A THEOREM CONCERNING PARTITIONS AND ITS CONSEQUENCE IN THE THEORY OF LIE ALGEBRAS

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1. Introduction. In the first part of this paper we state and prove a theorem concerning the partition $(j; l, i)$ of an integer j into at most l integers k_p, none of which exceed i; l and i being themselves integers. $(j; l, i)$ is thus the number of distinct solutions of the equations

$$j = k_1 + \ldots + k_l, \tag{1.1}$$

where the k_p satisfy the inequalities

$$i \geqslant k_1 \geqslant k_2 \geqslant \ldots \geqslant k_l \geqslant 0. \tag{1.2}$$

In the second part a consequence of this theorem in the theory of representation of the Lie algebra of the unitary unimodular group, $SU(n)$, is noted.

2. Before stating the theorem, some well-known properties of $(j; l, i)$ are noted.

(a) For fixed l, i, it is clear that the maximum value that j may have is il, and that

$$(il; l, i) = 1. \tag{2.1}$$

(b) One may trivially show that

$$(j; l, i) = (il - j; l, i). \tag{2.2}$$

(c) The following formula is given by Dickson **(1)**:

$$(j; l+1, i) - (j-1; l+1, i) = (j-1; l, i) - (j-1-i; l, i). \tag{2.3}$$

(d) One may also trivially show that

$$(j; l, i) = (j; i, l).$$

THEOREM. *$(j; l, i) - (j-1; l, i) \geqslant 0$ for integers $j \leqslant t_l$, where*

$$t_l = 1 + [\tfrac{1}{2}il].$$

Proof. We prove this theorem by induction on l. Now, clearly,

$$(j; 1, i) = \begin{cases} 1, & j \leqslant i, \\ 0, & j > i; \end{cases}$$

hence for $j \leqslant t_1 = 1 + [\frac{1}{2}i]$, we have

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$$(j; 1, i) - (j - 1, 1, i) = 1 - 1 = 0.$$

Thus the theorem is true for $l = 1$. Suppose it to be true for some $l \geqslant 1$; we try to infer it for $l + 1$. Thus we need to prove that, for $j \leqslant t_{(l+1)}$,

$$(j; l + 1, i) - (j - 1; l + 1, i) \geqslant 0$$

or, using equation (2.3), that

$$(2.4) \qquad (j - 1; l, i) - (j - 1 - i; l, i) \geqslant 0.$$

For $j \leqslant t_l + 1$, i.e., for $j - 1 \leqslant t_l$, this follows immediately from the statement of the theorem for l, since we may write

$$(j - 1; l, i) - (j - 1 - i; l, i) = [(j - 1; l, i) - (j - 2; l, i)] + [(j - 2; l, i) - (j - 3; l, i)] + \ldots + [(j - i; l, i) - (j - i - 1; l, i)].$$

All the bracketed terms are non-negative since if $j - 1 \leqslant t_l$, then so are $j - 2$, $j - 3$, etc. Hence we have shown that for $j \leqslant t_l + 1$,

$$(j; l + 1, i) - (j - 1; l + 1, i) \geqslant 0.$$

For those cases where $t_l + 1 \geqslant t_{(l+1)}$ the theorem for $(l + 1)$ is already proved. This occurs when $i = 1, 2$ and, for l an even integer, $i = 3$. In the following we therefore restrict ourselves to the cases where $t_l + 1 < t_{(l+1)}$. It therefore remains to prove equation (2.4) for $t_l + 1 < j \leqslant t_{(l+1)}$ or, writing $j' = j - 1$, that for $t_l < j' < t_{(l+1)}$,

$$(j'; l, i) - (j' - i; l, i) \geqslant 0.$$

Using equation (2.2) this becomes

$$(2.5) \qquad (il - j'; l, i) - (j' - i; l, i) \geqslant 0.$$

Equation (2.5) will follow from the statement of the theorem for l if

(a) $il - j' \geqslant j' - i$ and

(b) $il - j' \leqslant t_l$.

(a) $(il - j') - (j' - i) = i(l + 1) - 2j'$.

Now $j' \leqslant t_{(l+1)} - 1 \leqslant \frac{1}{2}i(l + 1)$ or $i(l + 1) - 2j' \geqslant 0$ so that $(il - j') - (j' - i) \geqslant 0$,

(b) $$il - j' \leqslant il - (t_l + 1) \leqslant 2t_l - (t_l + 1) = t_l - 1$$

and therefore condition (b) is also satisfied. Thus the inequality (2.5) follows from the statement of the theorem for l and therefore

$$(j; l + 1, i) - (j - 1; l + 1, i) \geqslant 0$$

for $t_l + 1 < j \leqslant t_{(l+1)}$ as well as for $j \leqslant t_l + 1$. Therefore, truth of the theorem for l implies truth of the theorem for $l + 1$, so that, by induction, the theorem is true.

3. Consider the Lie algebra, $A(n-1)$, of the unitary unimodular group in n dimensions, $SU(n)$. This is an $(n-1)$-rank algebra and therefore has $(n-1)$ inequivalent fundamental representations, which we denote* by Π^i, $i = 1, \ldots, n-1$. The jth "level" of an irreducible representation ϕ of $A(n-1)$ is defined to be the set of weights of ϕ which are obtainable by subtracting j simple roots from the highest weight of ϕ. Now it can be shown (see Hughes **(4)**) that the number of weights on the jth level of the ith fundamental representation, Π^i, of $A(n-1)$ is equal to the partition $(j; n-i, i)$.

Thus a consequence of property (a) of §2, with $l = n - i$, is that Π^i has $i(n-i)$ levels. Property (b) states that the number of weights of the jth and $\{i(n-i) - j\}$th levels of Π^i are equal, so that the weights of Π^i are distributed in a symmetrical manner about its middle level. Property (d) is intimately related to the fact that Π^i and $\Pi^{(n-i-1)}$ are contragredient representations.

A consequence of the theorem proved here is that Π^i is "spindle-shaped", i.e., that the dimensions of its levels increase monotonically until the middle level (or levels, depending on whether $i(n-i)$ is even or odd), after which they decrease again monotonically. This is a very special instance of a theorem proved by algebraic techniques by Dynkin **(3)**, namely, that all irreducible representations of all semi-simple Lie algebras are spindle-shaped.

References

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2. E. B. Dynkin, *Maximal sub-groups of the classical groups*, Supplement, Amer. Math. Soc. Transl., Ser. 2, *6* (1957), 319.

3. ——— *Some properties of the system of weights of a linear representation of a semisimple Lie group*, Dokl. Akad. Nauk SSSR (N.S.), *71* (1950), 221.

4. J. W. B. Hughes, *Theory of unitary groups*, University College, London, Department of Physics Review paper (September, 1965).

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*See, for example, Dynkin **(2)** for an account of the properties of $A(n-1)$, and for the notation employed here.