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FORMULAS FOR THE NEHARI COEFFICIENTS OF BOUNDED UNIVALENT FUNCTIONS

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1. Introduction. The Grunsky inequalities [**6**] and their generalizations (e.g., [**5**; **14**; **17**]) have become an increasingly important tool for the study of the coefficients of normalized univalent functions defined on the unit disc. In particular, proofs based upon the Grunsky inequalities have now settled the Bieberbach conjecture for the fifth [**15**] and sixth [**13**] coefficients. For bounded univalent functions the situation is similar, although the Grunsky inequalities go over to those of Nehari [**11**]. These inequalities and their generalizations [**1**; **2**; **3**; **4**; **12**; **18**] provide a fruitful approach to the study of coefficient problems for various subclasses of bounded univalent functions.

One difficulty in applying the conditions is the actual determination of the required Grunsky-Nehari coefficients. To overcome this in the case of the Grunsky coefficients, Hummel [**8**] has derived some formulas which greatly ease their calculation, and moreover machine computed tables of these coefficients are now available (Miller [**10**], Ross [**16**]). Corresponding formulas would be useful in working with inequalities of the Nehari type, and it is our purpose here to develop such formulas. While our main attention will be to the classes S_1 and D_1 which are defined below, we indicate in the concluding section how our techniques and formulas apply to other function classes as well, including the Bieberbach-Eilenberg functions as one important example.

Class S_1. The functions $f(z)$ which are regular analytic and univalent in $U = \{z: |z| < 1\}$, have a Taylor series expansion about the origin of the form

$$(1) \qquad f(z) = b_1 z + b_2 z^2 + \ldots + b_n z^n + \ldots, \quad b_1 > 0,$$

and satisfy $|f(z)| < 1$ in U.

Class D_1. The functions $F(z)$ which are regular analytic and univalent in U, have a Taylor series expansion about the origin of the form

$$(2) \qquad F(z) = \beta + \beta_1 z + \beta_2 z^2 + \ldots + \beta_n z^n + \ldots, \quad \beta > 0,$$

and satisfy the conditions $|F(z)| < 1$ and $F(z) + F(\zeta) \neq 0$ for all $z, \zeta \in U$.

For both these classes the Nehari inequalities can be written in the form

$$(3) \qquad \operatorname{Re} \sum_{\mu,\nu=0}^{N} [a_{\mu\nu}\lambda_\mu\lambda_\nu + b_{\mu\nu}\lambda_\mu\bar{\lambda}_\nu] \leqslant \sum_{\mu=1}^{N} \frac{|\lambda_\mu|^2}{\mu}, \quad N = 1, 2, \ldots,$$

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where λ_0 is an arbitrary real number and $\lambda_1, \lambda_2, \ldots$ are arbitrary complex numbers. The Nehari coefficients $a_{\mu\nu}$ and $b_{\mu\nu}$ are defined differently in the two classes, however.

For a function f of the form (1) the symmetric Nehari coefficients $a_{\mu\nu}(f)$ are obtained from the series expansion

$$\text{(4)} \qquad \log \frac{f(z) - f(\zeta)}{z - \zeta} = \sum_{\mu,\nu=0}^{\infty} a_{\mu\nu}(f) z^\mu \zeta^\nu,$$

convergent near the origin. Likewise the function has hermitean Nehari coefficients $b_{\mu\nu}(f)$ which are obtained from

$$\text{(5)} \qquad -\log [1 - f(z)\overline{f(\zeta)}] = \sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}(f) z^\mu \bar{\zeta}^\nu.$$

Then a regular analytic function f of the form (1) will be in S_1 if and only if (3) holds, where $a_{\mu\nu} = a_{\mu\nu}(f)$ and $b_{\mu\nu} = b_{\mu\nu}(f)$ (Nehari [**11**], Schiffer and Tammi [**18**]).

For a function F of the form (2) the symmetric Nehari coefficients $A_{\mu\nu}(F)$ are defined by

$$\text{(6)} \qquad \log \frac{F(z) - F(\zeta)}{[F(z) + F(\zeta)](z - \zeta)} = \sum_{\mu,\nu=0}^{\infty} A_{\mu\nu}(F) z^\mu \zeta^\nu$$

and the hermitean Nehari coefficients $B_{\mu\nu}(F)$ are defined by

$$\text{(7)} \qquad \log \frac{1 + F(z)\overline{F(\zeta)}}{1 - F(z)\overline{F(\zeta)}} = \sum_{\mu,\nu=0}^{\infty} B_{\mu\nu}(F) z^\mu \bar{\zeta}^\nu,$$

these series being convergent in some neighborhood of the origin. Then a regular function F of the form (2) will belong to D_1 if and only if (3) holds for $a_{\mu\nu} = A_{\mu\nu}(F)$ and $b_{\mu\nu} = B_{\mu\nu}(F)$ (DeTemple [**2**]).

In all cases then, the Nehari coefficients can be defined implicitly by means of the relations (4), (5), (6), (7). For applications, however, the coefficients must be computed explicitly, and it is our goal to simplify these otherwise tedious calculations.

2. Preliminaries. Let r be any real number and let $s_1, s_2, \ldots, s_k$ be non-negative integers, not all zero. If $s = s_1 + \ldots + s_k$ then the multinomial coefficients are defined

$$\text{(8)} \qquad \binom{r}{s_1, \ldots, s_k} = \frac{r(r-1)\ldots(r-s+1)}{s_1! \ldots s_k!}.$$

In the case that r is a positive integer and $s \leqq r$ we note the identity

$$\text{(9)} \qquad \binom{r+s}{s_1, \ldots, s_k} = \binom{r+s}{r, s_1, \ldots, s_k}.$$

Now if $f(z) = c_0 + c_1 z + c_2 z^2 + \ldots$ is convergent in some disc about the origin, then for non-negative integers r we have in that same disc the expansion (multinomial theorem)

$$(10)\quad [f(z)]^r = \sum_{k=0}^{\infty} c_k^{(r)} z^k$$

where $c_0^{(r)} = c_0^r$ and for $k = 1, 2, \ldots$

$$(11)\quad c_k^{(r)} = \sum_{\substack{(s_1,\ldots,s_k)\\ \in \mathscr{S}_k}} \binom{r}{s_1, \ldots, s_k} c_0^{r-s} c_1^{s_1} \ldots c_k^{s_k},$$

and where

$$(12)\quad \mathscr{S}_k = \{(s_1, \ldots, s_k) : s_j \geqq 0, s_1 + 2s_2 + \ldots. + ks_k = k\}.$$

Moreover, in the case $c_0 > 0$, the expansion remains valid near the origin even when r is a rational number, positive or negative. Note that $c_0^{(0)} = 1$, $c_k^{(0)} = 0$ and $c_k^{(1)} = c_k$ for $k = 1, 2, \ldots.$

It will also be convenient to introduce the sets of k-tuples

$$(13)\quad \mathscr{S}_k^n = \{(s_1, \ldots, s_k) : s_j \geqq 0, s \leqq k + 1 - n, s_1 + 2s_2 + \ldots + ks_k = n\},$$

where $k = 1, 2, \ldots, 0 \leqq n \leqq k$, and $s = s_1 + \ldots + s_k$. In particular $\mathscr{S}_k^0 = \{(0, \ldots, 0)\}$ and $\mathscr{S}_k^k = \{(0, \ldots, 0, 1)\}$.

Suppose a map τ is defined as follows: for $(s_1, \ldots, s_k) \in \mathscr{S}_k^n$ let $\tau(s_1, \ldots, s_k) = (k + 1 - n - s, s_1, \ldots, s_k)$. Now $k + 1 - n - s \geqq 0$ and $(k + 1 - n - s) + 2s_1 + \ldots + (k + 1)s_k = k + 1$ so τ maps $\bigcup_{n=0}^k \mathscr{S}_k^n$ into $\mathscr{S}_{k+1}$. Moreover, the mapping can be uniquely inverted. If $(s_0, s_1, \ldots, s_k) \in \mathscr{S}_{k+1}$ and if n is defined by $n = k + 1 - s_0 - s_1 - \ldots - s_k$, then $0 \leqq n \leqq k$ and $n + s \leqq k + 1$, where $s = s_1 + \ldots + s_k$. But $s_0 + 2s_1 + \ldots + (k + 1)s_k = k + 1$ so $s_1 + 2s_2 + \ldots + ks_k = k + 1 - s - s_0 = n$. That is, $(s_1, \ldots, s_k) \in \mathscr{S}_k^n$ and $\tau(s_1, \ldots, s_k) = (s_0, s_1, \ldots, s_k)$. Thus τ defines an isomorphism between $\bigcup_{n=0}^k \mathscr{S}_k^n$ and $\mathscr{S}_{k+1}$. From this fact we obtain the following lemma.

LEMMA. *For any functions* $\varphi_{k,n}(s_1, \ldots, s_k)$, $0 \leqq n \leqq k$, $k = 1, 2, \ldots,$ *we have*

$$\sum_{n=0}^{k} \sum_{\substack{(s_1,\ldots,s_k)\\ \in \mathscr{S}_k^n}} \varphi_{k,n}(s_1, \ldots, s_k) = \sum_{\substack{(s_0,s_1,\ldots,s_k)\\ \in \mathscr{S}_{k+1}}} \varphi_{k,k+1-s-s_0}(s_1, \ldots, s_k)$$

where $s = s_1 + \ldots + s_k$.

In a more compact notation we will write this in the form

$$(14)\quad \sum_{\bigcup_{n=0}^k \mathscr{S}_k^n} \varphi_{k,n}(s_1, \ldots, s_k) = \sum_{Q_{k+1}} \varphi_{k,k+1-q}(q_2, \ldots, q_{k+1}),$$

where

$$Q_l = \{(q_1, \ldots, q_l) : q_j \geqq 0, q_1 + 2q_2 + \ldots + lq_l = l\}$$

and

$$q = q_1 + \ldots + q_l.$$

Of course $Q_l = \mathscr{S}_l$, the only change being in the generic letter used to describe the set. This notational convention will be used numerous times in our development.

3. Nehari coefficient formulas for Class S_1. It is well-known that if $f(z) \in S_1$, then so is $[f(z^p)]^{1/p} = b_1^{1/p}z + \ldots$ for each $p = 1, 2, \ldots$. Due to the additional utility of the Nehari inequalities when p is greater than 1, we shall derive Nehari coefficient formulas for the function $[f(z^p)]^{1/p}$, for which we define

$$a_{\mu\nu}^{(p)}(f) = a_{\mu\nu}(f(z^p)^{1/p}) \quad \text{and} \quad b_{\mu\nu}^{(p)}(f) = b_{\mu\nu}(f(z^p)^{1/p}).$$

Since $b_1 \neq 0$, we may write $f(z) = b_1(z + a_2z^2 + \ldots) = b_1g(z)$, where $a_n = b_1^{-1}b_n$. Then according to (4) we have

$$\begin{aligned}(15)\quad \sum_{\mu,\nu=0}^{\infty} a_{\mu\nu}^{(p)}(f)z^\mu\zeta^\nu &= \log\frac{[f(z^p)]^{1/p} - [f(\zeta^p)]^{1/p}}{z - \zeta} \\ &= \frac{1}{p}\log b_1 + \log\frac{[g(z^p)]^{1/p} - [g(\zeta^p)]^{1/p}}{z - \zeta}.\end{aligned}$$

Thus, for $\mu, \nu \neq 0$, $a_{\mu\nu}^{(p)}(f) = c_{\mu\nu}^{(p)}(g)$, where $c_{\mu\nu}^{(p)}(g)$ are the Grunsky coefficients for the normalized univalent function g. Hummel [**8**] has derived formulas for these coefficients in the case $\mu, \nu \geqq 1$. It remains to derive formulas for the coefficients $a_{\mu 0}^{(p)}(f)$ for $\mu \geqq 0$. Setting $\zeta = 0$ in (15), we obtain

$$\begin{aligned}\sum_{\mu=0}^{\infty} a_{\mu 0}^{(p)}(f)z^\mu &= \log\frac{[f(z^p)]^{1/p}}{z} \\ &= \frac{1}{p}\log\frac{f(z^p)}{z^p} \\ &= \frac{1}{p}\log b_1 + \frac{1}{p}\log(1 + a_2z^p + \ldots).\end{aligned}$$

For z sufficiently small we can write this as a power series in $(a_2z^p + \ldots)$ and then apply the multinomial theorem (10), where we set $c_m = a_{m+2}$, $m = 0, 1, 2, \ldots$. We find

$$\begin{aligned}(16)\quad \sum_{\mu=0}^{\infty} a_{\mu 0}^{(p)}(f)z^\mu &= \frac{1}{p}\log b_1 + \frac{1}{p}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}(a_2z^p + a_3z^{2p} + \ldots)^n \\ &= \frac{1}{p}\log b_1 + \frac{1}{p}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}z^{np}(c_0 + c_1z^p + \ldots)^n \\ &= \frac{1}{p}\log b_1 + \frac{1}{p}\sum_{n=1}^{\infty}\sum_{m=0}^{\infty}\frac{(-1)^{n+1}}{n}c_m^{(n)}z^{(n+m)p}.\end{aligned}$$

Now set $k = m + n$. Then $k \geqq 1$ and $1 \leqq n = k - m$ so $0 \leqq m \leqq k - 1$.

Thus (16) takes the form

$$\sum_{\mu=0}^{\infty} a_{\mu 0}{}^{(p)}(f) z^{\mu} = \frac{1}{p} \log b_1 + \frac{1}{p} \sum_{k=1}^{\infty} \sum_{m=0}^{k-1} \frac{(-1)^{k-m+1}}{k-m} c_m{}^{(k-m)} z^{kp}.$$

Clearly $a_{\mu 0}{}^{(p)}(f) = 0$ unless μ is an integral multiple of p. In the case $\mu = kp$ for some integer $k = 1, 2, \ldots$, we have by (9), (11) and (14) and recalling $c_m = a_{m+2}$,

$$\begin{aligned} a_{k_p,0}{}^{(p)}(f) &= \frac{1}{p} \sum_{m=0}^{k-1} \frac{(-1)^{k-m+1}}{k-m} \sum_{\mathscr{S}_m} \binom{k-m}{s_1, \ldots, s_m} a_2{}^{k-m-s} a_3{}^{s_1} \ldots a_{m+2}{}^{s_m} \\ &= \frac{1}{p} \sum_{m=0}^{k-1} \frac{(-1)^{k-m+1}}{k-m} \sum_{\mathscr{S}_{k-1}^m} \binom{k-m}{s_1, \ldots, s_{k-1}} a_2{}^{k-m-s} a_3{}^{s_1} \ldots a_{k+1}{}^{s_{k-1}} \\ &= \frac{1}{p} \sum_{m=0}^{k-1} \sum_{\mathscr{S}_{k-1}^m} \frac{(-1)^{k-m+1}}{k-m} \binom{k-m}{s_1, \ldots, s_{k-1}} a_2{}^{k-m-s} a_3{}^{s_1} \ldots a_{k+1}{}^{s_{k+1}} \\ &= \frac{1}{p} \sum_{Q_k} \frac{(-1)^{q+1}}{q} \binom{q}{q_1, \ldots, q_k} a_2{}^{q_1} a_3{}^{q_2} \ldots a_{k+1}{}^{q_k}. \end{aligned}$$

Summarizing, we obtain the following theorem:

THEOREM 1. *Let $f(z) = b_1(z + a_2 z^2 + \ldots) \in S_1$. Then $a_{\mu 0}{}^{(p)}(f) = 0$ if μ is not divisible by p. If $\mu = kp$, $k = 0, 1, 2, \ldots$, then*

$$(17) \quad \begin{aligned} a_{00}{}^{(p)}(f) &= \frac{1}{p} \log b_1 \\ a_{kp,0}{}^{(p)}(f) &= \frac{1}{p} \sum_{Q_k} \frac{(-q)^{q+1}}{q} \binom{q}{q_1, \ldots, q_k} a_2{}^{q_1} \ldots a_{k+1}{}^{q_k}, \ k \geqslant 1, \end{aligned}$$

where $Q_k = \{(q_1, \ldots, q_k) : q_j \geqq 0,\ q_1 + 2q_2 + \ldots + kq_k = k\}$ and $q = q_1 + \ldots + q_k$.

COROLLARY 1. *For all $N = 1, 2, \ldots$, and all $\mu, \nu = 0, 1, 2, \ldots$, the symmetric Nehari coefficients satisfy the formula*

$$a_{N\mu,N\nu}{}^{(Np)} = \frac{1}{N} a_{\mu\nu}{}^{(p)}.$$

Proof. The case for $\mu, \nu \geqq 1$ follows from the similar result for the Grunsky coefficients [**8**, Theorem 5]. It is obvious for $\mu \geqq 0$, $\nu = 0$ by examination of (17).

The symmetric Nehari coefficients for Class S_1, as given by equation (17) are listed in Table I for the cases $\mu = kp$, $k = 0, 1, 2, 3, 4, 5$ and $\nu = 0$. This table complements the table of Hummel [**8**, p. 149].

In order to determine the formula for the hermitean Nehari coefficients, we

TABLE 1. Symmetric Nehari coefficients $a_{\mu 0}^{(p)}(f)$, for Class S_1, $\mu = kp$, $k = 0, 1, 2, 3, 4, 5$.

$$a_{00}^{(p)}(f) = \frac{1}{p}\log b_1$$

$$a_{p0}^{(p)}(f) = \frac{1}{p} a_2$$

$$a_{2p,0}^{(p)}(f) = \frac{1}{p}\left(a_3 - \frac{1}{2}a_2^2\right)$$

$$a_{3p,0}^{(p)}(f) = \frac{1}{p}\left(a_4 - a_2a_2 + \frac{1}{3}a_2^3\right)$$

$$a_{4p,0}^{(p)}(f) = \frac{1}{p}\left(a_5 - a_2a_4 - \frac{1}{2}a_3^2 + a_2^2a_3 - \frac{1}{4}a_2^4\right)$$

$$a_{5p,0}^{(p)}(f) = \frac{1}{p}\left(a_6 - a_2a_5 - a_3a_4 + a_2^2a_4 + a_2a_3^2 - a_2^3a_3 + \frac{1}{5}a_2^5\right)$$

where $f(z) = b_1z + b_2z^2 + \ldots \in S_1$ and $a_n = b_1^{-1}b_n$.

Note: for $a_{\mu\nu}^{(p)}(f)$, $\mu, \nu \geqq 1$ (see Hummel [8]).

use essentially the same techniques as before, this time beginning with equation (5):

$$\sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}^{(p)}(f)z^{\mu}\bar{\zeta}^{\nu} = -\log\{1 - [f(z^p)\overline{f(\zeta^p)}]^{1/p}\}$$

For z and ζ small, and now taking $c_k = b_{k+1}$ in the multinomial theorem (10), we then have

$$\begin{aligned}\sum_{\mu,\nu=1}^{\infty} b_{\mu\nu}^{(p)}(f)z^{\mu}\bar{\zeta}^{\nu} &= \sum_{n=1}^{\infty}\frac{1}{n}[f(z^p)\overline{f(\zeta^p)}]^{n/p}\\ &= \sum_{n=1}^{\infty}\frac{z^n\bar{\zeta}^n}{n}(b_1 + b_2z^p + \ldots)^{n/p}(b_1 + \bar{b}_2\bar{\zeta}^p + \ldots)^{n/p}\\ &= \sum_{n=1}^{\infty}\sum_{k,l=0}^{\infty}\frac{1}{n}\overline{c_k^{(n/p)}}c_l^{(n/p)}z^{n+pk}\bar{\zeta}^{n+pl}.\end{aligned}$$

Now let $i = n + pk$ and $j = n + pl$. Then $n \geqq 1$ implies $k \leqq (i-1)/p$ and $l \leqq (j-1)/p$. But $i - j = p(k - l)$ so unless $\mu - \nu$ is divisible by p we must have $b_{\mu\nu}^{(p)}(f) = 0$. In the case $\mu = \nu + mp$, $m = 0, 1, 2, \ldots$, we have $k = l + m$ and

$$b_{\nu+mp,\nu}^{(p)}(f) = \overline{b_{\nu,\nu+mp}^{(p)}}(f) = \sum_{l=0}^{[(\nu-1)/p]}\frac{1}{\nu - pl}c_{l+m}^{(\nu/p-l)}\overline{c_l^{(\nu/p-l)}}$$

for $\nu = 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$. Using (11), and recalling $c_k = b_{k+1}$, this

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formula becomes

$$
\begin{aligned}
b_{\nu+mp,\nu}{}^{(p)}(f) &= \sum_{l=0}^{[(\nu-1)/p]} \frac{1}{\nu - pl} \\
&\quad \times \left(\sum_{\mathscr{S}_{l+m}} \binom{\nu/p - l}{s_1, \ldots, s_{l+m}} b_1{}^{\nu/p-l-s} b_2{}^{s_1} \ldots b_{l+m+1}{}^{s_{l+m}} \right) \\
&\quad \cdot \left(\sum_{\mathscr{R}_l} \binom{\nu/p - l}{r_1, \ldots, r_l} b_1{}^{\nu/p-l-r} \overline{b_2{}^{r_1} \ldots b_{l+1}{}^{r_l}} \right) \\
&= \sum_{l=0}^{[(\nu-1)/p]} \frac{(b_1{}^2)^{\nu/p-l}}{\nu - pl} \sum_{\mathscr{S}_{l+m}} \sum_{\mathscr{R}_l} \binom{\nu/p - l}{s_1, \ldots, s_{l+m}} \binom{\nu/p - l}{r_1, \ldots, r_l} \\
&\qquad \times (a_2{}^{s_1} \ldots a_{l+m+1}{}^{s_{l+m}}) (\overline{a_2{}^{r_1} \ldots a_{l+1}{}^{r_l}})
\end{aligned} \tag{18}
$$

when $a_n = b_1{}^{-1} b_n$. Due to the dependence of $b_{\mu\nu}{}^{(p)}(f)$ on both a_t and $\bar{a}_t$, this formula cannot be reduced further. Thus we have obtained the following theorem:

THEOREM 2. *Let f be a function in Class S_1. Then whenever $\mu - \nu$ is not divisible by p, $b_{\mu\nu}{}^{(p)}(f) = 0$. If $\mu = \nu + mp$, $\nu = 1, 2, \ldots,$ and $m = 0, 1, 2, \ldots,$ then*

$$
\begin{aligned}
b_{\nu+mp,\nu}{}^{(p)}(f) &= \sum_{l=0}^{[(\nu-1)/p]} \frac{(b_1{}^2)^{\nu/p-l}}{\nu - pl} \sum_{\mathscr{S}_{l+m}} \sum_{\mathscr{R}_l} \binom{\nu/p - l}{s_1, \ldots, s_{l+m}} \binom{\nu/p - l}{r_1, \ldots, r_l} \\
&\qquad \cdot (a_2{}^{s_1} \ldots a_{l+m+1}{}^{s_{l+m}}) (\overline{a_2{}^{r_1} \ldots a_{l+1}{}^{r_l}})
\end{aligned} \tag{19}
$$

where $a_n = b_1{}^{-1} b_n$, $n = 1, 2, \ldots$.

COROLLARY 2. *For all $N = 1, 2, \ldots,$ and all $\mu, \nu = 1, 2, \ldots,$ the hermitean Nehari coefficients satisfy the formula*

$$b_{N\mu,N\nu}{}^{(Np)}(f) = \frac{1}{N} b_{\mu,\nu}{}^{(p)}(f).$$

Proof. $\mu - \nu = mp$ if and only if $N_\mu - N_\nu = m(Np)$. Also the interval $(\nu - 1, \nu - 1/N)$ contains no integer so neither does the interval $((\nu - 1)/p, (N\nu - 1)/Np)$; that is, $[(\nu - 1)/p] = [(N\nu - 1)/Np]$. But then (19) is identical in the two cases except for the denominator terms $\nu - pl$ and $N(\nu - pl)$.

In order to tabulate the hermitean Nehari coefficients, it is convenient to define the following quantities. For $\nu, p = 1, 2, \ldots$ let $x = \nu/p$. Then define

$$\Omega_0{}^{(x)} = 1/x$$

and for $0 \leqq l \leqq [x - 1/p]$ and $l + m = 1, 2, \ldots,$

$$\Omega_{l+m}{}^{(x-l)} = \frac{1}{x - l} \sum_{\mathscr{S}_{l+m}} \binom{x - l}{s_1, \ldots, s_{l+m}} a_2{}^{s_1} \ldots a_{l+m+1}{}^{s_{l+m}}. \tag{20}$$

Then we can write (19) in the form

$$b_{\nu+mp,\nu}(f) = \sum_{l=0}^{[(x-1)/p]} \frac{x - l}{p} (b_1{}^2)^{x-l} \Omega_{l+m}{}^{(x-l)} \overline{\Omega_l{}^{(x-l)}} \tag{21}$$

for $\nu = 1, 2, \ldots$ and $m = 0, 1, 2, \ldots$. The quantities (20) and coefficients (21) are listed in Table 2. In order to use this table it is first necessary to determine

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TABLE 2. Hermitean Nehari coefficients for Class S_1.

$$\Omega_0^{(r)} = 1/r$$
$$\Omega_1^{(r)} = a_2$$
$$\Omega_2^{(r)} = a_3 + \frac{1}{2}(r-1)a_2^2$$
$$\Omega_3^{(r)} = a_4 + (r-1)a_2a_3 + \frac{1}{6}(r-1)(r-2)a_2^3$$
$$\Omega_4^{(r)} = a_5 + (r-1)a_2a_4 + \frac{1}{2}(r-1)a_3^2 + \frac{1}{2}(r-1)(r-2)a_2^2a_3 + \frac{1}{24}(r-1)(r-2)(r-3)a_2^4$$

m	$\left[x-\frac{1}{p}\right]=0$	$\left[x-\frac{1}{p}\right]=1$	$\left[x-\frac{1}{p}\right]=2$	$\left[x-\frac{1}{p}\right]=3$	$\left[x-\frac{1}{p}\right]=4$
0	$\frac{x}{p}b_1^{2x}\Omega_0^{(x)}\overline{\Omega_0^{(x)}}$	$\frac{x-1}{p}b_1^{2(x-1)}\Omega_1^{(x-1)}\overline{\Omega_1^{(x-1)}}$	$\frac{x-2}{p}b_1^{2(x-2)}\Omega_2^{(x-2)}\overline{\Omega_2^{(x-2)}}$	$\frac{x-3}{p}b_1^{2(x-3)}\Omega_3^{(x-3)}\overline{\Omega_3^{(x-3)}}$	$\frac{x-4}{p}b_1^{2(x-4)}\Omega_4^{(x-4)}\overline{\Omega_4^{(x-4)}}$
1	$\frac{x}{p}b_1^{2x}\Omega_1^{(x)}\overline{\Omega_0^{(x)}}$	$\frac{x-1}{p}b_1^{2(x-1)}\Omega_2^{(x-1)}\overline{\Omega_1^{(x-1)}}$	$\frac{x-2}{p}b_1^{2(x-2)}\Omega_3^{(x-2)}\overline{\Omega_2^{(x-2)}}$	$\frac{x-3}{p}b_1^{2(x-3)}\Omega_4^{(x-3)}\overline{\Omega_3^{(x-3)}}$	
2	$\frac{x}{p}b_1^{2x}\Omega_2^{(x)}\overline{\Omega_0^{(x)}}$	$\frac{x-1}{p}b_1^{2(x-1)}\Omega_3^{(x-1)}\overline{\Omega_1^{(x-1)}}$	$\frac{x-2}{p}b_1^{2(x-2)}\Omega_4^{(x-2)}\overline{\Omega_2^{(x-2)}}$		
3	$\frac{x}{p}b_1^{2x}\Omega_3^{(x)}\overline{\Omega_0^{(x)}}$	$\frac{x-1}{p}b_1^{2(x-1)}\Omega_4^{(x-1)}\overline{\Omega_1^{(x-1)}}$			
4	$\frac{x}{p}b_1^{2x}\Omega_4^{(x)}\overline{\Omega_0^{(x)}}$				

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$m = (\mu - \nu)/p$, $x = \nu/p$ and $[x - 1/p]$. The desired coefficient is then found by adding terms in the row corresponding to m (indicated in the left hand column) through the column headed by the value of $[x - 1/p]$. Table 2 has been constructed so it can be used for values of ν and m for which $\nu + m \leqq 5$. For example, in order to determine $b_{7,4}^{(3)}(f)$, we find that $m = 1$, $x = 4/3$ and $[x - 1/p] = 1$. Thus,

$$b_{7,4}^{(3)}(f) = \frac{4}{9} b_1^{8/3}\Omega_1^{(4/3)}\overline{\Omega_0^{(4/3)}} + \frac{1}{9} b_1^{2/3}\Omega_2^{(1/3)}\overline{\Omega_1^{(1/3)}}$$
$$= \frac{1}{3} b_1^{8/3} a_2 + \frac{1}{9} b_1^{2/3}\left(a_3 - \frac{1}{3} a_2^2\right)\overline{a_2}.$$

4. Nehari coefficient formulas for Class D_1. In order to determine the symmetric Nehari coefficients it is convenient to write $F(z) = \beta(1 + 2f(z))$ where

$$f(z) = \alpha_1 z + \alpha_2 z^2 + \ldots, \alpha_i = \beta_i/2\beta.$$

According to (6), we have

$$\sum_{\mu,\nu=0}^{\infty} A_{\mu\nu}(F) z^\mu \zeta^\nu = \log \frac{F(z) - F(\zeta)}{[F(z) + F(\zeta)](z - \zeta)}$$
$$= \log \frac{f(z) - f(\zeta)}{z - \zeta} - \log (1 + f(z) + f(\zeta))$$
$$= \sum_{\mu,\nu=0}^{\infty} a_{\mu\nu}(f) z^\mu \zeta^\nu - \log (1 + f(z) + f(\zeta))$$

where $a_{\mu\nu}(f)$ is the symmetric Nehari coefficient whose formula (in the case $p = 1$) was derived in Section 3.

For sufficiently small z and ζ we may expand the logarithm as a power series in $[f(z) + f(\zeta)]$ and can then apply the binomial theorem. Defining

(22) $\mathscr{A}_{\mu\nu}(F) = A_{\mu\nu}(F) - a_{\mu\nu}(f)$

we obtain

$$\sum_{\mu,\nu=0}^{\infty} \mathscr{A}_{\mu\nu}(F) z^\mu \zeta^\nu = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sum_{m=0}^{n} \binom{n}{m} f(z)^{n-m} f(\zeta)^m.$$

We now apply the multinomial theorem (10) to obtain a power series in z and ζ, taking $c_l = a_{l+1}$.

$$\sum_{\mu,\nu=0}^{\infty} \mathscr{A}_{\mu\nu}(F) z^\mu \zeta^\nu = \sum_{n=1}^{\infty} \sum_{m=0}^{n} \sum_{l=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^n}{n} \binom{n}{m} c_l^{(n-m)} c_k^{(m)} z^{l+m-n} \zeta^{k+m}. \tag{23}$$

Now let $\mu = l + n - m$ and $\nu = k + m$. Then $\mu \geqq n - m$ and $\nu \geqq m$. It is convenient to deal separately with two cases: μ or ν zero and $\mu, \nu \geqq 1$. If both $\mu, \nu = 0$, we have

$$\mathscr{A}_{00}(F) = 0.$$

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For $\mu \geqq 1$ and $\nu = 0$ we must have $k = m = 0$. Hence, from (23),

$$\begin{aligned}
\mathscr{A}_{\mu 0}(F) &= \sum_{n=1}^{\mu} \frac{(-1)^n}{n} c_{\mu-n}{}^{(n)} c_0{}^{(0)} \\
&= \sum_{l=0}^{\mu-1} \sum_{\mathscr{S}_l} \binom{\mu - l}{s_1, \ldots, s_l} \frac{(-1)^{\mu-l}}{\mu - l} \alpha_1{}^{\mu-l-s} \alpha_2{}^{s_1} \ldots \alpha_{l+1}{}^{s_l} \\
&= \sum_{l=0}^{\mu-1} \sum_{\mathscr{S}^l_{\mu-1}} \binom{\mu - l}{s_1, \ldots, s_{\mu-1}} \frac{(-1)^{\mu-l}}{\mu - l} \alpha_1{}^{\mu-l-s} \alpha_2{}^{s_1} \ldots \alpha_\mu{}^{s_{\mu-1}} \qquad (24)\\
&= \sum_{Q_\mu} \binom{q}{q_2, \ldots, q_\mu} \frac{(-1)^q}{q} \alpha_1{}^{q_1} \alpha_2{}^{q_2} \ldots \alpha_\mu{}^{q_\mu} \\
&= \sum_{Q_\mu} \binom{q}{q_1, \ldots, q_\mu} \frac{(-1)^q}{q} \alpha_1{}^{q_1} \ldots \alpha_\mu{}^{q_\mu}
\end{aligned}$$

where formulas (9), (11) and (14) have been used.

Next consider the case $\mu, \nu \geqq 1$. Since $\alpha_k{}^{(0)} = 0$ if $k > 0$, $m = 0$ or $m = n$ respectively imply that k and l in (23) are zero and hence $\nu = 0$ or $\mu = 0$. But this is not the case and so when the order of summation on n and m is interchanged we must have $\nu \geqq m \geqq 1$ and $\mu + m \geqq n \geqq m + 1$. Then (23) yields

$$\begin{aligned}
\mathscr{A}_{\mu\nu}(F) &= \sum_{m=1}^{\nu} \sum_{n=m+1}^{\mu+m} \frac{(-1)^n}{n} \binom{n}{m} c_{\mu-(n-m)}{}^{(n-m)} c_{\nu-m}{}^{(m)} \\
&= \sum_{k=0}^{\nu-1} \sum_{l=0}^{\mu-1} \frac{(-1)^{\mu+\nu-k-l}}{\mu + \nu - k - l} \binom{\mu + \nu - k - l}{\mu - l} c_l{}^{(\mu-l)} c_k{}^{(\nu-k)} \\
&= \sum_{k=0}^{\nu-1} \sum_{\mathscr{R}_k} \sum_{l=0}^{\mu-1} \sum_{\mathscr{S}_l} \binom{\mu + \nu - k - l}{\mu - l} \binom{\mu - l}{s_1, \ldots, s_l} \binom{\nu - k}{r_1, \ldots, r_k} \\
&\quad \times \frac{(-1)^{\mu+\nu-l-k}}{\mu + \nu - l - k} \alpha_1{}^{\mu+\nu-k-l-r-s} (\alpha_2{}^{s_1} \ldots \alpha_{l+1}{}^{s_l}) (\alpha_2{}^{r_1} \ldots \alpha_{k+1}{}^{r_k}) \\
&= \sum_{k=0}^{\nu-1} \sum_{\mathscr{R}^k_{\nu-1}} \sum_{l=0}^{\mu-1} \sum_{\mathscr{S}^l_{\mu-1}} \binom{\mu + \nu - k - l}{\mu - l} \binom{\mu - l}{s_1, \ldots, s_{\mu-1}} \qquad (25)\\
&\quad \times \binom{\nu - k}{r_1, \ldots, r_{\nu-1}} \frac{(-1)^{\mu+\nu-l-k}}{\mu + \nu - l - k} \alpha_1{}^{\mu+\nu-k-l-r-s} (\alpha_2{}^{s_1} \ldots \alpha_\mu{}^{s_{\mu-1}}) \\
&\quad \times (\alpha_2{}^{r_1} \ldots \alpha_\nu{}^{r_{\nu-1}}) = \sum_{\mathscr{R}_\nu} \sum_{\mathscr{S}_\mu} \binom{s + r}{r} \binom{s}{s_2, \ldots, s_\mu} \binom{r}{r_2, \ldots, r_\nu} \\
&\quad \times \frac{(-1)^{s+r}}{s + r} \alpha_1{}^{s_1+r_1} (\alpha_2{}^{s_2} \ldots \alpha_\mu{}^{s_\mu}) (\alpha_2{}^{r_2} \ldots \alpha_\nu{}^{r_\nu}) \\
&= \sum_{\mathscr{S}_\mu} \sum_{\mathscr{R}_\nu} \frac{(-1)^{s+r}}{s + r} \binom{s + r}{s_1, \ldots, s_\mu, r_1, \ldots, r_\nu} (\alpha_1{}^{s_1} \ldots \alpha_\mu{}^{s_\mu}) \\
&\qquad \times (\alpha_1{}^{r_1} \ldots \alpha_\nu{}^{r_\nu}).
\end{aligned}$$

In this form the symmetry is obvious, but further simplification is possible if we assume $\nu \leqq \mu$ and then let $h = \mu + \nu$. Further, we set $s_i = 0$ for $i > \mu$ and $r_i = 0$ for $i > \nu$, and define $q_i = s_i + r_i$ for $i = 1, 2, \ldots h$. Then

$$q_1 + 2q_2 + \ldots + hq_h = (s_1 + 2s_2 + \ldots + hq_h) \\ + (r_1 + 2r_2 + \ldots + \nu r_\nu) = \mu + \nu = h$$

whenever $(s_1, \ldots, s_\mu) \in \mathscr{S}_\mu$ and $(r_1, \ldots, r_\nu) \in \mathscr{R}_\nu$, and so $(q_1, \ldots, q_h) \in Q_h$. Not all of Q_h can be generated in this manner since $q_{\mu+1}, \ldots, q_h$ are all automatically zero. However, we claim (25) can be replaced by sums over Q_h and $\mathscr{R}_h^{\nu}$; that is,

$$(26)\quad \mathscr{A}_{\mu\nu}(F) = \sum_{Q_h} \sum_{\mathscr{R}_h^\nu} \binom{q}{q_1 - r_1, \ldots, q_h - r_h, r_1, \ldots, r_h} \frac{(-1)^q}{q} \alpha_1^{q_1} \ldots \alpha_h^{q_h}.$$

To check this we first note that under the mapping $(r_1, \ldots, r_\nu) \rightarrow (r_1, \ldots, r_\nu, 0, \ldots, 0)$, recalling $\nu \leqq \mu$, there is a natural identification between $\mathscr{R}_\nu$ and $\mathscr{R}_h^{\nu}$. Moreover, if for some i_0 we have $q_{i_0} - r_{i_0} < 0$, then the corresponding multinomial coefficient in (26) will vanish and thus make no contribution. We need only show, then, that in the case $(q_1, \ldots, q_h) \in Q_h$, $(r_1, \ldots, r_h) \in \mathscr{R}_h^{\nu}$, and $q_1 - r_1 \geqq 0, \ldots, q_h - r_h \geqq 0$, the contribution made in (26) has a corresponding term in (25) which makes an equal contribution. But this is clear; just let $s_i = q_i - r_i$. Then $s_i \geqq 0$ and $s_1 + 2s_2 + \ldots + hs_h = (q_1 - r_1) + \ldots + h(q_h - r_h) = h - \nu = \mu$; that is, $(s_1, \ldots, s_h) \in \mathscr{S}_\mu$. Although (26) was derived under the assumption $\nu \geqq 1$, we notice that if we formally set $\nu = 0$, so $h = \mu$, then the same formula as (24) for $\mathscr{A}_{\mu 0}$ obtains.

THEOREM 3. *Let F be a function in Class D_1. If $F(z) = \beta(1 + 2f(z))$, where $f(z) = \alpha_1 z + \alpha_2 z^2 + \ldots$ and $\alpha_i = (2\beta)^{-1}\beta_i$, then the symmetric Nehari coefficients are given by*

$$(27)\quad A_{\mu\nu}(F) = a_{\mu\nu}(f) + \mathscr{A}_{\mu\nu}(F)$$

where $a_{\mu\nu}(f)$ is the symmetric S_1 Nehari coefficient and

$$\mathscr{A}_{00}(F) = 0$$

$$(28)\quad \mathscr{A}_{\mu\nu}(F) = \sum_{Q_h} \left\{ \sum_{\mathscr{R}_h^\nu} \binom{q}{q_1 - r_1, \ldots, q_h - r_h, r_1, \ldots, r_h} \right\} \\ \times \frac{(-1)^q}{q} \alpha_1^{q_1} \ldots \alpha_h^{q_h}$$

where $h = \mu + \nu$, $v \leqq \mu$, $\mu, \nu = 1, 2, \ldots, (\mu, \nu) \neq (0, 0)$.

The $\mathscr{A}_{\mu\nu}(F)$ components of the Nehari coefficients for Class D_1, (28), are listed in Table 3.

TABLE 3. The $\mathscr{A}_{\mu\nu}(F)$ components of the symmetric Nehari coefficients for Class D_1, $0 \leqq \nu \leqq \mu \leqq 3$.

$\mathscr{A}_{00}(F)$	$= 0$
$\mathscr{A}_{10}(F)$	$= -\alpha_1$
$\mathscr{A}_{20}(F)$	$= \frac{1}{2}\alpha_1^2 - \alpha_2$
$\mathscr{A}_{30}(F)$	$= -\frac{1}{3}\alpha_1^3 + \alpha_1\alpha_2 - \alpha_3$
$\mathscr{A}_{11}(F)$	$= \alpha_1^2$
$\mathscr{A}_{21}(F)$	$= \alpha_1\alpha_2 - \alpha_1^3$
$\mathscr{A}_{31}(F)$	$= \alpha_1\alpha_3 - 2\alpha_1^2\alpha_2 + \alpha_1^4$
$\mathscr{A}_{22}(F)$	$= \alpha_2^2 - 2\alpha_1^2\alpha_2 + \frac{3}{2}\alpha_1^4$
$\mathscr{A}_{32}(F)$	$= \alpha_2\alpha_3 - \alpha_1^2\alpha_3 - 2\alpha_1\alpha_2^2 + 4\alpha_1^3\alpha_2 - \alpha_1^5$
$\mathscr{A}_{33}(F)$	$= \alpha_3^2 - 4\alpha_1\alpha_2\alpha_3 + 2\alpha_1^3\alpha_3 + 6\alpha_1^2\alpha_2^2 - 8\alpha_1^4\alpha_2 - \frac{10}{3}\alpha_1^6$

We now turn to the hermitean Nehari coefficients $B_{\mu\nu}(F)$, defined by

$$\sum_{\mu,\nu=0}^{\infty} B_{\mu\nu}(F)z^\mu\bar{\zeta}^\nu = \log\left(\frac{1 + F(z)\overline{F(\zeta)}}{1 - F(z)\overline{F(\zeta)}}\right).$$

Since $|F(z)| < 1$ we may expand this logarithm as a power series in $F(z)\overline{F(\zeta)}$ and then apply the multinomial theorem to obtain

$$\sum_{\mu,\nu=0}^{\infty} B_{\mu\nu}(F)z^\mu\bar{\zeta}^\nu = 2\sum_{n=0}^{\infty} \frac{[F(z)\overline{F(\zeta)}]^{2n+1}}{2n+1}$$
$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}\sum_{\mu=0}^{\infty}\sum_{\mathscr{S}_\mu}\binom{2n+1}{s_1,\ldots,s_\mu}\beta^{2n+1-s}\beta_1^{s_1}\ldots\beta_\mu^{s_\mu}z^\mu$$
$$\times\sum_{\nu=0}^{\infty}\sum_{\mathscr{R}_\nu}\binom{2n+1}{r_1,\ldots,r_\nu}\beta^{2n+1-r}\overline{\beta_1^{r_1}\ldots\beta_\nu^{r_\nu}\zeta^\nu}.$$

Thus in terms of the $\alpha_i = \beta_i/2\beta$ we have

$$B_{\mu\nu}(F) = 2\sum_{n=0}^{\infty}\frac{\beta^{4n+2}}{2n+1}\sum_{\mathscr{S}_\mu}\sum_{\mathscr{R}_\nu}2^{r+s}\binom{2n+1}{s_1,\ldots,s_\mu}\binom{2n+1}{r_1,\ldots,r_\nu}$$
$$\times(\alpha_1^{s_1}\ldots\alpha_\mu^{s_\mu})(\overline{\alpha_1^{r_1}\ldots\alpha_\nu^{r_\nu}}).$$

Noting that

$$\binom{2n+1}{s_1,\ldots,s_\mu} = \binom{2n+1}{s}\binom{s}{s_1,\ldots,s_\mu}$$

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we can then write

$$(29)\quad B_{\mu\nu}(F) = \sum_{\mathscr{S}_\mu} \sum_{\mathscr{R}_\nu} 2^{r+s+1}\beta^2\sigma_{rs}\binom{s}{s_1,\ldots,s_\mu}\binom{r}{r_1,\ldots,r_\nu} \times (\alpha_1^{s_1}\ldots\alpha_\mu^{s_\mu})\overline{(\alpha_1^{r_1}\ldots\alpha_\nu^{r_\nu})}$$

where

$$(30)\quad \sigma_{rs} = \sum_{n=0}^{\infty}\binom{2n+1}{s}\binom{2n+1}{r}\frac{\beta^{4n}}{2n+1}.$$

We now wish to express the σ_{rs} as finite sums. Considering them as power series in the parameter $y = \beta^2$, a short computation shows

$$(31)\quad (r+1)\sigma_{r+1,s} = \frac{d}{dy}(y\sigma_{rs}) - r\sigma_{rs}, \quad r, s = 0, 1, 2, \ldots.$$

An induction argument from (31) then shows

$$(32)\quad r!\sigma_{rs} = y^{r-1}\frac{d^{r-1}}{dy^{r-1}}\sigma_{1s}, \quad s = 0, 1, 2, \ldots, r = 1, 2, \ldots.$$

Now $\sigma_{rs} = \sigma_{sr}$ and so if in (32) we first set $s = 0$ and then replace r with s, we find

$$(33)\quad s!\sigma_{0s} = y^{s-1}\frac{d^{s-1}}{dy^{s-1}}\sigma_{10}, \quad s = 1, 2, \ldots.$$

But from (31) we see

$$(34)\quad \sigma_{1s} = \frac{d}{dy}(y\sigma_{0s}), \quad s = 0, 1, 2, \ldots.$$

and so combining (32), (33), (34) we get

$$\sigma_{rs} = \frac{1}{r!s!}y^{r-1}\frac{d^r}{dy^r}y^s\frac{d^{s-1}}{dy^{s-1}}\sigma_{10}, \quad r, s = 1, 2, \ldots$$

Applying Leibniz's formula for the rth derivative of a product we obtain the symmetric form

$$(35)\quad \sigma_{rs} = \frac{1}{r!s!}\sum_{k=0}^{\min(r,s)}\binom{r}{k}\binom{s}{k}k!y^{r+s-k-1}\frac{d^{r+s-k-1}}{dy^{r+s-k-1}}\sigma_{10}.$$

From (30) we see

$$\sigma_{10} = \frac{1}{1-y^2} = \frac{1}{2}\left(\frac{1}{1-y} + \frac{1}{1+y}\right)$$

and hence

$$\frac{d^{n-1}}{dy^{n-1}}\frac{1}{1-y^2} = \frac{1}{2}(n-1)![(1-y)^{-n} - (-1)^n(1+y)^{-n}]$$
$$= \frac{(n-1)!}{(1-y^2)^n}\sum_{j=0}^{n}\binom{n}{2j+1}y^{n-2j-1}.$$

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This allows (35) to be written as

$$\sigma_{rs} = \frac{1}{r!s!}\sum_{k=0}^{r+s-1}\binom{r}{k}\binom{s}{k}k!y^{r+s-k-1}\frac{(r+s-k-1)!}{(1-y^2)^{r+s-k}}\sum_{j=0}^{r+s-k}\binom{r+s-k}{2j+1}$$

$$(36)\qquad \times y^{r+s-k-2j-1} = \frac{1}{r!s!(1-y^2)^{r+s}}\sum_{k=0}^{r+s-1}\binom{r}{k}\binom{s}{k}$$

$$\times k!(r+s-k-1)!\sum_{j=0}^{r+s-k}\sum_{l=0}^{k}\binom{k}{l}\binom{r+s-k}{2j+1}y^{2(r+s-l-j-1)}.$$

Next suppose we change the summation indices according to

$$(37)\qquad \begin{aligned} m &= r+s-l-j-1,\\ p &= r+s-k,\\ q &= r+s-l. \end{aligned}$$

Now $2j+1 \leqq r+s-k$ and $l \leqq k$ so $j \leqq r+s-l-j-1 = m$; we thus have the limits $0 \leqq m \leqq r+s-1$. Making these substitutions into (36) and simplifying we get the final form

$$(38)\qquad \begin{aligned} \sigma_{rs} &= \frac{1}{(1-y^2)^{r+s}}\sum_{m=0}^{r+s-1} y^{2m}\\ &\times \left\{\sum_{p=1}^{r+s}\sum_{q=p}^{r+s}\frac{(-1)^{p+q}}{p}\binom{p}{p-s,p-r,r+s-q,q-p}\binom{p}{2q-2m-1}\right\}. \end{aligned}$$

Although (38) was derived under the assumption $r, s \geqq 1$, it nevertheless is valid even if one of r or s is zero, as may be checked using equation (33). Introducing the functions $\chi_{rs} = 2^{r+s+1}\beta^2\sigma_{rs}$, we have derived the following result.

THEOREM 4. *Let $F(z) = \beta(1 + 2\alpha_1 z + 2\alpha_2 z^2 + \ldots) \in D_1$. Then the hermitean Nehari coefficients $B_{\mu\nu}(F)$ are given by*

$$(39)\qquad B_{00}(F) = \log\frac{1+\beta^2}{1-\beta^2}$$

and, for $\mu + \nu \geqq 1$, by

$$(40)\qquad B_{\mu\nu}(F) = \sum_{\mathscr{S}_\mu}\sum_{\mathscr{R}_\nu}\chi_{rs}\binom{s}{s_1,\ldots,s_\mu}\binom{r}{r_1,\ldots,r_\nu} \times (\alpha_1^{s_1}\ldots\alpha_\mu^{s_\mu})\overline{(\alpha_1^{r_1}\ldots\alpha_\nu^{r_\nu})}$$

where

$$(41)\qquad \begin{aligned} \chi_{rs} &= \frac{2^{r+s+1}\beta^2}{(1-\beta^4)^{r+s}}\sum_{m=0}^{r+s-1}\beta^{4m}\\ &\times \left\{\sum_{p=1}^{r+s}\sum_{q=p}^{r+s}\frac{(-1)^{p+q}}{p}\binom{p}{p-s,p-r,r+s-q,q-p}\binom{p}{2q-2m-1}\right\}. \end{aligned}$$

Tables 4 and 5 list respectively χ_{rs} for $0 \leqq s \leqq r \leqq 3$ and $B_{\mu\nu}(F)$ for $0 \leqq \nu \leqq \mu \leqq 3$.

TABLE 4. χ_{rs}, $0 \leqq s \leqq r \leqq 3$, $1 \leqq r$.

$$\chi_{10} = 4\beta^2(1-\beta^4)^{-1}$$

$$\chi_{20} = 8\beta^2(1-\beta^4)^{-2}\beta^4$$

$$\chi_{30} = 16\beta^2(1-\beta^4)^{-3}\left(\frac{1}{3}\beta^4+\beta^8\right)$$

$$\chi_{11} = 8\beta^2(1-\beta^4)^{-2}(1+\beta^4)$$

$$\chi_{21} = 16\beta^2(1-\beta^4)^{-3}(3\beta^4+\beta^8)$$

$$\chi_{31} = 32\beta^2(1-\beta^4)^{-4}(\beta^4+6\beta^8+\beta^{12})$$

$$\chi_{22} = 32\beta^2(1-\beta^4)^{-4}(3\beta^4+8\beta^8+\beta^{12})$$

$$\chi_{32} = 64\beta^2(1-\beta^4)^{-5}(\beta^4+15\beta^8+15\beta^{12}+\beta^{16})$$

$$\chi_{33} = 128\beta^2(1-\beta^4)^{-6}\left(\frac{1}{3}\beta^4+17\beta^8+60\beta^{12}+\frac{82}{3}\beta^{16}+\beta^{20}\right)$$

TABLE 5. Hermitean Nehari coefficients $B_{\mu\nu}(F)$ for Class D_1 $0 \leqq \nu \leqq \mu \leqq 3$m $1 \leqq \mu$.

$$B_{10}(F) = \chi_{10}\alpha_1$$

$$B_{20}(F) = \chi_{10}\alpha_2 + \chi_{20}\alpha_1^2$$

$$B_{30}(F) = \chi_{10}\alpha_3 + 2\chi_{20}\alpha_1\alpha_2 + \chi_{30}\alpha_1^3$$

$$B_{11}(F) = \chi_{11}|\alpha_1|^2$$

$$B_{21}(F) = \chi_{11}\alpha_2\bar{\alpha}_1 + \chi_{21}\alpha_1^2\bar{\alpha}_1$$

$$B_{31}(F) = \chi_{11}\alpha_3\bar{\alpha}_1 + 2\chi_{21}\alpha_1\alpha_2\bar{\alpha}_1 + \chi_{31}\alpha_1^3\bar{\alpha}_1$$

$$B_{22}(F) = \chi_{11}|\alpha_2|^2 + 2\chi_{21}\,\mathrm{Re}\,(\alpha_1^2\bar{\alpha}_2) + \chi_{22}|\alpha_1|^4$$

$$\begin{aligned} B_{32}(F) = {} & \chi_{11}\alpha_3\bar{\alpha}_3 + \chi_{21}(\alpha_3\bar{\alpha}_1^2 + 2\alpha_1\alpha_2\bar{\alpha}_2) \\ & + 2\chi_{22}\alpha_1\alpha_2\bar{\alpha}_1^2 + \chi_{31}\alpha_1^3\bar{\alpha}_2 + \chi_{32}\alpha_1^3\bar{\alpha}_1^2 \end{aligned}$$

$$\begin{aligned} B_{33}(F) = {} & \chi_{11}|\alpha_3|^2 + 4\chi_{21}\,\mathrm{Re}\,(\alpha_1\alpha_2\bar{\alpha}_3) + 2\chi_{31}\,\mathrm{Re}\,(\alpha_1^3\bar{\alpha}_3) \\ & + 4\chi_{22}|\alpha_1\alpha_2|^2 + 4\chi_{32}|\alpha_1|^2\,\mathrm{Re}\,(\alpha_1^2\bar{\alpha}_2) \\ & + \chi_{33}|\alpha_1|^6 \end{aligned}$$

5. Modification of the formulas for application to other function classes. While the formulas that have been derived apply to the Nehari coefficients appropriate to the classes S_1 and D_1, with suitable modification they also yield the Nehari (or Grunsky) coefficients relevant to many other function classes as well. In this section we briefly illustrate such modifications for some of these classes.

Class S_1^R. This consists of the function $f(z) = b_1 z + b_2 z^2 + \ldots \in S_1$ with real coefficents b_n. As before the Nehari inequalities (3) apply, with the Nehari coefficients $a_{\mu\nu}(f)$ and $b_{\mu\nu}(f)$ defined by (4) and (5). The formulas of Hummel [**8**] and Theorem 1 for the symmetric coefficients is unchanged, but simplifications are now possible for the $b_{\mu\nu}(f)$ formulas. Letting $\mu = \nu + m$, $m \geqq 0$, and taking the case $p = 1$, then the first equality of (18) is now

$$(42)\quad b_{\nu+m,\nu}(f) = \sum_{l=0}^{\nu-1} \frac{1}{\nu - l} \sum_{\mathscr{S}_{l+m}} \sum_{\mathscr{R}_l} \binom{\nu - l}{s_1, \ldots, s_{l+m}} \binom{\nu - l}{r_1, \ldots, r_l} \times (b_1^{\nu-l-s} b_2^{s_1} \ldots b_{l+m+1}^{s_{l+m}}) \cdot (b_1^{\nu-l-r} b_2^{r_1} \ldots b_{l+1}^{r_l}),$$

where we recall $s = s_1 + s_2 + \ldots + s_{l+m}$, $r = r_1 + r_2 + \ldots + r_l$. Setting $s_j = 0$ for $l + m + 1 \leqq j \leqq \mu - 1$ and $r_k = 0$ for $l + 1 \leqq k \leqq \mu - 1$, then the integers $v_1, \ldots, v_\mu, w_1, \ldots, w_\mu$ can be defined by

$$(43)\quad \begin{aligned} v_1 &= 2(\nu - l) - (s_1 + s_2 + \ldots) - (r_1 + r_2 + \ldots) \\ v_2 &= s_1 + r_1 \\ &\ldots \\ v_\mu &= s_{\mu-1} + r_{\mu-1} \\ w_1 &= \nu - l - (r_1 + r_2 + \ldots) \\ w_2 &= r_1 \\ &\ldots \\ w_\mu &= r_{\mu-1}. \end{aligned}$$

Now the multinomial coefficients which appear in (42) will be zero unless $s_1 + s_2 + \ldots \leqq \nu - l$ and $r_1 + r_2 + \ldots \leqq \nu - l$. Thus for the terms in (42) which give rise to a non-zero contribution we have $(v_1, \ldots, v_\mu) \in V_{\nu-l}$ and $(w_1, \ldots, w_\mu) \in W_{\nu-l}$, where we define

$$(44)\quad V_k = \{(v_1, \ldots, v_\mu) : v_j \geqq 0, v_1 + 2v_2 + \ldots + \mu v_\mu = h, \; v_1 + v_2 + \ldots + v_\mu = 2k\}$$

$$(45)\quad W_k = \{(w_1, \ldots, w_\mu) : 0 \leqq w_j \leqq v_j, w_1 + 2w_2 + \ldots + \mu w_\mu = \nu, \; w_1 + w_2 + \ldots + w_\mu = k\}.$$

Because the transformation in (43) is invertible, the following theorem follows by a short computation from equation (42).

THEOREM 5. *Let* $f = b_1 z + b_2 z^2 + \ldots \in S_1^R$. *Then*

$$(46)\quad b_{\mu\nu}(f) = \sum_{k=1}^{\nu} \frac{1}{k} \sum_{V_k} \sum_{W_k} \binom{k}{w_1, \ldots, w_\mu} \binom{k}{v_1 - w_1, \ldots, v_\mu - w_\mu} b_1^{v_1} \ldots b_\mu^{v_\mu},$$

where V_k *and* W_k *are given by* (44) *and* (45).

Class E (Bieberbach-Eilenberg). An analytic function $f(z) = b_1 z + b_2 z^2 + \ldots$ is in Class E if $f(z)f(\zeta) \neq 1$ for all pairs $z, \zeta \in U$. Hummel and Schiffer [**9**] have shown that the univalent Bieberbach-Eilenberg functions are characterized by the Grunsky-type inequalities

$$(47)\quad \operatorname{Re} \sum_{\mu,\nu=0}^{N} \gamma_{\mu\nu}(f)\lambda_\mu\lambda_\nu \leqq \sum_{\mu=1}^{N} \frac{|\lambda_\mu|^2}{\mu}, \quad N = 1, 2, 3, \ldots,$$

where λ_0 is real and $\lambda_1, \lambda_2, \ldots$ are arbitrary. The Grunsky coefficients $\gamma_{\mu\nu}(f)$ are defined by

$$(48)\quad \log \frac{f(z) - f(\zeta)}{(z - \zeta)[1 - f(z)f(\zeta)]} = \sum_{\mu,\nu=0}^{\infty} \gamma_{\mu\nu}(f) z^\mu \zeta^\nu.$$

Comparing (48) with (4) and (5) the following result is immediate.

THEOREM 6. *For* $f = b_1 z + b_2 z^2 + \ldots \in E$ *let* $\gamma_{\mu\nu}(f)$ *be defined by* (48). *Then*

$$\gamma_{00}(f) = \log b_1;$$

$$\gamma_{\mu 0}(f) = \sum_{Q_\mu} \frac{(-1)^{q+1}}{q} \binom{q}{q_1, \ldots, q_\mu} b_1^{-q} b_2^{q_1} \ldots b_{\mu+1}^{q_\mu}, \quad \mu \geqq 1,$$

$$(49)\quad \gamma_{\mu\nu}(f) = \sum_{Q_h} \sum_{k=0}^{h-1} \sum_{\mathscr{S}_k'} \frac{(-1)^{q+1}}{h - k + q - s} \binom{h - k + q - s}{s_1, \ldots, s_h, q_1 - s_1, \ldots, q_h - s_h}$$
$$\times b_1^{-q} b_2^{q_1} \ldots b_{h+1}^{q_h}$$
$$+ \sum_{k=1}^{\nu} \frac{1}{k} \sum_{V_k} \sum_{W_k} \binom{k}{w_1, \ldots, w_\mu} \binom{k}{v_1 - w_1, \ldots, v_\mu - w_\mu}$$
$$\times b_1^{v_1} \ldots b_\mu^{v_\mu}, \quad \mu \geqq \nu \geqq 1;$$

here $h = \mu + \nu$, Q_h, V_k, W_k *are given respectively by* (12), (44), (45), *and*

$$\mathscr{S}_k' = \{(s_1, \ldots, s_h) : 0 \leqq s_j \leqq q_j, s_1 + 2s_2 + \ldots + hs_h = k\},$$
$$s = s_1 + s_2 + \ldots + s_h,$$
$$q = q_1 + q_2 + \ldots + q_h.$$

Class Γ. An analytic function $f = b_1 z + b_2 z^2 + \ldots$ is in class Γ if $f(z)\overline{f(\zeta)} \neq -1$ for all pairs $z, \zeta \in U$. Śladkowska [**19**] has recently shown that the inequality (3) holds when $a_{\mu\nu} = a_{\mu\nu}(f)$ as given by (4) and $b_{\mu\nu} = \hat{b}_{\mu\nu}(f)$ where

$$(50)\quad -\log [1 + f(z)\overline{f(\zeta)}] = \sum_{\mu,\nu=1}^{\infty} \hat{b}_{\mu\nu}(f) z^\mu \bar{\zeta}^\nu.$$

Comparison of (5) and (50) shows that the derivation of formula (19) need only be modified by the insertion of a factor $(-1)^{\nu-l}$. That is (in the case $p = 1$),

$$\hat{b}_{\nu+m,\nu}(f) = \sum_{l=0}^{\nu-1} \frac{(-b_1^{\,2})^{\nu-l}}{\nu-l} \sum_{\mathscr{S}_{l+m}} \sum_{\mathscr{R}_l} \binom{\nu-l}{s_1,\ldots,s_{l+m}} \binom{\nu-l}{r_1,\ldots,r_l} \times (a_2^{\,s_1}\ldots a_{l+m+1}^{\;\;s_{l+m}}) (\overline{a_2^{\,r_1}\ldots a_{l+1}^{\;\;r_l}}) \tag{51}$$

where as usual $a_j = b_1^{-1}b_j$.

Class D. (Guelfer [**7**]). An analytic function $F(z) = 1 + 2\alpha_1 z + 2\alpha_2 z^2 + \ldots$ is a Guelfer function if $F(z) + F(\zeta) \neq 0$ for all pairs $z, \zeta \in U$. De Temple [**2**] has shown that the univalent Guelfer functions satisfy the Grunsky-type inequalities (47), where $\gamma_{\mu\nu}(f)$ is replaced by $A_{\mu\nu}(F)$ as defined by (6). Hence formulas for $A_{\mu\nu}(F)$ are obtained directly from Theorem 3 upon setting $\beta = 1$.

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