_i>\left(\sqrt{\frac{2}{c}}-\varepsilon'\right)\sqrt{c\log n}\right\}\right]$$
$$\le e^{-\sum_{i=1}^{k_n}P\left\{Z_i>\left(\sqrt{\frac{2}{c}}-\varepsilon'\right)\sqrt{c\log n}\right\}}.$$

Now using the standard estimate ([4], page 175) for the upper tail of the normal distribution it is easy to see that $P\{Z_i>(\sqrt{2/c}-\varepsilon')\sqrt{2}\log n\}$ is, for large n, greater than $n^{-1+\delta'}$ for some $\delta'>0$. Also $k_n>n^{1-\delta'/2}=$ for large n. Hence the probability in (7) is less than $e^{-n^{\delta}}$ for large n where $\delta=\delta'/2$.

Next we estimate Δ_n which stands for the difference between the probabilities in (6) and (7). For this we use the following

Lemma (Berman (1964)). *Let the random variables $X_1, X_2, \ldots, X_n$ have joint Gaussian distribution with zero means, unit variances and correlations $\{r(i,j): 1\le i\le j\le n\}$. Also let $Z_1, Z_2, \ldots, Z_n$ be independent standard normal.*

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Then for any $a>0$,

$$\left|P\left(\max_{1\le i\le n} X_i<a\right)-P\left(\max_{1\le i\le n} Z_i<a\right)\right|$$
$$\le \sum_{1\le i\le j<n}[1-r^2(i,j)]^{-1/2}\cdot|r(i,j)|\exp\left\{-\frac{a^2}{1+|r(i,j)|}\right\}.$$

To apply this lemma let $r_n(i,j)$ denote the correlation coefficient between $Y_{n,i}$ and $Y_{n,j}$. Under our hypothesis (1) it is a straightforward verification that $\exists$ finite positive constant B independent of n, i, j such that

$$|r_n(i,i+1)|<B(\log\log n)^{-\beta},\qquad 1\le i\le k_n-1;\quad\text{and}$$
$$|r_n(i,i+k)|<B(k-1)^{-\beta},\qquad k>1,\qquad 1\le i+k\le k_n. \tag{8}$$

Thus $r_n(i,j)\to 0$ as $n\to\infty$ uniformly in i, j. Also note that $\{(\sqrt{2/c}-\varepsilon')\sqrt{c}\log n\}^2>(2-\varepsilon'')\log n$, where $\varepsilon''=2\sqrt{2}\varepsilon' c$ and $\varepsilon''>0$ can be made arbitrarily small by making ε' and hence ε small enough. Hence applying the lemma we have, for all large n,

$$\Delta_n\le\text{const}\sum_{1\le i\le j<k_n}|r_n(i,j)|\cdot n^{-(2-\varepsilon''/1+|r_n(i,j)|)}$$
$$\le\text{const}\sum_{1\le i\le j<k_n}|r_n(i,j)|\,n^{-2+(\varepsilon''/2)}$$
$$\le\text{const}\left\{\sum_{k=2}^{n}(n-k)(k-1)^{-\beta}n^{-2+(\varepsilon''/2)}+(n-1)n^{-2+(\varepsilon''/2)}\right\}$$

where in the last step we have used (8) and the fact that $k_n<n$. Note that $\sum_{k=2}^{n}(n-k)(k-1)^{-\beta}\le n\sum_{k=2}^{n}(k-1)^{-\beta}\le\text{const}\,n^{2-\beta}$. Hence, we get (assuming $\beta<1$ without loss of generality),

$$\Delta_n\le\text{const}\,n^{2-\beta-2+(\varepsilon''/2)}=\text{const}\,n^{-\beta+(\varepsilon''/2)}. \tag{9}$$

Here $\beta>0$ is fixed and ε'' can be taken to be less than 2β by choosing our initial $\varepsilon>0$ small enough which is permissible since, if (4) holds for some ε, it also holds for all smaller ε. Thus from (9) we can conclude that

$$\Delta_n\le n^{-\nu},\quad\text{for some}\quad\gamma>0,\quad\text{for all large } n. \tag{10}$$

Combining (10) and (7) we get (4).

It now follows from (4) and the first Borel–Cantelli lemma that $\Theta([e^{k/c}]-1,k-1)<\sigma(\sqrt{2/c}-\varepsilon)$ for only finitely many k's with probability one. Note that for n with $[e^{k/c}]\le n<[e^{(k+1)/c}]$, the numerator is the definition of $\Theta(n,[c\log n])$ is less than or equal to the numerator in the definition of $\Theta([e^{(k+1)/c}]-1,k)$ whereas the denominators are asymptotically equal. Thus $\Theta(n,[c\log n])<\sigma(\sqrt{2/c}-\varepsilon)$ only finitely often with probability one which proves (3).

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The proof that $\overline{\lim}_{n\to\infty} \Theta(n, [c \log n]) \le \sigma\sqrt{2/c}$ is much simpler. Indeed, for $\varepsilon > 0$,

$$(11)\quad P\left\{\Theta(n, [c \log n]) > \sigma\left(\sqrt{\frac{2}{c}} + \varepsilon\right)\right\} \le nP\left\{\frac{S[c \log n]}{\sigma[c \log n]} > \left(\sqrt{\frac{2}{c}} + c\right)\sqrt{c} \log n\right\}$$

Again using the standard estimate ([4], page 175) of the upper tail of the normal distribution it is easy to see that the right side of (11) is dominated by $n^{-\alpha}$ for some $\alpha > 0$. In conjunction with the first Borel–Cantelli lemma this implies $\Theta([e^{k/c}], k) > \sigma(\sqrt{2/c} + \varepsilon)$ only finitely often with probability one. By an approximation similar to the one already used this means $\Theta(n, [c \log n]) > (\sigma\sqrt{2/c} + \varepsilon)$ only finitely often with probability one. This completes the proof of the theorem.

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