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THE CLASS $L(\log L)^\alpha$ AND SOME LACUNARY SETS

SANJIV KUMAR GUPTA, SHOBHA MADAN and U. B. TEWARI

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Abstract

A well-known result of Zygmund states that if $f \in L\left(\log^+ L\right)^{1/2}$ on the circle group $\mathbb{T}$ and E is a Hadamard set of integers, then $\hat{f}\mid_E \in \ell_2(E)$. In this paper we investigate similar results for the classes $B_\alpha = L\left(\log^+ L\right)^\alpha$, $\alpha > 0$ on an arbitrary infinite compact abelian group G and Sidon subsets E of the dual Γ. These results are obtained as special cases of more general results concerning a new class of lacunary sets $S_{\alpha,\beta}$, $0 < \alpha \le \beta$, where a subset E of Γ is an $S_{\alpha,\beta}$ set if $\hat{B}_\alpha \mid_E \subseteq \ell_{2\beta/\alpha}(E)$. We also prove partial results on the distinctness of the $S_{\alpha,\beta}$ sets in the index β.

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1. Introduction

Zygmund ([6, Ch. XII, 7.6]) proved that if f is a function on the circle group $\mathbb{T}$ such that $|f|\left(\log^+|f|\right)^{1/2} \in L^1(\mathbb{T})$ and E is a Hadamard set of positive integers then $\sum_{n\in E}\left|\hat{f}(n)\right|^2 < \infty$. Hewitt and Ross ([2, p. 446]) pointed out that this phenomenon has not been explored for Sidon sets and groups other than $\mathbb{T}$. In this paper we investigate this and prove the following generalization of Zygmund's result:

Let G be a compact abelian group and let Γ be its dual group. Let

$$B_\alpha = \left\{f : |f|\left(\log^+|f|\right)^\alpha \in L^1(G)\right\}, \quad \alpha > 0.$$

If E is a Sidon subset of Γ and $0 < \alpha \le 1/2$ then $\hat{B}_\alpha\mid_E \subseteq \ell_{1/\alpha}(E)$ and there exists a Sidon subset E of Γ such that $\hat{B}_\alpha\mid_E \not\subseteq \ell_r(E)$ for $r < 1/\alpha$ (Corollary 4.3).

We then use this result to derive some results about multiplier spaces of certain subspaces of L^1.

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The background to this paper was an investigation into the existence of sets E satisfying $\hat{B}_\alpha\mid_E \subseteq \ell_{1/\alpha}(E)$ but which were not Sidon. Our initial example was $E_k = E + E + \cdots + E$ (k times), where $E = \{2^k : k \in \mathbb{N}\}$. For $0 < \alpha \leq k/2$, $\hat{B}_\alpha\mid_{E_k} \subseteq \ell_{k/\alpha}(E_k)$ but $\hat{B}_\alpha\mid_{E_k} \not\subseteq \ell_r(E_k)$, $r < k/\alpha$. Also E_k is not a Sidon set. (Note that $F + F$ is never Sidon when F is an infinite set.) These considerations led us to define and study a new class of lacunary sets which we call $S_{\alpha,\beta}$ sets:

DEFINITION. A subset $E \subseteq \Gamma$ is called an $S_{\alpha,\beta}$ set if $\hat{B}_\alpha\mid_E \subseteq \ell_{2\beta/\alpha}(E)$, $0 < \alpha \leq \beta$.

In view of this definition, the above mentioned result of Zygmund states that a Hadamard set of positive integers is an $S_{1/2,1/2}$ set and our generalization of the result states that a Sidon subset of Γ is an $S_{\alpha,1/2}$ set, where $0 < \alpha \leq 1/2$.

In Section 3, we give a characterization of $S_{\alpha,\beta}$ sets in Theorem 3.3. This is the main result of that section. As a corollary, we get the following new characterization of Sidon-sets:

THEOREM. *A subset $E \subseteq \Gamma$ is a Sidon set if and only if $\hat{B}_{1/2}\mid_E \subseteq \ell_2(E)$.*

Furthermore, we use Theorem 3.3 to give some examples of $S_{\alpha,\beta}$ sets. We have also included some applications to certain multiplier problems.

In Section 4, we provide a partial answer to the problem of deciding whether the class of $S_{\alpha,\beta}$ sets are distinct for distinct indices β. We prove that for each $k \in \mathbb{N}$, there exists a subset $E \subseteq \Gamma$ which is an $S_{\alpha,k/2}$ set for $0 < \alpha \leq k/2$, but not an $S_{\alpha,\beta}$ set for $0 < \alpha \leq \beta < k/2$. This is a consequence of Theorem 4.1, whose proof takes up all of Section 4. We have not been able to prove the distinctness of $S_{\alpha,\beta}$ sets in the index α.

2. Notation and Terminology

Throughout G will be an infinite compact abelian group and Γ will denote its dual group. The following definitions and results from Krasnosel'skii [3] will be helpful in dealing with the spaces B_α mentioned in the introduction.

DEFINITION. A function ϕ defined on $[0, \infty)$ is said to be a *Young's function* if it is increasing, continuous, convex, and satisfies

$$(1) \qquad \lim_{t\to 0} \frac{\phi(t)}{t} = 0 \quad \text{and}$$

$$(2) \qquad \lim_{t\to \infty} \frac{\phi(t)}{t} = \infty.$$

Let ϕ be a Young's function. If f is a measurable function on G, we define

$$N_\phi(f) = \int_G \phi(|f(x)|)\, dx,$$

where $N_\phi(f)$ is defined to be ∞ if $\phi \circ |f|$ is not integrable. The *Orlicz space* $L^\phi(G)$ is defined as follows:

$L^\phi(G) = \{f : f \text{ measurable on } G \text{ with } N_\phi(\lambda f) < \infty \text{ for some } \lambda > 0\}$.
For $f \in L^\phi$, we define

$$\|f\|_{L^\phi} = \inf_{\lambda > 0} 1/\lambda \left\{N_\phi(\lambda f) + 1\right\}.$$

Then $(L^\phi, \|\cdot\|_\phi)$ is a Banach space.

Two Young's functions ϕ_1 and ϕ_2 are *equivalent* if there exist positive constants k_1, k_2 such that

$$\phi_1(k_1 t) \le \phi_2(t) \le \phi_1(k_2 t), \quad t \ge t_0.$$

If ϕ_1 and ϕ_2 are two equivalent Young's functions then $L^{\phi_1} = L^{\phi_2}$ and the norms defined by ϕ_1 and ϕ_2 are equivalent.

We say that a Young's function ϕ satisfies the Δ_2-*condition* if there exists a constant $C > 0$ and $t_0 \ge 0$ such that

$$\phi(2t) \le C\phi(t) \quad \text{for all } t \ge t_0.$$

If ϕ satisfies the Δ_2-condition then the dual of L^ϕ is isomorphic to an Orlicz function space L^ψ, where ψ is also a Young's function given by $\psi(s) = \sup_{t \ge 0}(st - \phi(t))$, $s \ge 0$.

We shall be particularly concerned with the following Young's function: Letting $\alpha > 0$, we define

$$\phi_\alpha(t) = \begin{cases} t^{1+\alpha}/e^\alpha, & 0 \le t \le e, \\ t(\log t)^\alpha, & t > e. \end{cases}$$

It is easy to see that ϕ_α is a Young's function satisfying the Δ_2-condition. In fact,

$$\phi_\alpha(2t) \le 2^{1+\alpha}\phi_\alpha(t) \quad \text{for all} \quad t \ge e.$$

The spaces B_α mentioned in the introduction are nothing but L^{ϕ_α}. It is not difficult to show that the dual space $(B_\alpha)^*$ is given by L^{ψ_α}, where

$$\psi_\alpha(t) = \begin{cases} e^{2\alpha}t^2/(2\alpha)^{2\alpha}, & 0 \le t \le (2\alpha)^\alpha, \\ e^{t^{1/\alpha}}, & t > (2\alpha)^\alpha. \end{cases}$$

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If E is a subset of Γ and $S \subseteq L^1(G)$, then we define

$$S_E = \left\{ f \in S : \hat{f} = 0 \text{ outside } E \right\}.$$

T will denote the space of trigonometric polynomials on G. We write $|E|$ for the cardinality of a set E. In the following, $C, C_1, C_2, \ldots$ are constants which may vary from one line to the next. For other notation we refer to Hewitt and Ross [1, 2] and Lopez and Ross [4].

Section 3

Let us recall that a subset $E \subseteq \Gamma$ is called an $S_{\alpha,\beta}$ set if $\hat{B}_\alpha\,|_E \subseteq \ell_{2\beta/\alpha}(E)$, where $0 < \alpha \le \beta$. In this section we prove the main theorem of this paper giving a characterization of $S_{\alpha,\beta}$ sets. First we prove two simple lemmas: Lemma 3.1 below will be needed in the proof of Theorem 3.3 and we use Lemma 3.2 to construct examples of $S_{\alpha,\beta}$ sets.

LEMMA 3.1. *Let E be a subset of Γ, $\alpha > 0$ and $1 < r \le \infty$. Then $\hat{B}_\alpha\,|_E \subseteq \ell_r(E)$ if and only if $\ell_{r'}(E) \subseteq \left(B_\alpha^*\right)\hat{}\,|_E$, where $1/r + 1/r' = 1$.*

PROOF. If $r = \infty$, the result is obvious, so assume $r < \infty$. Suppose $\hat{B}_\alpha\,|_E \subseteq \ell_r(E)$. Then by the closed graph theorem, there exists a constant $C > 0$ such that

$$\left\| \hat{f}\,|_E \right\|_{\ell_r(E)} \le C \left\| f \right\|_{B_\alpha}, \quad \forall f \in B_\alpha.$$

Given $\phi \in \ell_{r'}(E)$, we define a linear functional on B_α by

$$K_\phi(f) = \sum_{\gamma \in E} \phi(\gamma) \hat{f}(\gamma).$$

Then

$$\left| K_\phi(f) \right| \le \left\| \phi \right\|_{\ell_{r'}(E)} \left\| \hat{f}\,|_E \right\|_{\ell_r(E)} \le C \left\| \phi \right\|_{\ell_{r'}(E)} \left\| f \right\|_{B_\alpha}, \quad \forall f \in B_\alpha.$$

It follows that for some $g \in B_\alpha^*$,

$$K_\phi(f) = \int_G f(x) g(-x)\, dx, \quad \forall f \in B_\alpha.$$

In particular, taking $f(x) = \gamma(x)$, $\gamma \in E$ we get $\hat{g}(\gamma) = \phi(\gamma)$. Hence $\phi \in \left(B_\alpha^*\right)\hat{}\,|_E$.

Conversely, suppose $\ell_{r'}(E) \subseteq (B_\alpha^*)\hat{}$. Again, by the closed graph theorem, there exists a constant $C > 0$ such that

$$\|g\|_{B_\alpha^*} \le C\|\hat{g}\|_{\ell_{r'}(E)}, \quad \forall \hat{g} \in \ell_{r'}(E).$$

Now let $f \in B_\alpha$ and put $\phi = \hat{f}\,|_E$. Define a linear functional on the class of functions on Γ with finite support by

$$K_\phi(\psi) = \sum_{\gamma \in E} \phi(\gamma)\psi(\gamma).$$

For each such ψ, there exists a trigonometric polynomial g with $\hat{g}(\gamma) = \psi(\gamma)$, so that

$$K_\phi(f) = \int_G f(x)g(-x)\,dx$$

and

$$|K_\phi(\psi)| \le \|f\|_{B_\alpha}\|g\|_{B_\alpha^*} \le C\|f\|_{B_\alpha}\|\hat{g}\|_{\ell_{r'}(E)} = C\|f\|_{B_\alpha}\|\psi\|_{\ell_{r'}(E)}.$$

Hence K_ϕ extends to a continuous linear functional on $\ell_{r'}(E)$ and so $\phi \in \ell_r(E)$. The following lemma is essentially an interpolation result.

LEMMA 3.2. *Let E be a subset of Γ, $\beta > 0$ and $1 \le p < \infty$. Suppose there exists a constant $C > 0$ such that*

$$\|f\|_{L^p} \le Cp^\beta\|\hat{f}\|_{\ell_2}, \quad \forall f \in T_E.$$

Then if $0 < \alpha \le \beta$ and $q = p\beta/\alpha$, there exists a constant $C_{\alpha,\beta} > 0$ such that

$$\|f\|_{L^q} \le C_{\alpha,\beta}q^\alpha\|\hat{f}\|_{\ell_r}, \quad \forall f \in T_E,$$

where $r = 2\beta/(2\beta - \alpha)$.

PROOF. Consider the linear map U, defined on $\hat{T}_E$ by $U\hat{f} = f$, $\forall f \in T_E$. The hypothesis implies that U extends to a bounded linear map from $\ell_2(E)$ to L_E^p with norm at most Cp^β. Clearly U also extends to a bounded linear map from $\ell_1(E)$ to L_E^∞ with norm at most 1. If $q = p\beta/\alpha$, let $\delta = 1 - \alpha/\beta$. Then $0 \le \delta < 1$ and $\delta/1 + (1-\delta)/2 = (2\beta - \alpha)/2\beta = 1/r$. By the Riesz-Thorin Convexity Theorem, U extends to a bounded linear map from $\ell_r(E)$ to L_E^q with norm at most

$$(Cp^\beta)^{\alpha/\beta} = C^{\alpha/\beta}(\alpha/\beta)^\alpha q^\alpha = C_{\alpha,\beta}q^\alpha.$$

This completes the proof of the lemma.

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THEOREM 3.3. *Let G be a compact abelian group and Γ its dual group. Let $E \subseteq \Gamma$ and $0 < \alpha \leq \beta$. Then E is an $S_{\alpha,\beta}$ set if and only if there exists a constant C depending only on α and β and not on q such that*

$$\|f\|_{L^q} \leq Cq^\alpha \|\hat{f}\|_{\ell_{r'}} \quad \forall f \in T_E, \quad \forall q \geq 2\beta/\alpha \tag{3.4}$$

where $r = 2\beta/\alpha$ (and thus $r' = 2\beta/(2\beta - \alpha)$).

PROOF. SUFFICIENCY From Lemma 3.1, E is an $S_{\alpha,\beta}$ set if and only if $\ell_{r'}(E) \subseteq (B_\alpha^*)\hat{}|_E$.

Now suppose $E \subseteq \Gamma$ and (3.4) holds. Let $\phi \in \ell_{r'}(E)$. Since $2 \leq r < \infty$, we have $1 < r' \leq 2$ and $\ell_{r'}(E) \subseteq \ell_2(E)$. Hence there exists $f \in L^2(G)$ such that $\hat{f}(\gamma) = \phi(\gamma)$ if $\gamma \in E$, and $\hat{f}(\gamma) = 0$ if $\gamma \notin E$. We claim that $f \in B_\alpha^*$. Let $\lambda > 0$ and consider

$$\int_G \exp\left(\lambda^{1/\alpha} |f(x)|^{1/\alpha}\right) dx = \sum_{k=0}^{\infty} \frac{\lambda^{k/\alpha}}{k!} \|f\|_{L^{k/\alpha}}^{k/\alpha} = \left(\sum_{k \leq 2\beta} + \sum_{k > 2\beta}\right) \frac{\lambda^{k/\alpha}}{k!} \|f\|_{L^{k/\alpha}}^{k/\alpha}.$$

Using (3.4) for $q = k/\alpha$ in the second summation, we get

$$\begin{aligned}\sum_{k>2\beta} \frac{\lambda^{k/\alpha}}{k!} \|f\|_{L^{k/\alpha}}^{k/\alpha} &\leq \sum_{k>2\beta} \frac{\lambda^{k/\alpha}}{k!} \left[C(k/\alpha)^\alpha \|\hat{f}\|_{\ell_{r'}}\right]^{k/\alpha} = \sum_{k>2\beta} C^{k/\alpha} \frac{\lambda^{k/\alpha}}{\alpha^k} \frac{k^k}{k!} \|\hat{f}\|_{\ell_{r'}}^{k/\alpha} \\ &\leq \sum_{k>2\beta} \left[C^{1/\alpha} \frac{\lambda^{1/\alpha}}{\alpha} e \|\hat{f}\|_{\ell_{r'}}^{1/\alpha}\right]^k\end{aligned}$$

which is finite for a suitable choice of λ, that is, λ such that the expression in the square bracket is < 1.

NECESSITY First we note that (3.4) is equivalent to saying that there exists a constant C_1 depending only on α and β such that

$$\|f\|_{L^{k/\alpha}} \leq C_1(k/\alpha)^\alpha \|\hat{f}\|_{\ell_{r'}}, \quad \forall f \in T_E, \quad k \in \mathbb{N} \quad \text{and} \quad k \geq 2\beta. \tag{3.5}$$

Clearly (3.4) implies (3.5). On the other hand, if (3.5) holds and $q \geq 2\beta/\alpha$, let m be the unique integer such that $(m-1)/\alpha < q \leq m/\alpha$. Then

$$\begin{aligned}\|f\|_{L^q} \leq \|f\|_{L^{m/\alpha}} &\leq C_1(m/\alpha)^\alpha \|\hat{f}\|_{\ell_{r'}} \leq C_1(q + 1/\alpha)^\alpha \|\hat{f}\|_{\ell_{r'}} \\ &\leq C_1(1 + 1/2\beta)^\alpha q^\alpha \|\hat{f}\|_{\ell_{r'}} = Cq^\alpha \|\hat{f}\|_{\ell_{r'}}.\end{aligned}$$

Now suppose $E \subseteq \Gamma$ is an $S_{\alpha,\beta}$ set and (3.4), or equivalently (3.5), does not hold. Then for each $n \in \mathbb{N}$, there exists $f_n \in T_E$ and an integer $k_n \geq 2\beta$ such that if $q_n = k_n/\alpha$, we have

$$\|f_n\|_{L^{q_n}} > n(q_n)^\alpha \|\hat{f}_n\|_{\ell_{r'}}.$$

Let $g_n = f_n/\|\hat{f}_n\|_{\ell_{r'}}$. Then $\|g_n\|_{L^{q_n}} \geq C_\alpha n k_n^\alpha$. We now estimate the norm $\|g_n\|_{B_\alpha^*}$. From the definition of the B_α^* norm, there exists $\lambda_n > 0$ such that

$$\|g_n\|_{B_\alpha^*} + C_n > \frac{1}{\lambda_n}\left[1 + \int_G \exp\left(\lambda_n^{1/\alpha}\,|g_n(x)|^{1/\alpha}\right)dx\right]$$

where $0 < C_n \leq e^{2\alpha}$.

If some subsequence of $\{\lambda_n\}$ tends to zero, we get a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\|g_{n_k}\|_{B_\alpha^*} \longrightarrow \infty$.

If not, then there exists a $\delta > 0$ such that $\lambda_n \geq \delta\ \forall n$. Then

$$\begin{aligned}\|g_n\|_{B_\alpha^*} + C_n &> \frac{1}{\lambda_n}\int_G \exp\left(\lambda_n^{1/\alpha}\,|g_n(x)|^{1/\alpha}\right)dx \\ &\geq \frac{1}{\lambda_n}\frac{\lambda_n^{q_n}}{k_n!}\|g_n\|_{L^{q_n}}^{q_n} \geq \frac{\lambda_n^{(q_n)-1}}{k_n!}(C_\alpha n k_n^\alpha)^{q_n} \\ &\geq \frac{1}{\delta}(\delta C_\alpha n)^{q_n}\frac{(k_n^{k_n})}{k_n!} \to \infty \quad \text{as} \quad n \to \infty.\end{aligned}$$

Hence $\ell_{r'}(E) \not\subseteq (B_\alpha^*)\hat{}\,|_E$, so that by Lemma 3.1, E cannot be an $S_{\alpha,\beta}$ set. This completes the proof of the theorem.

As a consequence of Theorem 3.3 we get the following characterization of Sidon subsets of Γ.

COROLLARY 3.6. *Let E be a subset of Γ. Then E is a Sidon set if and only if $\hat{B}_{1/2}|_E \subseteq \ell_2(E)$.*

PROOF. Let E be a Sidon subset of Γ. Then as is well known ([4, p. 59]), we have

$$(*) \qquad \|f\|_{L^p} \leq C p^{1/2}\|f\|_{\ell^2}, \quad \forall f \in T_E, \quad \forall p,\ 2 < p < \infty,$$

and by Theorem 3.3, this is equivalent to $\hat{B}_{1/2}|_E \subseteq \ell_2(E)$. For the converse, Pisier [5] has shown that every $E \subseteq \Gamma$ satisfying (*) is a Sidon set.

3.7 EXAMPLES OF $S_{\alpha,\beta}$ SETS.

(i) If $E \subseteq \Gamma$ is a Sidon set and $1 < p < \infty$, then

$$\begin{aligned}\|f\|_{L^p} &\le Cp^{1/2}\|\hat{f}\|_{\ell_2}, && \forall f \in T_E,\\ &\le Cp^{\beta}\|\hat{f}\|_{\ell_2}, && \text{if} \quad \beta \ge 1/2.\end{aligned}$$

Therefore by Lemma 3.2 and Theorem 3.3, E is an $S_{\alpha,\beta}$ set for all $\beta \ge 1/2$ and $0 < \alpha \le \beta$.

(ii) Recall ([4, p. 15]) that a subset $E \subseteq \Gamma$ is called an *asymmetric set* if $1 \notin E$ and whenever $\gamma \in E$ with $\gamma^2 \ne 1$, then $\gamma^{-1} \notin E$. We call E a *dissociate set* if $1 \notin E$ and for every finite subset $F \subseteq E$ and mapping $m : F \longrightarrow \{0, \pm 1, \pm 2\}$ such that $\prod_{\gamma \in F} \gamma^{m(\gamma)} = 1$, we have $\gamma^{m(\gamma)} = 1, \quad \forall \gamma \in F$.

If G is an infinite compact abelian group, then Γ always contains infinite dissociate sets ([4, p. 21]).

Now let E be an infinite dissociate set and $k \in N$. Define

$$E_k = \left\{ \prod_{\gamma \in S} \gamma : S \text{ is an asymmetric subset of } E \cup E^{-1} \text{ with } |S| = k \right\}.$$

Then ([4, p. 65]) there exists a constant $A_k > 0$ such that

$$\|f\|_{L^q} \le A_k q^{k/2}\|\hat{f}\|_{\ell_2}, \quad \forall f \in T_{E_k}, \quad 2 < q < \infty.$$

Therefore, by Lemma 3.2 and Theorem 3.3, E_k is an $S_{\alpha,\beta}$ set for $\beta \ge k/2$ and $0 < \alpha \le \beta$.

(iii) The examples of $S_{\alpha,\beta}$ sets given above require $\beta \ge 1/2$. We now show that for $\beta < 1/2$, $S_{\alpha,\beta}$ sets need not exist.

Let $G = \prod_A Z_p$, where A is an infinite index set and $\Gamma = \prod_A^* Z_p$ its dual group. Suppose $E \subseteq \Gamma$ is an infinite subset and F a finite subset of E. The subgroup H of Γ generated by F has cardinality at most $p^{|F|}$. Let $V = H^{\perp}$. Then V is an open subgroup of G, so $m(V) > 0$.

Put $h = \chi_V / m(V)$. Then $\hat{h} = \chi_H$ and so $\|\hat{h}\,|_E\|_{\ell_r} \ge |F|^{1/r}$, $1 \le r < \infty$.

Next we estimate $\|h\|_{B_\alpha}$. By the Plancherel theorem,

$$\|h\|_{L^2}^2 = 1/m(V) = \|\hat{h}\|_{L^2}^2 = |H|,$$

so that $m(V)\,|H| = 1$. Therefore

$$\begin{aligned}\|h\|_{B_\alpha} &\le 1 + e + \int_G |h(x)|\,(\log^+ |h(x)|)^{\alpha} dx \le 4 + \frac{1}{m(V)} \int_V (\log^+ |H|)^{\alpha} dx\\ &= 4 + (\log^+ |H|)^{\alpha} \le C(\log^+ |H|)^{\alpha} = C\,|F|^{\alpha}\,(\log p)^{\alpha}.\end{aligned}$$

Taking $\beta < 1/2$ and $r = 2\beta/\alpha < 1/\alpha$, we have

$$\left\|\hat{h}\,|_{E}\right\|_{\ell_r} / \|h\|_{B_\alpha} \to \infty \quad \text{as} \quad |F| \to \infty.$$

Hence $\hat{B}_\alpha\,|_E \not\subseteq \ell_r(E)$, which proves that E is not an $S_{\alpha,\beta}$ set.

Now we include an application of the preceding results to some multiplier problems. For the following definition we refer to Hewitt and Ross ([2, p. 368]).

DEFINITION 3.8. Let S_1 and S_2 be subsets of $L^1(G)$. A bounded function ϕ on Γ is said to define a *multiplier* from S_1 to S_2 if $\phi\hat{f} \in \hat{S}_2$ for every $f \in S_1$.

The set of multipliers from S_1 to S_2 is denoted by (S_1, S_2). For $1 \leq r < \infty$ and $S \subseteq L^1$, let $S_r = \left\{f \in S : \hat{f} \in \ell_r(\Gamma)\right\}$.

Using the Plancherel theorem and Hölder's inequality, we have,

$$\ell_{rp/(r-p)} \subseteq (L_r^1, L_p^1), \tag{3.9}$$

provided $1 \leq p \leq 2 < r < \infty$.

It is not known if the inclusion in (3.9) is proper. We are not able to settle this issue. Instead, we prove the proper inclusion in the larger space $((B_\alpha)_r, L_p^1)$. (In fact for any $S \subseteq L^1$, $(L_r^1, L_p^1) \subseteq (S_r, L_p^1)$.)

THEOREM 3.10. *Let* $0 < \alpha \leq 1/2$ *and* $B_{\alpha,r} = \left\{f \in B_\alpha : \hat{f} \in \ell_r(\Gamma)\right\}$. *Then* $\ell_{rp/(r-p)} \not\subseteq (B_{\alpha,r}, L_p^1)$, *provided* $1 \leq p \leq 2$ *and* $1/\alpha < r < \infty$.

PROOF. Let E be an infinite $S_{\alpha,1/2}$ subset of Γ. Then $\hat{B}_\alpha\,|_E \subseteq \ell_{1/\alpha}(E)$. Hence using the Plancherel theorem and Hölder's inequality we see that

$$\ell_{p/(1-p\alpha)}(E) \subseteq (B_\alpha, L_p^1) \subseteq (B_{\alpha,r}, L_p^1).$$

Since $r > 1/\alpha$, we observe that $p/(1-p\alpha) > rp/(r-p)$ and therefore

$$\ell_{rp/(r-p)}(E) \not\subseteq \ell_{p/(1-p\alpha)}(E).$$

Theorem (3.10) now follows from this observation.

REMARK. If $\alpha > 1/2$ then $(B_{\alpha,r}, L_p^1) \supseteq (B_{1/2,r}, L_p^1)$. It follows from Theorem 3.10 that $\ell_{rp/(r-p)} \not\subseteq (B_{1/2,r}, L_p^1)$, provided $1 \leq p \leq 2$ and $r > 2$. Hence, $\ell_{rp/(r-p)} \not\subseteq (B_{\alpha,r}, L_p^1)$, provided $\alpha > 1/2$ and $1 \leq p \leq 2 < r < \infty$.

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Section 4

In Section 3, we gave a characterization of $S_{\alpha,\beta}$ sets. It is easy to see that if $\beta_1 < \beta_2$ then every S_{α,β_1} set is also an S_{α,β_2} set. It is natural to ask whether the class of S_{α,β_1} sets is a proper subclass of S_{α,β_2} sets. In this section we show that for each positive integer k there exists a subset $E \subseteq \Gamma$ which is an $S_{\alpha,k/2}$ set for $0 < \alpha \leq k/2$ but not an $S_{\alpha,\beta}$ set for any $\beta < k/2$.

The main result of this section is Theorem 4.1 from which the above result about $S_{\alpha,\beta}$ sets follows as an immediate consequence. We would like to mention that the problem of deciding whether the classes of $S_{\alpha,\beta}$ sets are distinct for distinct indices α remains unresolved.

THEOREM 4.1. *Let G be an infinite compact abelian group and $k \in \mathbb{N}$. Then there exists a subset $E_k \subseteq \Gamma$ and a constant C_k such that*

(i) $\|f\|_{L^q} \leq C_k q^{k/2} \|\hat{f}\|_{\ell_2}, \quad \forall f \in T_{E_k}, \quad$ *and* $\quad 2 < q < \infty$,

(ii) $\hat{B}_\alpha|_{E_k} \not\subseteq \ell_r(E_k), \quad 0 < \alpha \leq k/2 \quad$ *and* $\quad r < k/\alpha$.

COROLLARY 4.2. *Let $k \in \mathbb{N}$. Then there exists a subset $E \subseteq \Gamma$ which is an $S_{\alpha,k/2}$ set for $0 < \alpha \leq k/2$, but not an $S_{\alpha,\beta}$ set for $0 < \alpha \leq \beta < k/2$.*

PROOF. This is an immediate consequence of Theorems 3.3 and 4.1.

In addition, we now get the result mentioned in the introduction.

COROLLARY 4.3. *Let G be an infinite compact abelian group, E a Sidon subset of Γ, and $0 < \alpha \leq 1/2$. Then $\hat{B}_\alpha|_E \subseteq \ell_{1/\alpha}(E)$. Further, there exists a Sidon subset $E \subseteq \Gamma$ for which $\hat{B}_\alpha|_E \not\subseteq \ell_r(E)$ for every $r < 1/\alpha$.*

PROOF. If E is a Sidon subset, then it is an $S_{\alpha,1/2}$ subset (Example 3.7(i)). The set E constructed in Theorem 4.1 for $k = 1$ is a Sidon subset (cf. Corollary 3.6) and $\hat{B}_\alpha|_E \not\subseteq \ell_r(E)$ for $r < 1/\alpha$, by (ii).

PROOF OF THEOREM 4.1. There are several steps in the proof of the above theorem. We first prove the theorem in three special cases, namely, when $\Gamma = Z$, $Z(p^\infty)$ and $\prod^*_{\alpha \in A} Z_{q_\alpha}$, where A is an infinite set. Then we prove three lemmas and finally using these and the structure theorem for compact abelian groups, we reduce the general case to these three special cases.

In the proof for each of the three groups mentioned above, we start with a dissociate set E, so that (cf. Example (ii) of Section 3) the set $E_k = E + E + \cdots + E$ (k-times) satisfies inequality (i) of the theorem. Next we construct a sequence $\{h_n\}$ of trigonometric polynomials such that $\|h_n\|_{B_\alpha}/\|\hat{h}_n\,|_{E_k}\|_{\ell_r} \longrightarrow 0$ as $n \longrightarrow \infty$, where $1 \le r < k/\alpha$ and $0 < \alpha \le k/2$. Conclusion (ii) then follows from the closed graph theorem.

4.4 THE CASE $\Gamma = Z$.

Let $E = \{3^\ell\}_{\ell=1}^{\infty}$. Then E is a Hadamard set with Hadamard constant 3. Hence E is a dissociate set ([4, p. 23]). As remarked above, the set

$$E_k = \{3^{\ell_1} + 3^{\ell_2} + \cdots + 3^{\ell_k} : \quad \ell_j \in \mathbb{N}, \quad j = 1, 2, \ldots, k\}$$

satisfies inequality (i) with a constant depending only on k.

Let K_n denote the nth Fejer kernel. For n large enough (so that $K_n(t) \le 1/2$ if $|t| \ge \pi/2$) we have

$$\|K_n\|_{B_\alpha} \le 1 + \pi e + 2\int_0^{\pi/2} K_n(x)(\log^+ K_n(x))^\alpha dx.$$

Since $K_n(x) \le \min(n+1, \pi^2/\left((n+1)x^2\right)$, we get

$$\begin{aligned}\|K_n\|_{B_\alpha} &\le 1 + e\pi + 2\int_0^{1/(n+1)} (n+1)(\log(n+1))^\alpha dx \\ &\quad + 2\int_{1/(n+1)}^{\pi/2} \frac{\pi^2}{(n+1)x^2}\left(\log\frac{\pi^2}{(n+1)x^2}\right)^\alpha dx \\ &\le 1 + e\pi + 2(\log(n+1))^\alpha + 2\frac{\pi^2}{(n+1)}(\log \pi^2(n+1))^\alpha(n+1-2/\pi) \\ &\le C_\alpha(\log(n+1))^\alpha.\end{aligned}$$

Now let $h_n = K_{3^n k}$. Then for n large,

$$\|h_n\|_{B_\alpha} \le C(\log 3^n k)^\alpha = C_k n^\alpha.$$

To estimate $\|\hat{h}_n\,|_{E_k}\|_{\ell_r}$ we need an upper bound on $|[1, 3^n k] \cap E_k|$. For this, we observe that if $m = 3^{\ell_1} + 3^{\ell_2} + \cdots + 3^{\ell_k}$ with $\ell_1 < \ell_2 < \cdots < \ell_k$, then the k-tuple $(\ell_1, \ell_2, \ldots, \ell_k)$ is uniquely determined by m. Hence

$$|[1, 3^n k] \cap E_k| \ge {}^nC_k \ge C_k n^k \quad \text{for } n \text{ large enough.}$$

Therefore $\|\hat{h}_n\,|_{E_k}\|_{\ell_r} \ge C_k n^{k/r}$ and if $1 \le r < k/\alpha$, $\|h_n\|_{B_\alpha}/\|\hat{h}_n\,|_{E_k}\|_{\ell_r} \le C_k n^{\alpha - k/r} \longrightarrow 0$ as $n \to \infty$.

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4.5 THE CASE $\Gamma = Z(p^\infty)$.

Recall that $Z(p^\infty) = \{j/p^n : j \in Z,\ n \in Z\}/Z$, where p is a prime. We write $[j/p^n]$ for the elements (equivalence classes) of $Z(p^\infty)$.

In the following we assume $p \neq 2$. For $p = 2$ some modification is needed which we indicate at the end.

Let $E = \{[1/p^n] : n \in Z\}$. Then E is a dissociate set, for if $\sum_{j=1}^m k_j \left[1/p^{n_j}\right] = 0$, where $n_1 < n_2 < \cdots < n_m$ and $k_j \in \{\pm 1, \pm 2\}$, then

$$k_m/p^{n_m} + \sum_{j=1}^{m-1} k_j/p^{n_j} \in Z \quad \text{or} \quad k_m p^{n_{m-1}}/p^{n_m} + p^{n_{m-1}} \sum_{j=1}^{m-1} k_j/p^{n_j} \in Z.$$

But then, $k_m p^{n_{m-1}}/p^{n_m} \in Z$ which is impossible since $p \neq 2$.

Now fix $k > 0$ and let $E_k = E + E + \cdots + E$ (k-times). The inequality (i) of the theorem holds for E_k (see Example (ii), Section 3).

Next let H_n be the subgroup generated by $[1/p^n]$, that is,

$$H_n = \{[j/p^n] : j = 1, 2, \ldots, p^n\}, \quad |H_n| = p^n.$$

Let $V_n = H_n^\perp$ and put $h_n = \chi_{V_n}/m(V_n)$. Then $\hat{h}_n = \chi_{H_n}$. We have seen in Example (iii), Section 3 that

$$m(V_n)\,|H_n| = 1 \quad \text{and} \quad \left\|h_n\right\|_{B_\alpha} \le C(\log p)^\alpha n^\alpha.$$

We now estimate $\left\|\hat{h}_n\,|_{E_k}\right\|_{\ell_r} = |H_n \cap E_k|^{1/r}$.

Clearly $|H_n \cap E_k| \le n^k$. If $m \in E_k$ has the representation

$$m = 1/p^{\ell_1} + 1/p^{\ell_2} + \cdots + 1/p^{\ell_k} \pmod{Z}, \quad \text{with} \quad \ell_1 < \ell_2 < \cdots < \ell_k,$$

then the k-tuple $(\ell_1, \ell_2, \ldots, \ell_k)$ is uniquely determined by m. For, if not, suppose

$$1/p^{\ell_1} + 1/p^{\ell_2} + \cdots + 1/p^{\ell_k} = 1/p^{j_1} + 1/p^{j_2} + \cdots + 1/p^{j_k} \pmod{Z}$$

with $\ell_1 < \ell_2 < \cdots < \ell_k$ and $j_1 < j_2 < \cdots < j_k$.

Since both terms are less than 1, equality holds without mod Z. Then after cancellation we may assume $\ell_1 < j_1$. Multiplying by p^{ℓ_1} we get,

$$1 + p^{\ell_1 - \ell_2} + \cdots + p^{\ell_1 - \ell_k} = p^{\ell_1 - j_1} + p^{\ell_1 - j_2} + \cdots + p^{\ell_1 - j_k}$$

which is a contradiction since the left side is greater than 1, while the right is less than 1.

We conclude from this that $|H_n \cap E_k| \ge {}^nC_k \ge C_k n^k$ for large n, so that

$$C_k n^{k/r} \le \left\|\hat{h}_n\,|_{E_k}\right\|_{\ell_r} \le n^{k/r}$$

and the conclusion (ii) of the theorem follows if $1 \le r < k/\alpha$.

For the case $p = 2$, we start with the set $E = \left\{\left[1/2^{2n}\right] : n \in Z\right\}$. Then the above proof with p replaced by 2^2 works in exactly the same way.

4.6 THE CASE $\Gamma = \prod^*_{\alpha\in A} Z_{q_\alpha}$.

Here A is an infinite set, each q_α is a prime and $Z_{q_\alpha} = \{1, \omega_\alpha, \omega_\alpha^2, \dots, \omega_\alpha^{q_\alpha-1}\}$, where ω_α is a primitive q_αth root of unity. Γ is the dual of the compact abelian group $G = \prod_{\alpha\in A} Z_{q_\alpha}$. We consider two cases:

Case 1 $\sup_{\alpha\in A} q_\alpha < \infty$. In this case there exists a countably infinite set $\{\alpha_n\} \subseteq A$ such that for some prime p, $q_{\alpha_n} = p$ for all n. Let γ_n be a character on G defined by $\gamma_n(\omega) = \omega_{\alpha_n}$, $\omega \in G$.

Then each γ_n has order p. Let $E = \{\gamma_n\}_{n=1}^\infty$. Now E is an independent set, hence a dissociate set. Therefore the set $E_k = E + E + \cdots + E$ (k-times) satisfies inequality (i) of Theorem 4.1.

Let H_n be the subgroup generated by $\{\gamma_j\}_{j=1}^n$ and let $V_n = H^\perp$. Put $h_n = \chi_{V_n}/m(V_n)$. Then $\hat{h}_n = \chi_{H_n}$ and $|H_n| = p^n$.

By the same argument as in the case $\Gamma = Z(p^\infty)$, we see that

$$\|h_n\|_{B_\alpha} \le C_\alpha n^\alpha$$

and $C_k n^{k/r} \le \|\hat{h}_n \mid_{E_k}\|_{\ell_r} \le n^{k/r}$, and conclusion (ii) follows.

Case 2 $\sup\limits_{\alpha\in A} q_\alpha = \infty$.

Choose $\{q_{\alpha_n}\}$ such that $9k^2 < q_{\alpha_1} < q_{\alpha_2} < \cdots < q_{\alpha_n} \dots$ and $\lim\limits_{n\to\infty} q_{\alpha_n} = \infty$. As in case 1, define $\gamma_n(\omega) = \omega_{\alpha_n}$, $\omega \in G$. Then γ_n is of order q_{α_n}.

We let H_n be the subgroup generated by $\{\gamma_j\}_{j=1}^n$ and let $V_n = H_n^\perp$ and put $h_n = \chi_{V_n}/m(V_n)$.

Then $\hat{h}_n = \chi_{H_n}$, $|H_n| = \prod_{j=1}^n q_{\alpha_j}$, and

$$\|h_n\|_{B_\alpha} \le C(\log|H_n|)^\alpha \le C(\log\prod_{j=1}^n q_{\alpha_j})^\alpha.$$

In this case, constructing the set E_k from the set $E = \{\gamma_n\}_{n=1}^\infty$ as in the earlier cases does not give the required estimate for $\|\hat{h}_n \mid_{E_k}\|_{\ell_r}$. We overcome this difficulty as follows:

For each n, choose m_n as the largest integer for which $(3k)^{m_n+1} \le q_{\alpha_n}$ (where we fix $k \in \mathbb{N}$ as in the statement of the theorem).

Let $E = \{\gamma_n^{(3k)^j} : n \in \mathbb{N},\quad j = 0, 1, 2, \dots, m_n - 1\}$. To see that E is a dissociate set, it is enough to show that for each n, the subset $\{\gamma_n, \gamma_n^{3k}, \dots, \gamma_n^{(3k)^{m_n-1}}\}$ is a dissociate set, since the set $\{\gamma_n\}$ is independent. For this, suppose

$$(\gamma_n)^{p_1(3k)^{\ell_1}}(\gamma_n)^{p_2(3k)^{\ell_2}}\cdots(\gamma_n)^{p_m(3k)^{\ell_m}} = 1 \quad \text{with} \quad p_j = \pm1, \pm2$$

and $0 \le \ell_1 < \ell_2 < \cdots < \ell_m < m_n$. Then, since p_1 is not divisible by 3,

$$p_1(3k)^{\ell_1} + \cdots + p_m(3k)^{\ell_m} = (3k)^{\ell_1}(p_1 + p_2(3k)^{\ell_2-\ell_1} + \cdots + p_m(3k)^{\ell_m-\ell_1}) \ne 0.$$

Moreover

$$\left|\sum_{j=1}^{m} p_j(3k)^{\ell_j}\right| \leq 2\sum_{j=0}^{m_n-1}(3k)^j < (3k)^{m_n} < q_{\alpha_n},$$

so that we obtain the contradiction

$$(\gamma_n)^{\sum_1^m p_j(3k)^{\ell_j}} \neq 1.$$

Now if we take $E_k = E+E+\cdots+E$ (k-times), the inequality (i) of the theorem holds. It remains to estimate $\left\|\hat{h}_n\mid_{E_k}\right\|_{\ell_r} = |H_n \cap E_k|^{1/r}$. It is clear that $|H_n \cap E_k| \leq {}^{\sum_1^n m_j}C_k$.

We actually show that $|H_n \cap E_k| = {}^{\sum_1^n m_j}C_k$. For this, we show that each of the elements $(\gamma_{j_1})^{(3k)^{\ell_1}}(\gamma_{j_2})^{(3k)^{\ell_2}}\cdots(\gamma_{j_k})^{(3k)^{\ell_m}}$ (where $1 \leq j_1 \leq \cdots \leq j_k \leq n$ and $0 \leq \ell_i < m_{j_i}$) are distinct.

This will follow, if we show that $(\gamma_j)^{\sum_1^m s_i(3k)^{\ell_i}}$ never equals 1, where s_i and m are integers such that $0 < |s_i| \leq 2k$, $1 \leq m \leq 2k$ and $0 \leq \ell_1 < \ell_2 < \cdots < \ell_m < (3k)^{m_j}$. It is enough to show that $0 < \left|s_1(3k)^{\ell_1} + \cdots + s_m(3k)^{\ell_m}\right| < q_{\alpha_j}$. Now

$$\left|s_1(3k)^{\ell_1} + \cdots + s_m(3k)^{\ell_m}\right| < (2k)(2k)(3k)^{m_j-1} < (3k)^{m_j+1} \leq q_{\alpha_j}.$$

Next, suppose $s_1(3k)^{\ell_1} + \cdots + s_m(3k)^{\ell_m} = 0$. Then

$$s_1 = -(3k)^{\ell_2-\ell_1}\left\{s_2 + s_3(3k)^{\ell_3-\ell_2} + \cdots + s_m(3k)^{\ell_m-\ell_2}\right\},$$

which is not possible since $0 < |s_1| \leq 2k$.

Thus we have seen that

$$|H_n \cap E_k| = {}^{\sum_1^n m_j}C_k \geq C_k'\Big(\sum_1^n m_j\Big)^k \quad \text{for large} \quad n$$

$$\geq C_k'\Big(\log\prod_{j=1}^n q_{\alpha_j}\Big)^k.$$

Therefore $\left\|\hat{h}_n\mid_{E_k}\right\|_{\ell_r} \geq C\left(\log\prod_{j=1}^n q_{\alpha_j}\right)^{k/r}$, which is the required estimate.

4.7. To complete the proof of Theorem 4.1 for a compact abelian group, we need three short lemmas. We shall need the following definitions ([4, p. 24]).

4.8 DEFINITIONS. Let s be a non-negative integer, E_0 a subset of Γ and $\psi \in \Gamma$. $R_s(E_0, \psi)$ denotes the number (possibly ∞) of asymmetric subsets S of $E = E_0 \cup E_0^{-1}$ satisfying $|S| = s$ and $\prod_{\gamma\in S}\gamma = \psi$. Note that $R_s(E_0, \psi) = R_s(E, \psi)$ for all s and ψ.

A subset $E_0 \subseteq \Gamma$ is called a *Rider set* if there exists a constant $B > 0$ such that $R_s(E_0, 1) \leq B^s$ for all s. Note that if E_0 is a dissociate set, then E_0 is a Rider set since $R_s(E_0, 1) = 0$ for all $s \geq 1$.

If E is a Rider set, then inequality (i) of Theorem 4.1 holds for E_k ([4, p. 65]).

For convenience, we will say that the group G has *property* P_k if there exists a Rider set $E \subseteq \Gamma$ such that $\hat{B}_\alpha\,|_E \not\subseteq \ell_r(E_k)$ for $0 < \alpha \le k/2$ and $r < k/\alpha$.

Note that if a group G has property P_k, then Theorem 4.1 holds for G.

LEMMA 4.9. *Let H be a closed subgroup of G. Suppose G/H has property P_k. Then G has property P_k.*

PROOF. By hypothesis there exists a Rider subset $E \subseteq (G/H)\hat{} = H^\perp$ and a function $g \in B_\alpha(G/H)$ such that $\hat{g}\,|_{E_k} \notin \ell_r(E_k)$. We will show that the same set works for G also.

Since $g \in B_\alpha(G/H)$, $g \circ \pi \in B_\alpha(G)$, where $\pi : G \longrightarrow G/H$ is the quotient map. Furthermore $(g \circ \pi)\hat{}(\gamma) = \hat{g}(\gamma)$ for $\gamma \in H^\perp$. Therefore $(g \circ \pi)\hat{}\,|_{E_k} \notin \ell_r(E_k)$ and the lemma is proved.

LEMMA 4.10. *Let $\{G_\iota\}_{\iota\in A}$ be a family of infinite compact abelian groups and $G = \prod_{\iota\in A} G_\iota$. If for some $t_0 \in A$, G_{t_0} has property P_k, then G has property P_k.*

PROOF. We write $x = (x_\iota)_{\iota\in A}$ for elements of G with $e = (e_\iota)_{\iota\in A}$ as the identity element , and $\gamma = (\gamma_\iota)_{\iota\in A}$ for elements of the dual group Γ. By hypothesis, there exists a Rider set $E \subseteq \Gamma_{t_0}$ such that $B_\alpha(G_{t_0})\hat{}\,|_{E_k} \not\subseteq \ell_r(E_k)$.

Let $F = \prod_{\iota\neq t_0} e_{t_0} \times E$. Then it is easy to see that F is a Rider subset of Γ.

If $g \in B_\alpha(G_{t_0})$ is such that $\hat{g}\,|_{E_k} \notin \ell_r(E_k)$, define $g_1(x) = g(x_{t_0})$, $x \in G$. Then $g_1 \in B_\alpha(G)$ and $\hat{g}_1(\gamma) = \hat{g}(\gamma_{t_0})$. Hence $\hat{g}_1\,|_{F_k} \notin \ell_r(F_k)$.

LEMMA 4.11. *Let H be an open subgroup of G. If there exists a dissociate subset $\tilde{E} \subseteq \Gamma/H^\perp$ such that $B_\alpha(H)\hat{}\,|_{\tilde{E}_k} \not\subseteq \ell_r(\tilde{E}_k)$, then G has property P_k.*

PROOF. Suppose $\tilde{E} = \{\tilde{\gamma}_\alpha : \alpha \in A\} \subseteq \Gamma/H^\perp$ satisfies the hypothesis of the lemma. Let γ_α be any representative of the coset $\tilde{\gamma}_\alpha$ for each $\alpha \in A$, and $E = \{\gamma_\alpha : \alpha \in A\}$. We claim that E is the required Rider set.

If $\left\{\gamma_{\alpha_1}, \gamma_{\alpha_2}, \dots, \gamma_{\alpha_n}\right\}$ is an asymmetric subset of $E \cup E^{-1}$ such that $1 = \gamma_{\alpha_1} \cdot \gamma_{\alpha_2} \cdots \gamma_{\alpha_n}$, then $\tilde{1} = \tilde{\gamma}_{\alpha_1} \cdot \tilde{\gamma}_{\alpha_2} \cdots \tilde{\gamma}_{\alpha_n}$. Since $\tilde{E}$ is a dissociate set $\tilde{\gamma}_{\alpha_j} = \tilde{1}$ for each $j = 1, 2, \dots, n$, but this is not possible since $\tilde{1} \notin \tilde{E}$. Therefore $R_n(E, 1) = 0$ and so E is a Rider set.

Now let $f \in B_\alpha(H)$ be such that $\hat{f}\,|_{\tilde{E}_k} \notin \ell_r(\tilde{E}_k)$. Define

$$g(x) = \begin{cases} f(x) & \text{if } x \in H, \\ 0 & \text{if } x \notin H. \end{cases}$$

Then $g \in B_\alpha(G)$ and

$$\hat{g}(\gamma_\alpha) = \int_H (-x, \gamma_\alpha) g(x)\, dm_G(x) = C \int_H (-x, \gamma_\alpha) f(x)\, dm_H(x) = C\hat{f}(\gamma_\alpha)$$

where $m_G\mid_H = Cm_H$.

Therefore $\hat{g}\mid_{E_k} \notin \ell_r(E_k)$. This completes the proof of the lemma.

4.12 THE CONCLUSION OF THE PROOF OF THEOREM 4.1. We consider two cases. Firstly, suppose Γ is not a torsion group. Then G contains a closed subgroup H such that G/H is isomorphic to the circle group $\mathbb{T}$. By Lemma 4.9 and the case $\Gamma = Z$, the theorem is true.

Now if Γ is a torsion group, then Γ is a weak direct product of p-primary groups ([1, A.3]). By Lemma 4.10, we may assume that Γ is a p-primary group. Now there are two subcases:

Subcase 1. Γ is a p-primary divisible group. Then Γ is a weak direct product of groups of the form $Z(p_\alpha^\infty)$ ([1, A 14]). By Lemma 4.10 and the proof in the case $Z(p^\infty)$, the theorem holds in this case.

Subcase 2. Γ is a p-primary non-divisible group. Then Γ contains a subgroup $B = \prod_\alpha^* Z_{q_\alpha}$ such that Γ/B is divisible ([1, A 24]). Now $B = (G/B^\perp)\hat{}$, so that if B is infinite, the theorem holds for $G/B^\perp$, hence also for G, by Lemma 4.8.

Finally if B is finite, then $B^\perp$ is an open subgroup of G and $(B^\perp)\hat{} = \Gamma/B$ which is a divisible p-primary group. By case 1 above the theorem holds for $B^\perp$. Using Lemma 4.9 and the proof for the case $Z(p^\infty)$, we have in fact a dissociate set $\tilde{E} \subseteq \Gamma/B$ such that $B_\alpha(B^\perp)\hat{}\mid_{\tilde{E}_k} \not\subseteq \ell_r(\tilde{E}_k)$. By Lemma 4.11, there exists a Rider set $E \subseteq \Gamma$ such that $B_\alpha(G)\hat{}\mid_{E_k} \not\subseteq \ell_r(E_k)$ and the theorem holds for G.

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Department of Mathematics
Indian Institute of Technology Kanpur
Kanpur, 208016
India

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