History of a theorem in Elementary Geometry.

By J. S. Mackay, M.A., LL.D.

The theorem is

If the straight lines bisecting the angles at the base of a triangle and terminated by the opposite sides be equal, the triangle is isosceles.

This theorem was in the year 1840 communicated by Professor Lehmus of Berlin to Jacob Steiner with a request for a pure geometrical proof of it. The request was complied with at the time, but Steiner's proof was not published till some years later.

It was the first question proposed in the *Nouvelles Annales de Mathématiques*, Vol. I, p. 57 (1842), and two solutions are given in the same volume, pp. 138–139 and 311 by Messrs Rougevin and Grout de Saint-Paer. In this same journal see also Vol. XIII, pp. 192, 331 (1854), Vol. XIV, p. 32 (1855) and Vol. XVI, p. 102 (1857).

Steiner's own demonstration, which Sturm considered to be the most elementary one, and which must have been in circulation shortly after 1840, was given in Crelle's *Journal*, Vol. XXVIII, pp. 375–379 (1844). It is reproduced in Steiner's *Gesammelte Werke*, Vol. II, pp. 323–326 (1882). After giving his proof, Steiner considers also the case when the angles below the base are bisected; he generalises the theorem somewhat; and finally he discusses the case of the spherical triangle.

In Grunert's *Archiv der Mathematik*, Vol. IV, pp. 330–1 (1844) two proofs are given by L. Mossbrugger, Mathematical Master in the Canton school of Aarau. The theorem was communicated to him by Steiner, with the remark that notwithstanding its obvious insignificance, the proof was attended with some difficulty. C. Adams in his "Die merkwürdigsten Eigenschaften des geradlinigen Dreiecks," pp. 10–11 (1846) reproduces the second of Mossbrugger's proofs, and adds another of his own. In the course

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of years a considerable number of proofs have appeared in Grunert's *Archiv*. The following references may be given. See

Vol. XI	(1848)	pp. 444–445	(Franz Knopf)
XIII	(1849)	pp. 337–341	(Theodor Lange)
		pp. 341–344	(J. A. Grunert)
XV	(1850)	pp. 221–224	(Theodor Lange)
		pp. 225–226	(Prof. Lehmus)
		pp. 351–355	(Theodor Lange)
		pp. 359–360	(W. Mink)
XVI	(1851)	pp. 201–203	(R. Baltzer, A. Seebeck)
		pp. 259–260	(August)
		pp. 356–358	(Dr Zech)
XVIII	(1852)	pp. 357–359	(C. Schmidt)
XX	(1853)	pp. 459–461	(Dr Clausen)
XLI	(1864)	pp. 151–152	(A. Niegemann)
XLII	(1864)	pp. 232–236	(J. A. Grunert)

In 1852 in the *London, Edinburgh and Dublin Philosophical Magazine*, Vol. IV, Fourth Series, pp. 366–369, Prof. J. J. Sylvester has a paper entitled "On a Simple Geometrical Problem illustrating a conjectured Principle in the Theory of Geometrical Method." This paper contains two demonstrations, one by Mr B. L. Smith of Jesus College, Cambridge, and the other by himself. His own he confesses is less simple than the other, but it "has the advantage of lending itself to a considerable generalisation of the theorem." After some analytical discussion, Prof. Sylvester adds: "My reader will now be prepared to see why it is that all the geometrical demonstrations given of this theorem are indirect, I believe I may venture to say *necessarily* indirect. It is because the truth of the theorem depends on the necessary non-existence of real roots (between prescribed limits) of the analytical equation expressing the conditions of the question; and I believe that it may be safely taken as an axiom in geometrical method that whenever this is the case no other form of proof than that of the *reductio ad absurdum* is possible in the nature of things." Prof. Sylvester appends to his paper the following remarks: "If report may be believed, intellects capable of extending the bounds of the planetary system, and lighting up new regions of the universe with the torch of analysis

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have been baffled by the difficulties of the elementary problem stated at the outset of this paper, in consequence, it is to be presumed, of seeking a form of geometrical demonstration of which the question from its nature does not admit."

In the *Lady's and Gentleman's Diary* for 1856, p. 72, the theorem is again proposed by J. W. Elliott, and in the *Diary* for 1857, pp. 58–59, a geometrical proof is given by T. T. Wilkinson, and an analytical one by C. H. Brooks and others. In the *Diary* for 1859, pp. 87–88, T. T. Wilkinson gives another proof, and mentions that the theorem was originally proposed in the *Nouvelles Annales* for 1842. In the *Diary* for 1860, pp. 85–86, Wilkinson has some further remarks on the theorem, and among them he states that his proof in the *Diary* for 1857 is equally valid for the more general theorem : *If any two lines drawn from the base angles of a triangle meet on the bisector of the vertical angle and are equal to each other the triangle is isosceles.* He concludes his remarks with two other proofs by the Rev. William Mason.

The following letters appeared in the *Philosophical Magazine*, Fourth Series, Vol. XLVII, pp. 354–357 (1874). The solution spoken of in them is not reprinted here.

GONVILLE AND CAIUS COLLEGE,
CAMBRIDGE, *March 27, 1874.*

GENTLEMEN,

May I request the publication in your Magazine of the accompanying paper and letter which came from a lady in California? The paper, as you will see, is a solution of a geometrical problem which, more than twenty years ago, excited considerable interest by its discussion in your Magazine. The Vice-Chancellor of Cambridge, to whom these documents were addressed, placed them in my hands; and at his request I forwarded them to Professor Sylvester, who expresses the opinion, with which I entirely agree, that the solution is thoroughly sound, and authorises me to say that he has suggested the propriety of the publication of the papers in your Magazine. In this suggestion the Vice-Chancellor cordially concurs.

I am, Gentlemen,

Your obedient servant,

N. M. FERRERS.

To the Editors
of the Philosophical Magazine and Journal.

Oakland, *Feb.* 15, 1874.

Sir,

In a number of the *London, Edinburgh and Dublin Philosophical Magazine* for the year 1852 or 1862 is a discussion of a simple geometrical problem by J. J. Sylvester, in which this gentleman attempts to illustrate a conjectured principle in the Theory of Geometrical Method. The problem appears to have been given out at Cambridge many years before, and to have excited the attention of some of the first mathematicians of Europe for a number of years. A direct solution of the problem was demanded: and after a protracted discussion it was concluded that none was possible. Upon that some mathematicians lent their attention, by a study of the nature of the problem under question, to discovering some rule by which on inspection they might be able to tell whether a given problem admitted of only an indirect solution, a question of some importance in Astronomy.

I have not been able to discover that any direct solution has been brought forth as yet; so I send you this, given to me by a friend of mine, Mr Hesse, who solved it in 1842. I send it, thinking that every mistake cleared up in Science is a step toward truth.

If it would not be too much trespassing on your good nature, I should be pleased to hear of the fate of my solution.

Christine Chart.

A proof by Richard Jones will be found in a periodical, called *The Schoolmaster*, for 13th July 1878; another by Mr Descube in the *Journal de Mathématiques Elémentaires et Spéciales*, Vol. IV, pp. 538–539 (1880); and a third by Mr Soons, with a note by Professor Neuberg, in *Mathesis*, Second Series, Vol. V, pp. 261–262 (1895). In *Mathematical Questions and Solutions from the Educational Times*, Vol. 74, p. 73, proofs are given by Rev. T. Roach and Mr R. Chartres, and on pp. 74–75 a proof by Mr D. Biddle. In the same volume, pp. 105–107 (1901), another proof is given by Mr R. Chartres.

The following proof was sent me in 1866 by my friend the late Andrew Miller, M.A., Mathematical Master in the Dundee High School, to whom I had shown the theorem. My own proof, discovered shortly before, is hardly worth reproducing; it is substantially the same as that given by Professor M. C. Stevens, in an American journal *The Analyst*, Vol. VI, p. 89 (1879).

The reader will have no difficulty in making the figure.

Let ABC be the triangle, BD and CE the bisectors of the base angles.

If triangle ABC is not isosceles, let angle ACB be greater than angle ABC. Make angle ECF equal to half of angle ABC, and let CF intersect AB in F and BD in G.

Since angle FBG is equal to angle FCE, and angle BFG is equal to angle CFE, therefore triangles BFG, CFE are similar; and therefore

$$FB : BG = FC : CE.$$

But since angle FCB is greater than angle FBC, therefore FB is greater than FC.

Hence BG must be greater than CE, and consequently BD greater than CE, which is contrary to the hypothesis.