ON COMPACT PRIME RINGS AND THEIR RINGS OF QUOTIENTS

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1. In [10], it is defined that a right (or left) ideal I of a ring R is very large if the cardinality of R/I is finite. It is also proven in [10, Theorem 3.4] that if R is a prime ring with 1 such that its characteristic is zero, then R is a right order in a simple ring with the minimum condition on one sided ideals if every large right ideal of R is very large. In the present note, we shall prove that if R is a prime ring with 1 such that its characteristic is zero and R is also a compact topological ring, then R is a right and left order in a simple ring with the minimum condition on one sided ideals, which is also a non-discrete locally compact topological ring if and only if every large right ideal of R is open. In particular, if R is an integral domain with 1 (not necessarily commutative) such that its characteristic is zero, then R is <u>openly embeddable</u> [13, p. 58] in a locally compact (topological) division ring if and only if every large right ideal of R is open. Following S. Warner [13], we shall say R is openly embeddable in a quotient ring Q(R) if there is a topology on Q(R) which is compatible with its structure, which induces on R its given topology and for which R is an open subset.

2. Let R be a compact ring which is also a prime ring. If C is the component of 0 then $R \cdot C = 0$ by [7, Theorem 8, p. 161]. Since R is a prime ring, this implies that $C = \{0\}$. Hence by [3, (7.7) Theorem, p. 62] and by [7, Lemma 9, p. 160], R has a system of compact ideal neighbourhoods of 0. We shall use this non-elementary fact concerning compact rings in this paper. Consider R as a right regular R-module over itself. An R-homomorphism of a large right ideal M of R into R is called a semi-endomorphism of R_R [1] or [5].

LEMMA 2.1. <u>Let</u> R <u>be a compact prime ring</u>. <u>If every large right ideal of</u> R <u>is open then every semi-endomorphism</u> α <u>of</u> R_R is continuous.

<u>Proof</u>. Let M_α be a large right ideal on which α is defined. Let U(0) be an open set containing $\alpha(0) = 0$. Then there exists a two sided ideal S which is open such that $S \subseteq U(0) \cap M_\alpha$. If R is not a discrete space, then $S \neq (0)$ and $S^2 \neq (0)$ since R is a prime ring.

Now $f(S^2) \subseteq RS$. For if $x \in f(S^2)$ then $x = f(\Sigma a_i b_i) = \Sigma f(a_i)b_i$ for some $a_i, b_i \in S$, $i = 1, 2, \ldots, n$, and $x \in RS$. Hence $f(S^2) \subseteq RS \subseteq S \subseteq U(0)$. Since S^2 is a non-zero two sided ideal of a prime ring R, it is a large right ideal and hence it is an open set. Thus α is continuous at the point 0. Since the additive group of R is a topological group, this implies that α is continuous.

If every large right ideal of the non-discrete compact ring R is open, then every large right ideal I of R contains a non-zero ideal, say S. Since $aS \neq 0$ for every non-zero $a \in R$, $aI \neq 0$ and the right singular ideal of R is zero immediately. We also note that I is very large since R/I is a discrete compact group [3, (5.22), p. 38].

LEMMA 2.2. Let R be a compact prime ring such that every large right ideal of R is open. If a maximal right quotient ring Q(R) of a ring R is topologized by declaring a fundamental system of neighbourhoods of zero in Q(R) to be all neighbourhoods of zero in R, then Q(R) is a locally compact topological group under addition, multiplication is continuous at (0, 0), multiplication on the right by a is continuous at zero for each $a \in R$, and multiplication on the left by q is continuous at zero for each $q \in Q(R)$.

Proof. The assertions that Q(R) is a locally compact topological group under addition, multiplication is continuous at (0, 0) and the multiplication on the right by any element a in R follow at once from the definition of a topological ring. So it remains to show that multiplication on the left by q is continuous at zero for each $q \in Q(R)$. Let $q, x \in Q(R)$ and S_α be an open ideal in R. Let $Mq = \{r \in R \mid qr \in R\}$. Then Mq is a large right ideal of R [5], and it is open. By Lemma 2.1, there is an open set U such that $U \subseteq Mq \cap S_\alpha$ and $q(U) \subseteq S_\alpha$. U contains an open ideal of R, say S and $q(x + S) \subseteq qx + S_\alpha$. Thus the left multiplication determined by q is continuous.

COROLLARY. Let R be a compact prime ring such that every large right ideal of R is open. If the center of R is infinite then Q(R) is a finite dimensional topological vector space over its center.

Proof. Let F be the center of Q(R). Since Q(R) is a prime ring every non-zero element of F is a right and left regular element. Since Q(R) is a regular ring, this implies that F is a field. By [2, Proposition 7.1], the center of R is contained in F. Since the center of R is compact and infinite, F is not a discrete space as a subspace of Q(R) with the topology defined in Lemma 2.2. If $y \in R$, let $F_y = \{x \in Q(R) \mid yx = xy\}$. Let $L_y(x) = yx$ and $T_y(x) = xy$ ($\forall x \in Q(R)$). Then L_y, T_y are endomorphisms of the additive topological group Q(R) that are continuous at zero and hence everywhere. Consequently the set F_y where they coincide is closed, so the intersection of all the F_y's, $y \in R$, is also closed. Now by [2, Proposition 7.1], $F = \bigcap_{y \in R} F_y$. Thus F is a closed subset of $Q(R)$. Hence F is a non-discrete locally compact topological group with respect to $+$. Let $a \in F$ and $q \in Q(R)$. Let Σ^* be a system of ideal neighbourhoods of 0 and let $S_\alpha \in \Sigma^*$. Let $S_1 \in \Sigma^*$ such that $S_1 \subseteq S_\alpha$ and $qS_1 \subseteq S_\alpha$. Let $S_2 \in \Sigma^*$ such that $S_2 \subseteq S_1$ and $aS_2 \subseteq S_\alpha$.

Then

$$(q + S_2)(a + S_2) \subseteq qa + S_\alpha$$

since

$$(q + s_2)(a + s_2') = qa + qs_2' + s_2 a + s_2 s_2' = qa + qs_2' + as_2$$

$$+ s_2 s_2' \in qa + S_\alpha \text{ for all } s_2, s_2' \in S_2 .$$

This implies that $Q(R)$ is a topological space over F. Since $F \subseteq Q(R)$, the above proof also shows that $(a, b) \to ab$ from $F \times F$ into F is continuous. Hence by [8, Theorem 9] or by [12], $a \to a^{-1}$ is continuous for all non-zero $a \in F$. Thus F is a topological division ring and by [8, Lemma 9], $Q(R)$ is finite dimensional.

LEMMA 2.3. Let R be a compact prime ring with 1 such that its characteristic is zero. If every large right ideal of R is open then R is openly embeddable in a maximal right quotient prime regular ring $Q(R)$ which is also a left quotient ring of R.

Proof. By Lemma 2.2, all that needs to be shown is that multiplication on the right by q, $q \in Q(R)$, is continuous at zero. By [3, (4, 4), p. 17] and [3, (5.22), p. 38], $(qR + R)/R$ is finite. Hence $nqR \subseteq R$ for some integer n, whence $nq \in R$. Now, multiplication on the right by nq is continuous at zero by Lemma 2.2. Therefore, if S is a neighbourhood of zero, there is an open ideal S' such that $S'nq \subseteq S$; then $S'n$ is a non-zero ideal, hence open, and $(S'n)q \subseteq S$. Thus multiplication on the right by q is continuous at zero. $Q(R)$ being a prime regular ring follows from [5, Theorem 3] and [6, 2.7, p. 1388]. Now if $q \in Q(R)$ such that $q \neq 0$ then $Rq \neq (0)$. For if $Rq = (0)$, then the set $(R)^r = \{q \in Q(R) \mid Rq = (0)\}$ is a non-zero right ideal of $Q(R)$ and $Q(R) \cap R \neq (0)$. Hence Rq is a non-zero compact set since multiplication by q on the right is continuous and R is compact. As before, $(Rq + R)/R$ is a finite set and $rqn = rnq \in R$ for some integer n and for any $r \in R$. Thus $Q(R)$ is also a left quotient ring of R.

COROLLARY. *Let* R *be a compact prime ring with* 1 *such that the characteristic of* R *is zero. If every large right ideal of* R *is open then* Q(R) *is a finite dimensional topological algebra over its center.*

Proof. Since $1 \in R$ and the characteristic of R is zero, the center of R is not a finite set. Hence by Corollary of Lemma 2.2 and Lemma 2.3, Q(R) is a finite dimensional topological algebra over its center.

THEOREM 2.4. *Let* R *be a compact prime ring with zero characteristic such that* R *contains* 1. *Then* R *is openly embeddable in a locally compact ring* Q(R) *which is a simple ring with the minimum conditions on one sided ideals if and only if every large right ideal of* R *is open.* R *is also a right and left order in* Q(R) *if every large right ideal of* R *is open.*

Proof. Assume that every large right ideal of R is open. By Lemma 2.3, R is openly embeddable in Q(R) which is a prime regular ring with 1. Since every large right ideal of R is open, every large right ideal of R is very large. Thus by [10, Theorem 3.4], Q(R) is a simple ring with the minimum condition on one sided ideals. If Q(R) is a classical ring of right quotients of R by [6, 4.2, p. 1391] and hence every large right ideal of R contains a regular element [6, p. 1390] or [4, Lemma 8, p. 267]. Let I be a large right ideal of R and let a be a regular element in I. Then a is a unit of Q(R) by [11, Lemma 1, p. 110]. Furthermore, if $q \in Q(R)$ and $q \neq 0$ then there exists positive integers m and n such that $qma \neq 0 \in R$ and $naq \neq 0 \in R$. Since the characteristic of R is zero, ma and na are regular elements and they are units in Q(R). Thus R is a right and left order in Q(R). The converse statement follows from the facts that I contains a regular element, say a, which is a unit of Q(R) by [11, Lemma 1, p. 110] and hence the left multiplication by a is a homeomorphism.

COROLLARY A. *If* R *is a compact integral domain with zero characteristics, then* R *is openly embeddable in a locally compact division ring* Q(R) *if and only if every large right ideal of* R *is open.*

Proof. Assume every large right ideal of R is open. Then Q(R) is a simple ring with the minimum condition on one sided ideals by Theorem 2.4. Hence by Goldie's Theorem [4, p. 270], R is a right Ore domain and thus Q(R) is a division ring. Since Q(R) is a locally compact space and $(x, y) \to xy$ of $Q(R) \times Q(R)$ into Q(R) is continuous, by [8, Theorem 9] $x \to x^{-1}$ is continuous for all non-zero $x \in Q(R)$. The converse follows from Theorem 2.4.

COROLLARY B. *Let* R *be a compact integral domain with zero characteristic. If every large right ideal of* R *is open then* R *is an Ore domain.*

Proof. By Corollary A, Q(R) is a topological division ring and R is an open subset of Q(R). Hence if a, b are non-zero elements of R then aR and bR are open sets. Hence $aR \cap bR \neq 0$ unless R is a finite ring. Similarly, $Ra \cap Rb \neq 0$.

Remark. We note here that if R is a compact integral domain with zero characteristic such that every large right ideal is open, then R admits a valuation which preserves the topology. This follows from Corollary A and [9, Theorem 8].

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