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ON HOMOGENEOUS EXPANSIONS OF MIXED NORM SPACE FUNCTIONS IN THE BALL

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ABSTRACT. For f analytic in the complex ball having the homogeneous expansion $f(z) = \sum_{k=0}^{\infty} F_k(z)$, conditions for f to be of Hardy space H^p or of weighted Bergman spaces are expressed in terms of ℓ^p properties of the sequence $\{\|F_k\|_p\}$.

1. **Introduction.** Let $B = B_n$ be the open unit ball of $\mathbf{C}^n$ and let σ be the rotation invariant probability measure on the boundary S of B. In case $n = 1$, U and T will stand for B and S respectively. For $0 < p < \infty$, $0 < q \le \infty$, and $\beta > -1$, the spaces H^p and $A^{p,q,\beta}$ are defined to consist of those f holomorphic in B respectively for which

$$\|f\|_q = \sup_{0\le\rho<1} M_q(\rho,f) < \infty$$

and

$$\|f\|_{p,q,\beta} = \int_0^1 (1-\rho)^\beta M_q(\rho,f)^p \, d\rho < \infty,$$

where

$$M_q(\rho,f) = \left[\int_S |f(\rho\zeta)|^q \, d\sigma(\zeta)\right]^{1/q}, \quad q < \infty,$$

and

$$M_\infty(\rho,f) = \sup_{z\in\rho S} |f(z)|.$$

Our concern in this note is in the growth rates of Taylor coefficients of H^p or $A^{p,q,\beta}$ functions defined on B. There are three types of results in general on the growth of the Taylor coefficients of H^p functions defined on U: Coefficients results of Hardy-Littlewood is the first, Hausdorff-Young theorem is the next, and Paley type results on gap series is the last (see [7], [8], [9], and [13]).

Concerning our results, Section 2 deals with Hardy-Littlewood type extensions to B, Section 3 deals with Hausdorff-Young types, and Section 4 deals with Paley types.

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2. **Extension of Hardy-Littlewood theorem.** If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha_j \geq 0$, $1 \leq j \leq n$, is the multi-index then we denote $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ and $\alpha! = \alpha_1!\,\alpha_2! \cdots \alpha_n!$. If f is holomorphic in B then f can be representable by

$$f(z) = \sum_{k=0}^{\infty} F_k(z),$$

where F_k are homogeneous polynomials of degree k. Let I_m denote the set $\{k : 2^{m-1} \leq k < 2^m\}$ of integers if $m \geq 0$ and $I_0 = 0$. D. Kwak [10] deduced the following, which generalizes a classical one variable result of Hardy-Littlewood [7. Theorem 6.2].

THEOREM A [10. THEOREM 2.1]. *Let $0 < p \leq 2$, $q \geq 0$. Let $f(z) = \sum a_\alpha z^\alpha \in A^{p,p,q-1}$. Then*

$$\sum (|\alpha| + 1)^{(n+q/2)(p-2)} \left(\frac{\alpha!}{\Gamma(n + |\alpha| + q)} \right)^{p/2} |a_\alpha|^p \leq C \|f\|_{p,p,q-1}^p. \tag{2.1}$$

Here and throughout, $A^{p,p,-1} = H^p$ and C will denote a positive constant independent of particular function f. Note that $q = 0$ is the only interesting case of this result, since the case $q > 0$ easily obtained by integrating the estimates for $q = 0$. We improve this theorem in this section. We abuse obvious notations such as $\frac{1}{q} = 0$ if $q = \infty$ *etc.*

THEOREM 1. *Let $0 < p < \infty$, $1 \leq q \leq 2$, $\beta > -1$, and $\delta_2 = \frac{\beta+1}{p} + (n-1)(\frac{1}{q} - \frac{1}{2})$. Let $f(z) = \sum a_\alpha z^\alpha \in A^{p,q,\beta}(B)$. Then*

$$\sum_{m=0}^{\infty} \left[\sum_{|\alpha| \in I_m} \left(\frac{\alpha!}{\Gamma(n + |\alpha|)} \frac{|a_\alpha|^2}{(|\alpha| + 1)^{2\delta_2}} \right)^{q'/2} \right]^{p/q'} \leq C \|f\|_{p,q,\beta}^p \tag{2.2}$$

with the obvious understanding of (2.2) when $q = 1$.

To see that Theorem 1 is an improvement of Theorem A, we need the following imbedding theorem.

THEOREM B [6]. *Let $0 < p < r < \infty$, $p \leq s < \infty$, and $q \geq 0$. If $f \in A^{p,p,q-1}$, then*

$$\|f\|_{s,r,s\beta-1} \leq C \|f\|_{p,p,q-1}, \tag{2.3}$$

where $\beta = \frac{n+q}{p} - \frac{n}{r}$.

We now prove that Theorem 1 implies Theorem A: Suppose Theorem 1. Let $0 < p \leq 2$, $q \geq 0$, and let $f(z) = \sum a_\alpha z^\alpha \in A^{p,p,q-1}$. Then by (2.2),

$$\sum_{m=0}^{\infty} \left(\sum_{|\alpha| \in I_m} \frac{\alpha!}{\Gamma(n + |\alpha|)} \frac{|a_\alpha|^2}{(|\alpha| + 1)^{2\delta_2}} \right)^{p/2} < C \|f\|_{p,2,\beta}^p, \tag{2.4}$$

where $\delta_2 = \frac{\beta+1}{p}$. Set $\beta = p(\frac{n+q}{p} - \frac{n}{2}) - 1$, so that by (2.3),

$$\|f\|_{p,2,\beta} \leq C \|f\|_{p,p,q-1}. \tag{2.5}$$

Now since

$$\frac{\Gamma(n+|\alpha|)}{\Gamma(n+|\alpha|+q)} \le C(|\alpha|+1)^{-q}$$

by the Stirling's formula, we have

$$(2.6)\qquad \begin{aligned} &\sum_{|\alpha|\in I_m} (|\alpha|+1)^{(n+q/2)(p-2)} \Big(\frac{\alpha!}{\Gamma(n+|\alpha|+q)}\Big)^{p/2} |a_\alpha|^p \\ &\le C \sum_{|\alpha|\in I_m} (|\alpha|+1)^{np-2n-q} \Big(\frac{\alpha!}{\Gamma(n+|\alpha|)}\Big)^{p/2} |a_\alpha|^p, \end{aligned}$$

which is, by the Hölder's inequality, dominated by

$$(2.7)\qquad C\Big(\sum_{I_m} \frac{\alpha!}{\Gamma(n+|\alpha|)} \frac{|a_\alpha|^2}{(|\alpha|+1)^{2\delta_2}}\Big)^{p/2} \Big(\sum_{I_m} \frac{1}{(|\alpha|+1)^n}\Big)^{1-p/2}.$$

Since

$$(2.8)\qquad \sum_{|\alpha|\in I_m} \frac{1}{(|\alpha|+1)^n} \le C < \infty,$$

summation over m after combining (2.6), (2.7), and (2.8), we obtain

$$(2.9)\qquad \begin{aligned} &\sum_m (|\alpha|+1)^{(n+q/2)(p-2)} \Big(\frac{\alpha!}{\Gamma(n+|\alpha|+q)}\Big)^{p/2} |a_\alpha|^p \\ &\le C \sum_{m=0}^{\infty} \Big(\sum_{|\alpha|\in I_m} \frac{\alpha!}{\Gamma(n+|\alpha|)} \frac{|a_\alpha|^2}{(|\alpha|+1)^{2\delta_2}}\Big)^{p/2}. \end{aligned}$$

From (2.4), (2.5), and (2.9), we obtain (2.1).

Theorem 1 is an easy consequence of the following

THEOREM 2. *Let* $0 < p < \infty$, $1 \le q \le 2$, $q \le r \le \infty$, $\beta > -1$, *and* $\delta = \frac{\beta+1}{p} + (n-1)(\frac{1}{q} - \frac{1}{r})$. *Let* $f(z) = \sum_{k=0}^{\infty} F_k(z)$ *be the homogeneous polynomial expansion of an* f *in* $A^{p,q,\beta}(B)$. *Then*

$$(2.10)\qquad \sum_{m=0}^{\infty} \Big(\sum_{k\in I_m} \|(k+1)^{-\delta} F_k\|_r^{q'}\Big)^{p/q'} \le C\|f\|_{p,q,\beta}^p,$$

with the obvious understanding of the left side when $q = 1$.

PROOF OF THEOREM 1 USING THEOREM 2. Suppose

$$f(z) = \sum_{k=0}^{\infty} F_k(z) = \sum a_\alpha z^\alpha.$$

Noting that

$$\|F_k\|_2^2 = \Big\|\sum_{|\alpha|=k} a_\alpha z^\alpha\Big\|_2^2 = \sum_{|\alpha|=k} |a_\alpha|^2 \|z^\alpha\|_2^2 = \Gamma(n) \sum_{|\alpha|=k} |a_\alpha|^2 \frac{\alpha!}{\Gamma(n+|\alpha|)}$$

[14. Proposition 1.4.9], and taking $r = 2$ in (2.10), we obtain (2.2).

PROOF OF THEOREM 2. Let $f \in A^{p,q,\beta}(B)$, $1 \leq q \leq 2$. Let $f_\zeta(\lambda) = f(\zeta\lambda)$, $\zeta \in S$, $\lambda \in U$. We first prove the case $q = r$. We confine ourselves to the case $q > 1$ but the idea for the case $q = 1$ is identical except for notations.

It follows from the Hausdorff-Young theorem (see [7. Theorem 6.1], for example) that

$$\Big(\sum_{k=0}^{\infty} |F_k(\rho\zeta)|^{q'}\Big)^{1/q'} \leq \Big(\int_0^{2\pi} |f_\zeta(\rho e^{i\theta})|^q \frac{d\theta}{2\pi}\Big)^{1/q}, \quad \zeta \in S \tag{2.11}$$

(note that the dominating constant is 1). Integrating the q-power of (2.11) with respect to $d\sigma(\zeta)$, and then applying the Minkowski's inequality to the resulting left hand side, we obtain

$$\Big[\sum_{k=0}^{\infty} M_q(\rho, F_k)^{q'}\Big]^{1/q'} \leq M_q(\rho, f).$$

Therefore we have

$$\int_0^1 (1-\rho)^\beta M_q(\rho, f)^p \, d\rho \geq \int_0^1 (1-\rho)^\beta \Big[\sum_{k=0}^{\infty} M_q(\rho, F_k)^{q'}\Big]^{p/q'} d\rho.$$

Now, the last quantity is at least

$$\sum_{m=0}^{\infty} \int_{1-2^{-m}}^{1-2^{-(m+1)}} (1-\rho)^\beta \Big(\sum_{k \in I_m} \|F_k\|_q^{q'} \rho^{kq'}\Big)^{p/q'} d\rho,$$

which is, in turn, at least a positive constant times

$$\sum_{m=0}^{\infty} \Big(\sum_{k \in I_m} \|(k+1)^{(\beta+1)/p} F_k\|_q^{q'}\Big)^{p/q'}.$$

This completes the proof of Theorem 2 when $q = r$. The case $q < r$ is an easy combination of the following lemma with what we have just proven.

LEMMA. *Let $0 < p \leq r \leq \infty$. Let π be a homogeneous polynomial of degree k. Then there is a constant C depending only on p and n such that*

$$\|\pi\|_r \leq C(k+1)^{(n-1)(1/p-1/r)} \|\pi\|_p. \tag{2.12}$$

PROOF. See [2] and [5. pp. 8–9] for ideas similar to the following proof. Note first that it suffices to prove (2.12) for $r = \infty$. In fact, if we suppose (2.12) for $r = \infty$ then for $r < \infty$

$$\begin{aligned}
\|\pi\|_r &= \Big[\int_S |\pi(\zeta)|^r \, d\sigma(\zeta)\Big]^{1/r} \\
&\leq \Big[\int_S |\pi(\zeta)|^p \, d\sigma(\zeta)\Big]^{1/r} \|\pi\|_\infty^{(r-p)/r} \\
&\leq \|\pi\|_p^{p/r} [C(k+1)^{(n-1)/p} \|\pi\|_p]^{1-p/r} \\
&= C(k+1)^{(n-1)(1/p-1/r)} \|\pi\|_p.
\end{aligned}$$

Now we prove (2.12) for $r=\infty$. We may assume $\|\pi\|_\infty = |\pi(1,0,\ldots,0)|$. If we denote by ν_n the normalized Lebesgue measure on B_n, then by subharmonicity (see [14. 1.5.4])

$$\begin{aligned}\int_S |\pi(\zeta)|^p\,d\sigma(\zeta) &= \int_{B^{n-1}} d\nu_{n-1}(\zeta')\int_T \Big|\pi\big(\zeta',\sqrt{1-|\zeta'|^2}\lambda\big)\Big|^p \frac{|d\lambda|}{2\pi}\\ &\geq \int_{B^{n-1}} |\pi(\zeta',0)|^p\,d\nu_{n-1}(\zeta').\end{aligned} \tag{2.13}$$

If we set $\zeta''=(\zeta_2,\ldots,\zeta_{n-1})$ then by subharmonicity again

$$\int_{|\zeta''|^2<1-|\zeta_1|^2} |\pi(\zeta',0)|^p\,d\nu_{n-2}(\zeta'') \geq (1-|\zeta_1|^2)^{n-2}|\pi(\zeta_1,0,\ldots,0)|^p,$$

so that the last integral of (2.13) is at least

$$\begin{aligned}\int_{|\zeta_1|<1}(1-|\zeta_1|^2)^{n-2}|\pi(\zeta_1,0,\ldots,0)|^p\,d\nu_1(\zeta_1) &= \|\pi\|_\infty^p\int_0^1 2(1-r^2)^{n-2}r^{kp+1}\,dr\\ &\geq C(k+1)^{-(n-1)}\|\pi\|_\infty^p.\end{aligned}$$

The case $n=1$, $p\geq 1$ of Theorem 2 already appeared at [11]. We now turn our attention to H^p case.

THEOREM 3. *Let $0<p\leq 2$, $p<q$, $1\leq q\leq r\leq\infty$, $\delta=n(\frac1p-\frac1r)-(\frac1q-\frac1r)$ and let $\delta_2=n(\frac1p-\frac12)-(\frac1q-\frac12)$. Then there is a C such that*

$$\sum_{m=0}^\infty\Big(\sum_{k\in I_m}\|(k+1)^{-\delta}F_k\|_r^{q'}\Big)^{p/q'} \leq C\|f\|_p^p, \tag{2.14}$$

and

$$\sum_{m=0}^\infty\Bigg[\sum_{k\in I_m}\Big(\frac{\alpha!}{\Gamma(n+|\alpha|)}\frac{|a_\alpha|^2}{(|\alpha|+1)^{2\delta_2}}\Big)^{q'/2}\Bigg]^{p/q'} \leq C\|f\|_p^p, \tag{2.15}$$

for all $f(z)=\sum_{k=0}^\infty F_k(z)=\sum a^\alpha z^\alpha\in H^p$, with the obvious understanding of (2.14) and (2.15) when $q=1$.

PROOF. By an application of Hölder's inequality (fixing p and r) to the quantity on the left hand side of (2.14), we can see we may assume that $q\leq 2$ in proving (2.14). By Theorem B with $\beta=n(\frac1p-\frac1q)$, we have

$$\|f\|_{p,q,p\beta-1}\leq C\|f\|_p. \tag{2.16}$$

By Theorem 2,

$$\sum_{m=0}^\infty\Big(\sum_{k\in I_m}\|(k+1)^{-\delta}F_k\|_r^{q'}\Big)^{p/q'} \leq C\|f\|_{p,q,p\beta-1}^p, \tag{2.17}$$

where $\delta=\frac{\beta+1}{p}+(n-1)(\frac1q-\frac1r)=n(\frac1p-\frac1r)-(\frac1q-\frac1r)$. Combining (2.16) and (2.17) we obtain (2.14). (2.15) is an easy consequence of (2.14) with $r=2$ and [14. Proposition 1.4.9].

Theorem 3 breaks down when $p=q$. We shall see this in the last section. $n=1$ case of (2.14) appeared at [11].

3. **More on H^p coefficients.** We will consider the limiting case, that is, $q \longrightarrow p$, of Theorem 3.

THEOREM 4. *Let* $1 \le p \le 2$, $p \le r \le \infty$, *and* $\beta = (n-1)(\frac{1}{p} - \frac{1}{r})$. *Then there is a* C *such that*

$$\sum_{m=0}^{\infty}\Big(\sum_{k\in I_m} \|(k+1)^{-\beta}F_k\|_r^{p'}\Big)^{2/p'} \le C\|f\|_p^2 \tag{3.1}$$

for all $f = \sum_{k=0}^{\infty} F_k \in H^p(B)$, *with the obvious understanding of the left hand side norm when* $p = 1$.

PROOF. For notational convenience, we prove only when $1 < p \le 2$. Let $Rf(z)$ denote the radial derivative of holomorphic f: $Rf(z) = f(z) + \sum_j z_j \frac{\partial f}{\partial z_j}(z)$, $z \in B$. Then, for a fixed $\zeta \in S$, it follows from the Hausdorff-Young theorem applied to

$$Rf(\lambda\zeta) = \sum_k (k+1)F_k(\zeta)\lambda^k, \quad \lambda \in U,$$

that

$$\int_T |Rf(\lambda\zeta)|^p \frac{|d\lambda|}{2\pi} \ge \Big[\sum_k (k+1)^{p'}|F_k(\rho\zeta)|^{p'}\Big]^{p/p'}, \tag{3.2}$$

where $|\lambda| = \rho$. Now integrate (3.2) with respect to $d\sigma(\zeta)$ and apply the Minkowski's inequality to get

$$\int_S |Rf(\rho\zeta)|^p\, d\sigma(\zeta) \ge \Big[\sum_0^{\infty} (k+1)^{p'}\rho^{kp'}\|F_k\|_p^{p'}\Big]^{p/p'}. \tag{3.3}$$

On the other hand, the g-function

$$g(\zeta) = \Big(\int_0^1 (1-\rho)|Rf(\rho\zeta)|^2\, d\rho\Big)^{\frac{1}{2}}, \quad \zeta \in S, \tag{3.4}$$

satisfies the inequality [1. Theorem 3.1]

$$C\|f\|_p^p \ge \int_S g(\zeta)^p\, d\sigma(\zeta). \tag{3.5}$$

Therefore combining (3.3), (3.4), and (3.5), we have

$$C\|f\|_p^2 \ge \int_0^1 (1-\rho)\Big[\sum_k (k+1)^{p'}\rho^{kp'}\|F_k\|_p^{p'}\Big]^{2/p'}\, d\rho. \tag{3.6}$$

By the same way as in the proof of Theorem 2, the right hand side of (3.6) is at least a constant times

$$\sum_{m=0}^{\infty}\Big[\sum_{k\in I_m} \|F_k\|_p^{p'}\Big]^{2/p'}$$

and, by Lemma, this last quantity is at least a constant multiple of the left side of (3.1). This completes the proof.

When $n = 1$, (3.1) is known as C. N. Kellog's version of the classical Hausdorff-Young inequality [9]. By considering H^p as the dual of $H^{p'}$, $1/p + 1/p' = 1$ (see [4. 1.4] for example), we can easily deduce (by the standard duality argument as in [4. 5.9]) dual results of Theorem 3 and Theorem 4 when $2 \le p < \infty$. Also, Theorem 2 (so that Theorem 1 also) has its dual when $2 \le q \le \infty$. To prove this, first use the Hausdorff-Young theorem to reverse the inequality in (2.11). Then integrate both sides with respect to $d\sigma$ and use the Minkowski's inequality to dominate $M_q(\rho, f)$. Finally, use [12. Theorem 1] to dominate $A^{p,q,\beta}$ norm of f by the left side of (2.10) (with $r \le q \le \infty$).

4. **On Paley sets.** By definition, a set E of nonnegative integers is called a *Paley set* if the cardinality of the set $E_N = E \bigcap \{k : N \le k < 2N\}$ remains bounded as $N \longrightarrow \infty$. P. Ahern and W. Rudin ([3], [4]) fixed a certain type of homogeneous polynomials π and derived Paley type gap theorems of H^p functions on B that cannot happen on U:

THEOREM C [4. THEOREM 3.1 AND THEOREM 4.1]. *Let $1 \le p < 2$. Then the following are equivalent.*

(a) E is a Paley set.

(b) $\sum_{m \in E} \|f_m\|_p^p < \infty$ for every $f \in H^p(B)$, where f_m is the projection of f into the one-dimensional space spanned by π^m, that is,

$$f_m(z) = \left(\int_S f \bar{\pi}^m \, d\sigma\right) \frac{\pi^m}{\|\pi^m\|_2^2}(z), \quad z \in B.$$

We shall see below that this result is no longer true for general setting. Also in connection with this, one may ask if the exponent 2 in Theorem 4 can be improved. Our example shows that (3.1) breaks down when we replace the exponent 2 by a smaller one.

EXAMPLE. There is a sequence $\{P_k\}$ of homogeneous polynomials with $\deg P_k = k$ such that $\|P_k\|_\infty = 1$ and $\|P_k\|_2 \ge 2^{-n}\sqrt{\pi}$ for all k (see [16] or [15. p. 72]), so that if we take $f(z) = \sum_k F_k(z) = \sum_m a_m P_{2^m}(z)$, $z \in B$, with $\{a_m\} \in \ell^2 - \bigcup_{t<2} \ell^t$, then we have, by [16. Proposition 1.6],

$$\|f\|_p \le C\Big(\sum_m |a_m|^2\Big)^{\frac{1}{2}} < \infty,$$

but

$$\sum \|F_{2^m}\|_p^t \ge C \sum_m |a_m|^t = \infty$$

for all $t < 2$.

The following result characterizies Paley sets in the same vein.

THEOREM 5. *Let $0 < p < \infty$, $0 < q < 1$, and $\beta > -1$. Then the following are equivalent.*

(a) E is a Paley set.

(b) There is a C such that

$$\sum_{k\in E} \|(k+1)^{-(\beta+1)/p}F_k\|_1^p \le C\|f\|_{p,1,\beta}^p \tag{4.1}$$

for all $f = \sum F_k \in A^{p,1,\beta}$.

(c) There is a C such that

$$\sum_E \|F_k\|_1^2 \le C\|f\|_1^2$$

for all $f = \sum_{k=0}^{\infty} F_k \in H^1(B)$.

(d) There is a C such that

$$\sum_E \|(k+1)^{1-1/q}F_k\|_q^q \le C\|f\|_q^q$$

for all $f = \sum_{k=0}^{\infty} F_k \in H^q(B)$.

PROOF (a) $\Rightarrow$ (b). Suppose E is a Paley set. Then the cardinal number of $E\cap I_m$, $|E\cap I_m|$, remains bounded as $m \longrightarrow \infty$. Since

$$\sum_{k\in E} \|(k+1)^{-(\beta+1)/p}F_k\|_1^p = \sum_m \sum_{k\in E\cap I_m} \|(k+1)^{-(\beta+1)/p}F_k\|_1^p$$
$$\le \sup_m |E\cap I_m| \sum_m \sup_{k\in I_m} \|(k+1)^{-(\beta+1)/p}F_k\|_1^p,$$

follows (b) from Theorem 2.

(b) $\Rightarrow$ (a). Let us fix $\rho : 0 < \rho < 1$ and let

$$f(z) = (1-\rho z_1)^{-\gamma}, \quad \gamma = \frac{\beta+2}{p} + n, \quad z \in B.$$

Then by [14. Proposition 1.4.10]

$$\int_S |f(r\zeta)|\, d\sigma(\zeta) = O(1-\rho r)^{-(\beta+2)/p},$$

so that

$$\|f\|_{p,1,\beta}^p = O(1-\rho)^{-1}. \tag{4.2}$$

On the other hand,

$$f(z) = \sum F_k(z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+\gamma)}{\Gamma(\gamma)\Gamma(k+1)} \rho^k z_1^k,$$

so that, by the Stirling's formula,

$$\|(k+1)^{-(\beta+1)/p}F_k\|_1 \sim k^{n-1+1/p}\rho^k \|z_1^k\|_1. \tag{4.3}$$

Since $\|z_1^k\|_1 \sim (k+1)^{-(n-1)}$ (see, for example [4]), the last quantity of (4.3) is of $O(k^{1/p}\rho^k)$. Now fix a large enough N and set $\rho = 1 - \frac{1}{N}$ then it follows from (4.2), (4.3) and the hypothesis (4.1) that

$$N|E_N| = O(N).$$

Therefore E is a Paley set.

The proof that (a) $\Longleftrightarrow$ (c) and (a) $\Longleftrightarrow$ (d) is almost same to what we have just proven by the aid of Theorem 4 and (2.14) respectively. We omit rather obvious imitations.

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