Ergod. Th. & Dynam. Sys., (2024), **44**, 353–382
doi:10.1017/etds.2023.24

Phase transitions for non-singular Bernoulli actions

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(*Received* 25 *October* 2022 *and accepted in revised form* 21 *February* 2023)

Abstract. Inspired by the phase transition results for non-singular Gaussian actions introduced in [AIM19], we prove several phase transition results for non-singular Bernoulli actions. For generalized Bernoulli actions arising from groups acting on trees, we are able to give a very precise description of their ergodic-theoretical properties in terms of the Poincaré exponent of the group.

Key words: non-singular Bernoulli action, phase transition, strong ergodicity, Krieger type
2020 Mathematics Subject Classification: 37A40 (Primary); 20E08 (Secondary)

1. *Introduction*

When G is a countable infinite group and (X_0, μ_0) is a non-trivial standard probability space, the probability measure-preserving (pmp) action

$$G \curvearrowright (X_0, \mu_0)^G : \quad (g \cdot x)_h = x_{g^{-1}h}$$

is called a *Bernoulli action*. Probability measure-preserving Bernoulli actions are among the best-studied objects in ergodic theory and they play an important role in operator algebras [**Ioa10**, **Pop03**, **Pop06**]. When we consider a family of probability measures $(\mu_g)_{g\in G}$ on the base space X_0 that need not all be equal, the Bernoulli action

$$G \curvearrowright (X, \mu) = \prod_{g\in G} (X_0, \mu_g) \tag{1.1}$$

is in general no longer measure-preserving. Instead, we are interested in the case where $G \curvearrowright (X, \mu)$ is *non-singular*, that is, the group G preserves the *measure class* of μ. By Kakutani's criterion for equivalence of infinite product measures the Bernoulli action (1.1) is non-singular if and only if $\mu_h \sim \mu_g$ for every $h, g \in G$ and

$$\sum_{h\in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G. \tag{1.2}$$

Here $H^2(\mu_h, \mu_{gh})$ denotes the *Hellinger distance* between μ_h and μ_{gh} (see (2.2)).

It is well known that a pmp Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is mixing. In particular, it is ergodic and conservative. However, for non-singular Bernoulli actions, determining conservativeness and ergodicity is much more difficult (see, for instance, [**BKV19**, **Dan18**, **Kos18**, **VW17**]).

Besides non-singular Bernoulli actions, another interesting class of non-singular group actions comes from the *Gaussian construction*, as introduced in [**AIM19**]. If $\pi: G \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation of a locally compact second countable (lcsc) group on a real Hilbert space $\mathcal{H}$, and if $c: G \to \mathcal{H}$ is a 1-cocycle for the representation π, then the assignment

$$\alpha_g(\xi) = \pi_g(\xi) + c(g) \tag{1.3}$$

defines an *affine isometric action* $\alpha: G \curvearrowright \mathcal{H}$. To any affine isometric action $\alpha: G \curvearrowright \mathcal{H}$ Arano, Isono and Marrakchi associated a non-singular group action $\widehat{\alpha}: G \curvearrowright \widehat{\mathcal{H}}$, where $\widehat{\mathcal{H}}$ is the Gaussian probability space associated to $\mathcal{H}$. When $\alpha: G \curvearrowright \mathcal{H}$ is actually an orthogonal representation, this construction is well established and the resulting Gaussian action is pmp. As explained below [**BV20**, Theorem D], if G is a countable infinite group and $\pi: G \to \ell^2(G)$ is the left regular representation, the affine isometric representation (1.3) gives rise to a non-singular action that is conjugate with the Bernoulli action $G \curvearrowright \prod_{g \in G}(\mathbb{R}, \nu_{F(g)})$, where $F: G \to \mathbb{R}$ is such that $c_g(h) = F(g^{-1}h) - F(h)$, and $\nu_{F(g)}$ denotes the Gaussian probability measure with mean $F(g)$ and variance 1.

By scaling the 1-cocycle $c: G \to \mathcal{H}$ with a parameter $t \in [0, +\infty)$ we get a one-parameter family of non-singular actions $\widehat{\alpha}^t: G \curvearrowright \widehat{\mathcal{H}}^t$ associated to the affine isometric actions $\alpha^t: G \curvearrowright \mathcal{H}$, given by $\alpha_g^t(\xi) = \pi_g(\xi) + tc(g)$. Arano, Isono and Marrakchi showed that there exists a $t_{\text{diss}} \in [0, +\infty)$ such that $\widehat{\alpha}^t$ is dissipative up to compact stabilizers for every $t > t_{\text{diss}}$ and infinitely recurrent for every $t < t_{\text{diss}}$ (see §2 for terminology).

Inspired by the results obtained in [**AIM19**], we study a similar phase transition framework, but in the setting of non-singular Bernoulli actions. Such a phase transition framework for non-singular Bernoulli actions was already considered by Kosloff and Soo in [**KS20**]. They showed the following phase transition result for the family of non-singular Bernoulli actions of $G = \mathbb{Z}$ with base space $X_0 = \{0, 1\}$ that was introduced in [**VW17**, Corollary 6.3]. For every $t \in [0, +\infty)$ consider the family of measures $(\mu_n^t)_{n \in \mathbb{Z}}$ given by

$$\mu_n^t(0) = \begin{cases} 1/2 & \text{if } n \leq 4t^2, \\ 1/2 + t/\sqrt{n} & \text{if } n > 4t^2. \end{cases}$$

Then $\mathbb{Z} \curvearrowright (X, \mu_t) = \prod_{n \in \mathbb{Z}}(\{0, 1\}, \mu_n^t)$ is non-singular for every $t \in [0, +\infty)$. Kosloff and Soo showed that there exists a $t_1 \in (1/6, +\infty)$ such that $\mathbb{Z} \curvearrowright (X, \mu_t)$ is conservative for every $t < t_1$ and dissipative for every $t > t_1$ [**KS20**, Theorem 3]. In [**DKR20**, Example D] the authors describe a family of *non-singular Poisson suspensions* for which a similar phase transition occurs. These examples arise from dissipative essentially free actions of $\mathbb{Z}$, and thus they are non-singular Bernoulli actions. We generalize the phase transition result from [**KS20**] to arbitrary non-singular Bernoulli actions as follows.

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Suppose that G is a countable infinite group and let $(\mu_g)_{g\in G}$ be a family of equivalent probability measure on a standard Borel space X_0. Let ν also be a probability measure on X_0. For every $t \in [0, 1]$ we consider the family of equivalent probability measures $(\mu_g^t)_{g\in G}$ that are defined by

$$\mu_g^t = (1-t)\nu + t\mu_g. \tag{1.4}$$

Our first main result is that in this setting there is a phase transition phenomenon.

THEOREM A. *Let G be a countable infinite group and assume that the Bernoulli action $G \curvearrowright (X, \mu_1) = \prod_{g\in G}(X_0, \mu_g)$ is non-singular. Let $\nu \sim \mu_e$ be a probability measure on X_0 and for every $t \in [0, 1]$ consider the family $(\mu_g^t)_{g\in G}$ of equivalent probability measures given by (1.4). Then the Bernoulli action*

$$G \curvearrowright (X, \mu_t) = \prod_{g\in G}(X_0, \mu_g^t)$$

is non-singular for every $t \in [0, 1]$ and there exists a $t_1 \in [0, 1]$ such that $G \curvearrowright (X, \mu_t)$ is weakly mixing for every $t < t_1$ and dissipative for every $t > t_1$.

Suppose that G is a non-amenable countable infinite group. Recall that for any standard probability space (X_0, μ_0), the pmp Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is strongly ergodic. Consider again the family of probability measures $(\mu_g^t)_{g\in G}$ given by (1.4). In Theorem B below we prove that for t close enough to 0, the resulting non-singular Bernoulli action is strongly ergodic. This is inspired by [**AIM19**, Theorem 7.20] and [**MV20**, Theorem 5.1], which state similar results for non-singular Gaussian actions.

THEOREM B. *Let G be a countable infinite non-amenable group and suppose that the Bernoulli action $G \curvearrowright (X, \mu_1) = \prod_{g\in G}(X_0, \mu_g)$ is non-singular. Let $\nu \sim \mu_e$ be a probability measure on X_0 and for every $t \in [0, 1]$ consider the family $(\mu_g^t)_{g\in G}$ of equivalent probability measures given by (1.4). Then there exists a $t_0 \in (0, 1]$ such that $G \curvearrowright (X, \mu_t) = \prod_{g\in G}(X_0, \mu_g^t)$ is strongly ergodic for every $t < t_0$.*

Although we can prove a phase transition result in large generality, it remains very challenging to compute the critical value t_1. However, when $G \subset \mathrm{Aut}(T)$, for some locally finite tree T, following [**AIM19**, §10], we can construct *generalized Bernoulli actions* of which we can determine the conservativeness behaviour very precisely. To put this result into perspective, let us first explain briefly the construction from [**AIM19**, §10].

For a locally finite tree T, let $\Omega(T)$ denote the set of orientations on T. Let $p \in (0, 1)$ and fix a root $\rho \in T$. Define a probability measure μ_p on $\Omega(T)$ by orienting an edge towards ρ with probability p and away from ρ with probability $1-p$. If $G \subset \mathrm{Aut}(T)$ is a subgroup, then we naturally obtain a non-singular action $G \curvearrowright (\Omega(T), \mu_p)$. Up to equivalence of measures, the measure μ_p does not depend on the choice of root $\rho \in T$. The *Poincaré exponent* of $G \subset \mathrm{Aut}(T)$ is defined as

$$\delta(G \curvearrowright T) = \inf\left\{s > 0 \text{ for which } \sum_{w\in G\cdot v} \exp(-sd(v, w)) < +\infty\right\}, \tag{1.5}$$

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where $v \in V(T)$ is any vertex of T. In [**AIM19**, Theorem 10.4] Arano, Isono and Marrakchi showed that if $G \subset \mathrm{Aut}(T)$ is a closed non-elementary subgroup, the action $G \curvearrowright (\Omega(T), \mu_p)$ is dissipative up to compact stabilizers if $2\sqrt{p(1-p)} < \exp(-\delta)$ and weakly mixing if $2\sqrt{p(1-p)} > \exp(-\delta)$. This motivates the following similar construction.

Let $E(T) \subset V(T) \times V(T)$ denote the set of *oriented edges*, so that vertices v and w are adjacent if and only if $(v, w), (w, v) \in E(T)$. Suppose that X_0 is a standard Borel space and that μ_0, μ_1 are equivalent probability measures on X_0. Fix a root $\rho \in T$ and define a family of probability measures $(\mu_e)_{e \in E(T)}$ by

$$\mu_e = \begin{cases} \mu_0 & \text{if } e \text{ is oriented towards } \rho, \\ \mu_1 & \text{if } e \text{ is oriented away from } \rho. \end{cases} \tag{1.6}$$

Suppose that $G \subset \mathrm{Aut}(T)$ is a subgroup. Then the generalized Bernoulli action

$$G \curvearrowright \prod_{e \in E(T)} (X_0, \mu_e): \quad (g \cdot x)_e = x_{g^{-1} \cdot e} \tag{1.7}$$

is non-singular and up to conjugacy it does not depend on the choice of root $\rho \in T$. In our next main result we generalize [**AIM19**, Theorem 10.4] to non-singular actions of the form (1.7).

THEOREM C. *Let T be a locally finite tree with root $\rho \in T$ and let $G \subset \mathrm{Aut}(T)$ be a non-elementary closed subgroup with Poincaré exponent $\delta = \delta(G \curvearrowright T)$. Let μ_0 and μ_1 be equivalent probability measures on a standard Borel space X_0 and define a family of equivalent probability measures $(\mu_e)_{e \in E(T)}$ by (1.6). Then the generalized Bernoulli action (1.7) is dissipative up to compact stabilizers if* $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$ *and weakly mixing if* $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$.

2. *Preliminaries*

2.1. *Non-singular group actions.* Let $(X, \mu), (Y, \nu)$ be standard measure spaces. A Borel map $\varphi\colon X \to Y$ is called *non-singular* if the pushforward measure $\varphi_*\mu$ is equivalent to ν. If in addition there exist conull Borel sets $X_0 \subset X$ and $Y_0 \subset Y$ such that $\varphi\colon X_0 \to Y_0$ is a bijection we say that φ is a *non-singular isomorphism*. We write $\mathrm{Aut}(X, \mu)$ for the group of all non-singular automorphisms $\varphi\colon X \to X$, where we identify two elements if they agree almost everywhere. The group $\mathrm{Aut}(X, \mu)$ carries a canonical Polish topology.

A non-singular group action $G \curvearrowright (X, \mu)$ of an lcsc group G on a standard measure space (X, μ) is a continuous group homomorphism $G \to \mathrm{Aut}(X, \mu)$. A non-singular group action $G \curvearrowright (X, \mu)$ is called *essentially free* if the stabilizer subgroup $G_x = \{g \in G : g \cdot x = x\}$ is trivial for almost every (a.e.) $x \in X$. When G is countable this is the same as the condition that $\mu(\{x \in X : g \cdot x = x\}) = 0$ for every $g \in G \setminus \{e\}$. We say that $G \curvearrowright (X, \mu)$ is *ergodic* if every G-invariant Borel set $A \subset X$ satisfies $\mu(A) = 0$ or $\mu(X \setminus A) = 0$. A non-singular action $G \curvearrowright (X, \mu)$ is called *weakly mixing* if for any ergodic pmp action $G \curvearrowright (Y, \nu)$ the diagonal product action $G \curvearrowright X \times Y$ is ergodic. If G is not compact and $G \curvearrowright (X, \mu)$ is pmp, we say that $G \curvearrowright X$ is *mixing* if

$$\lim_{g \to \infty} \mu(g \cdot A \cap B) = \mu(A)\mu(B) \quad \text{for every pair of Borel subsets } A, B \subset X.$$

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Suppose that $G \curvearrowright (X, \mu)$ is a non-singular action and that μ is a probability measure. A sequence of Borel subsets $A_n \subset X$ is called *almost invariant* if

$$\sup_{g \in K} \mu(g \cdot A_n \triangle A_n) \to 0 \quad \text{for every compact subset } K \subset G.$$

The action $G \curvearrowright (X, \mu)$ is called *strongly ergodic* if every almost invariant sequence $A_n \subset X$ is trivial, that is, $\mu(A_n)(1 - \mu(A_n)) \to 0$. The strong ergodicity of $G \curvearrowright (X, \mu)$ only depends on the measure class of μ. When (Y, ν) is a standard measure space and ν is infinite, a non-singular action $G \curvearrowright (Y, \nu)$ is called strongly ergodic if $G \curvearrowright (Y, \nu')$ is strongly ergodic, where ν' is a probability measure that is equivalent to ν.

Following [**AIM19**, Definition A.16], we say that a non-singular action $G \curvearrowright (X, \mu)$ is *dissipative up to compact stabilizers* if each ergodic component is of the form $G \curvearrowright G/K$, for a compact subgroup $K \subset G$. By [**AIM19**, Theorem A.29] a non-singular action $G \curvearrowright (X, \mu)$, with $\mu(X) = 1$, is dissipative up to compact stabilizers if and only if

$$\int_G \frac{dg\mu}{d\mu}(x)\, d\lambda(g) < +\infty \quad \text{for a.e. } x \in X,$$

where λ denotes the left invariant Haar measure on G. We say that $G \curvearrowright (X, \mu)$ is *infinitely recurrent* if for every non-negligible subset $A \subset X$ and every compact subset $K \subset G$ there exists $g \in G \setminus K$ such that $\mu(g \cdot A \cap A) > 0$. By [**AIM19**, Proposition A.28] and Lemma 2.1 below, a non-singular action $G \curvearrowright (X, \mu)$, with $\mu(X) = 1$, is infinitely recurrent if and only if

$$\int_G \frac{dg\mu}{d\mu}(x)\, d\lambda(g) = +\infty \quad \text{for a.e. } x \in X.$$

A non-singular action $G \curvearrowright (X, \mu)$ is called *dissipative* if it is essentially free and dissipative up to compact stabilizers. In that case there exists a standard measure space (X_0, μ_0) such that $G \curvearrowright X$ is conjugate with the action $G \curvearrowright G \times X_0 : g \cdot (h, x) = (gh, x)$. A non-singular action $G \curvearrowright (X, \mu)$ decomposes, uniquely up to a null set, as $G \curvearrowright D \sqcup C$, where $G \curvearrowright D$ is dissipative up to compact stabilizers and $G \curvearrowright C$ is infinitely recurrent. When G is a countable group and $G \curvearrowright (X, \mu)$ is essentially free, we say that $G \curvearrowright X$ is *conservative* if it is infinitely recurrent.

LEMMA 2.1. *Suppose that G is an lcsc group with left invariant Haar measure λ and that (X, μ) is a standard probability space. Assume that $G \curvearrowright (X, \mu)$ is a non-singular action that is infinitely recurrent. Then we have that*

$$\int_G \frac{dg\mu}{d\mu}(x)\, d\lambda(g) = +\infty \quad \textit{for a.e. } x \in X.$$

Proof. Note that the set

$$D = \left\{ x \in X : \int_G \frac{dg\mu}{d\mu}(x)\, d\lambda(g) < +\infty \right\}$$

is G-invariant. Therefore, it suffices to show that $G \curvearrowright X$ is not infinitely recurrent under the assumption that D has full measure.

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Let $\pi : (X, \mu) \to (Y, \nu)$ be the projection onto the space of ergodic components of $G \curvearrowright X$. Then there exist a conull Borel subset $Y_0 \subset Y$ and a Borel map $\theta : Y_0 \to X$ such that $(\pi \circ \theta)(y) = y$ for every $y \in Y_0$.

Write $X_y = \pi^{-1}(\{y\})$. By [**AIM19**, Theorem A.29], for a.e. $y \in Y$ there exists a compact subgroup $K_y \subset G$ such that $G \curvearrowright X_y$ is conjugate with $G \curvearrowright G/K_y$. Let $G_n \subset G$ be an increasing sequence of compact subsets of G such that $\bigcup_{n\geq 1} \mathring{G}_n = G$. For every $x \in X$, write $G_x = \{g \in G : g \cdot x = x\}$ for the stabilizer subgroup of x. Using an argument as in [**MRV11**, Lemma 10], one shows that for each $n \geq 1$ the set $\{x \in X : G_x \subset G_n\}$ is Borel. Thus, for every $n \geq 1$ the set

$$U_n = \{y \in Y_0 : K_y \subset G_n\} = \{y \in Y_0 : G_{\theta(y)} \subset G_n\}$$

is a Borel subset of Y and we have that $\nu(\bigcup_{n\geq 1} U_n) = 1$. Therefore, the sets

$$A_n = \{g \cdot \theta(y) : g \in G_n, y \in U_n\}$$

are analytic and exhaust X up to a set of measure zero. So there exist an $n_0 \in \mathbb{N}$ and a non-negligible Borel set $B \subset A_{n_0}$. Suppose that $h \in G$ is such that $h \cdot B \cap B \neq \emptyset$. Then there exist $y \in U_{n_0}$ and $g_1, g_2 \in G_{n_0}$ such that $hg_1 \cdot \theta(y) = g_2 \cdot \theta(y)$, and we get that $h \in G_{n_0} K_y G_{n_0}^{-1} \subset G_{n_0} G_{n_0} G_{n_0}^{-1}$. In other words, for $h \in G$ outside the compact set $G_{n_0} G_{n_0} G_{n_0}^{-1}$ we have that $\mu(h \cdot B \cap B) = 0$, so that $G \curvearrowright X$ is not infinitely recurrent. □

We will frequently use the following result of Schmidt and Walters. Suppose that $G \curvearrowright (X, \mu)$ is a non-singular action that is infinitely recurrent and suppose that $G \curvearrowright (Y, \nu)$ is pmp and mixing. Then by [**SW81**, Theorem 2.3] we have that

$$L^\infty(X \times Y)^G = L^\infty(X)^G \mathbin{\overline{\otimes}} 1,$$

where $G \curvearrowright X \times Y$ acts diagonally. Although [**SW81**, Theorem 2.3] demands proper ergodicity of the action $G \curvearrowright (X, \mu)$, the infinite recurrence assumption is sufficient as remarked in [**AIM19**, Remark 7.4].

2.2. *The Maharam extension and crossed products.* Let (X, μ) be a standard measure space. For any non-singular automorphism $\varphi \in \mathrm{Aut}(X, \mu)$, we define its *Maharam extension* by

$$\widetilde{\varphi} : X \times \mathbb{R} \to X \times \mathbb{R} : \quad \widetilde{\varphi}(x, t) = (\varphi(x), t + \log(d\varphi^{-1}\mu/d\mu)(x)).$$

Then $\widetilde{\varphi}$ preserves the infinite measure $\mu \times \exp(-t)dt$. The assignment $\varphi \mapsto \widetilde{\varphi}$ is a continuous group homomorphism from $\mathrm{Aut}(X)$ to $\mathrm{Aut}(X \times \mathbb{R})$. Thus, for each non-singular group action $G \curvearrowright (X, \mu)$, by composing with this map, we obtain a non-singular group action $G \curvearrowright X \times \mathbb{R}$, which we call the *Maharam extension of* $G \curvearrowright X$. If $G \curvearrowright X$ is a non-singular group action, the translation action $\mathbb{R} \curvearrowright X \times \mathbb{R}$ in the second component commutes with the Maharam extension $G \curvearrowright X \times \mathbb{R}$. Therefore, we get a well-defined action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^G$, which is the *Krieger flow* associated to the action $G \curvearrowright X$. The Krieger flow is given by $\mathbb{R} \curvearrowright \mathbb{R}$ if and only if there exists a G-invariant σ-finite measure ν on X that is equivalent to μ.

Suppose that $M \subset B(\mathcal{H})$ is a von Neumann algebra represented on the Hilbert space $\mathcal{H}$ and that $\alpha \colon G \curvearrowright M$ is a continuous action on M of an lcsc group G. Then the *crossed product von Neumann algebra* $M \rtimes_\alpha G \subset B(L^2(G, \mathcal{H}))$ is the von Neumann algebra generated by the operators $\{\pi(x)\}_{x \in M}$ and $\{u_h\}_{h \in G}$ acting on $\xi \in L^2(G, \mathcal{H})$ as

$$(\pi(x)\xi)(g) = \alpha_{g^{-1}}(x)\xi(g), \quad (u_h\xi)(g) = \xi(h^{-1}g).$$

In particular, if $G \curvearrowright (X, \mu)$ is a non-singular group action, the crossed product $L^\infty(X) \rtimes G \subset B(L^2(G \times X))$ is the von Neumann algebra generated by the operators

$$(\pi(H)\xi)(g, x) = H(g \cdot x)\xi(g, x), \quad (u_h\xi)(g, x) = \xi(h^{-1}g, x),$$

for $H \in L^\infty(X)$ and $h \in G$. If $G \curvearrowright X$ is non-singular essentially free and ergodic, then $L^\infty(X) \rtimes G$ is a factor. Moreover, when G is a unimodular group, the Krieger flow of $G \curvearrowright X$ equals the flow of weights of the crossed product von Neumann algebra $L^\infty(X) \rtimes G$. For non-unimodular groups this is not necessarily true, motivating the following definition.

Definition 2.2. Let G be an lcsc group with modular function $\Delta \colon G \to \mathbb{R}_{>0}$. Let λ denote the Lebesgue measure on $\mathbb{R}$. Suppose that $\alpha \colon G \curvearrowright (X, \mu)$ is a non-singular action. We define the *modular Maharam extension* of $G \curvearrowright X$ as the non-singular action

$$\beta \colon G \curvearrowright (X \times \mathbb{R}, \mu \times \lambda) : \quad g \cdot (x, t) = (g \cdot x, t + \log(\Delta(g)) + \log(dg^{-1}\mu/d\mu)(x)).$$

Let $L^\infty(X \times \mathbb{R})^\beta$ denote the subalgebra of β-invariant elements. We define the *flow of weights* associated to $G \curvearrowright X$ as the translation action $\mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R})^\beta : (t \cdot H)(x, s) = H(x, s - t)$.

As we explain below, the flow of weights associated to an essentially free ergodic non-singular action $G \curvearrowright X$ equals the flow of weights of the crossed product factor $L^\infty(X) \rtimes G$, justifying the terminology. See also [Sa74, Proposition 4.1].

Let $\alpha \colon G \curvearrowright X$ be an essentially free ergodic non-singular group action with modular Maharam extension $\beta \colon G \curvearrowright X \times \mathbb{R}$. By [Sa74, Proposition 1.1] there is a canonical normal semifinite faithful weight φ on $L^\infty(X) \rtimes_\alpha G$ such that the modular automorphism group σ^φ is given by

$$\sigma_t^\varphi(\pi(H)) = \pi(H), \quad \sigma_t^\varphi(u_g) = \Delta(g)^{it} u_g \pi((dg^{-1}\mu/d\mu)^{it}),$$

where $\Delta \colon G \to \mathbb{R}_{>0}$ denotes the modular function of G.

For an element $\xi \in L^2(\mathbb{R}, L^2(G \times X))$ and $(g, x) \in G \times X$, write $\xi_{g,x}$ for the map given by $\xi_{g,x}(s) = \xi(s, g, x)$. Then by Fubini's theorem $\xi_{g,x} \in L^2(\mathbb{R})$ for a.e. $(g, x) \in G \times X$. Let $U \colon L^2(\mathbb{R}, L^2(G \times X)) \to L^2(G, L^2(X \times \mathbb{R}))$ be the unitary given on $\xi \in L^2(\mathbb{R}, L^2(G \times X))$ by

$$(U\xi)(g, x, t) = \mathcal{F}^{-1}(\xi_{g,x})(t + \log(\Delta(g)) + \log(dg^{-1}\mu/d\mu)(x)),$$

where $\mathcal{F}^{-1} \colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ denotes the inverse Fourier transform. One can check that conjugation by U induces an isomorphism

$$\Psi \colon (L^\infty(X) \rtimes_\alpha G) \rtimes_{\sigma^\varphi} \mathbb{R} \to L^\infty(X \times \mathbb{R}) \rtimes_\beta G.$$

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Let $\kappa : L^\infty(X \times \mathbb{R}) \to L^\infty(X \times \mathbb{R}) \rtimes_\beta G$ be the inclusion map and let $\gamma : \mathbb{R} \curvearrowright L^\infty(X \times \mathbb{R}) \rtimes_\beta G$ be the action given by

$$\gamma_t(\kappa(H))(x, s) = \kappa(H)(x, s - t), \quad \gamma_t(u_g) = u_g.$$

Then one can verify that Ψ conjugates the dual action $\widehat{\sigma^\varphi} : \mathbb{R} \curvearrowright (L^\infty(X) \rtimes_\alpha G) \rtimes_{\sigma^\varphi} \mathbb{R}$ and γ. Therefore, we can identify the flow of weights $\mathbb{R} \curvearrowright \mathcal{Z}((L^\infty(X) \rtimes_\alpha G) \rtimes_{\sigma^\varphi} \mathbb{R})$ with $\mathbb{R} \curvearrowright \mathcal{Z}(L^\infty(X \times \mathbb{R}) \rtimes_\beta G) \cong L^\infty(X \times \mathbb{R})^\beta$: the flow of weights associated to $G \curvearrowright X$.

Remark 2.3. It will be useful to speak about the *Krieger type* of a non-singular ergodic action $G \curvearrowright X$. In light of the discussion above, we will only use this terminology for countable groups G, so that no confusion arises with the type of the crossed product von Neumann algebra $L^\infty(X) \rtimes G$. So assume that G is countable and that $G \curvearrowright (X, \mu)$ is a non-singular ergodic action. Then the Krieger flow is ergodic and we distinguish several cases. If ν is atomic, we say that $G \curvearrowright X$ is of type I. If ν is non-atomic and finite, we say that $G \curvearrowright X$ is of type II_1. If ν is non-atomic and infinite, we say that $G \curvearrowright X$ is of type II_∞. If the Krieger flow is given by $\mathbb{R} \curvearrowright \mathbb{R}/\log(\lambda)\mathbb{Z}$ with $\lambda \in (0, 1)$, we say that $G \curvearrowright X$ is of type III_λ. If the Krieger flow is the trivial flow $\mathbb{R} \curvearrowright \{*\}$, we say that $G \curvearrowright X$ is of type III_1. If the Krieger flow is properly ergodic (that is, every orbit has measure zero), we say that $G \curvearrowright X$ is of type III_0.

2.3. *Non-singular Bernoulli actions.* Suppose that G is a countable infinite group and that $(\mu_g)_{g\in G}$ is a family of equivalent probability measures on a standard Borel space X_0. The action

$$G \curvearrowright (X, \mu) = \prod_{h \in G} (X_0, \mu_h) : \quad (g \cdot x)_h = x_{g^{-1}h} \tag{2.1}$$

is called the *Bernoulli action*. For two probability measures ν, η on a standard Borel space Y, the *Hellinger distance* $H^2(\nu, \eta)$ is defined by

$$H^2(\nu, \eta) = \frac{1}{2} \int_Y \left(\sqrt{d\nu/d\zeta} - \sqrt{d\eta/d\zeta}\right)^2 d\zeta, \tag{2.2}$$

where ζ is any probability measure on Y such that $\nu, \eta \prec \zeta$. By Kakutani's criterion for equivalence of infinite product measures [**Kak48**] the Bernoulli action (2.1) is non-singular if and only if

$$\sum_{h \in G} H^2(\mu_h, \mu_{gh}) < +\infty \quad \text{for every } g \in G.$$

If (X, μ) is non-atomic and the Bernoulli action (2.1) is non-singular, then it is essentially free by [**BKV19**, Lemma 2.2].

Suppose that I is a countable infinite set and that $(\mu_i)_{i\in I}$ is a family of equivalent probability measures on a standard Borel space X_0. If G is an lcsc group that acts on I, the action

$$G \curvearrowright (X, \mu) = \prod_{i \in I} (X_0, \mu_i) : \quad (g \cdot x)_i = x_{g^{-1} \cdot i} \tag{2.3}$$

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is called the *generalized Bernoulli action* and it is non-singular if and only if $\sum_{i\in I} H^2(\mu_i, \mu_{g\cdot i}) < +\infty$ for every $g \in G$. When ν is a probability measure on X_0 such that $\mu_i = \nu$ for every $i \in I$, the generalized Bernoulli action (2.3) is pmp and it is mixing if and only if the stabilizer subgroup $G_i = \{g \in G : g \cdot i = i\}$ is compact for every $i \in I$. In particular, if G is countable infinite, the pmp Bernoulli action $G \curvearrowright (X_0, \mu_0)^G$ is mixing.

2.4. *Groups acting on trees.* Let $T = (V(T), E(T))$ be a locally finite tree, so that the edge set $E(T)$ is a symmetric subset of $V(T) \times V(T)$ with the property that vertices $v, w \in V(T)$ are adjacent if and only if $(v, w), (w, v) \in E(T)$. When T is clear from the context, we will write E instead of $E(T)$. Also we will often write T instead of $V(T)$ for the vertex set. For any two vertices $v, w \in T$ let $[v, w]$ denote the smallest subtree of T that contains v and w. The distance between vertices $v, w \in T$ is defined as $d(v, w) = |V([v, w])| - 1$. Fixing a root $\rho \in T$, we define the *boundary* ∂T of T as the collection of all infinite line segments starting at ρ. We equip ∂T with a metric d_ρ as follows. If $\omega, \omega' \in \partial T$, let $v \in T$ be the unique vertex such that $d(\rho, v) = \sup_{v\in\omega\cap\omega'} d(\rho, v)$ and define

$$d_\rho(\omega, \omega') = \exp(-d(\rho, v)).$$

Then, up to homeomorphism, the space $(\partial T, d_\rho)$ does not depend on the chosen root $\rho \in T$. Furthermore, the Hausdorff dimension $\dim_H \partial T$ of $(\partial T, d_\rho)$ is also independent of the choice of $\rho \in T$.

Let $\mathrm{Aut}(T)$ denote the group of automorphisms of T. By [**Tit70**, Proposition 3.2], if $g \in \mathrm{Aut}(T)$, then either:

- g fixes a vertex or interchanges a pair of vertices (in this case we say that g is *elliptic*);
- or there exists a bi-infinite line segment $L \subset T$, called the *axis* of g, such that g acts on L by non-trivial translation (in this case we say that g is *hyperbolic*).

We equip $\mathrm{Aut}(T)$ with the topology of pointwise convergence. A subgroup $G \subset \mathrm{Aut}(T)$ is closed with respect to this topology if and only if for every $v \in T$ the stabilizer subgroup $G_v = \{g \in G : g \cdot v = v\}$ is compact. An action of an lcsc group G on T is a continuous homomorphism $G \to \mathrm{Aut}(T)$. We say that the action $G \curvearrowright T$ is *cocompact* if there is a finite set $F \subset E(T)$ such that $G \cdot F = E(T)$. A subgroup $G \subset \mathrm{Aut}(T)$ is called *non-elementary* if it does not fix any point in $T \cup \partial T$ and does not interchange any pair of points in $T \cup \partial T$. Equivalently, $G \subset \mathrm{Aut}(T)$ is non-elementary if there exist hyperbolic elements $h, g \in G$ with axes L_h and L_g such that $L_h \cap L_g$ is finite. If $G \subset \mathrm{Aut}(T)$ is a non-elementary closed subgroup, there exists a unique minimal G-invariant subtree $S \subset T$ and G is compactly generated if and only if $G \curvearrowright S$ is cocompact (see [**CM11**, §2]). Recall from (1.5) the definition of the Poincaré exponent $\delta(G \curvearrowright T)$ of a subgroup $G \subset \mathrm{Aut}(T)$. If $G \subset \mathrm{Aut}(T)$ is a closed subgroup such that $G \curvearrowright T$ is cocompact, then we have that $\delta(G \curvearrowright T) = \dim_H \partial T$.

3. *Phase transitions of non-singular Bernoulli actions: proof of Theorems A and B*

Let G be a countable infinite group and let $(\mu_g)_{g\in G}$ be a family of equivalent probability measures on a standard Borel space X_0. Let ν also be a probability measure on X_0. For $t \in [0, 1]$ we define the family of probability measures

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$$\mu_g^t = (1-t)\nu + t\mu_g, \quad g \in G. \tag{3.1}$$

We write μ_t for the infinite product measure $\mu_t = \prod_{g\in G} \mu_g^t$ on $X = \prod_{g\in G} X_0$. We prove Theorem 3.1 below, which is slightly more general than Theorem A.

THEOREM 3.1. *Let G be a countable infinite group and let $(\mu_g)_{g\in G}$ be a family of equivalent probability measures on a standard probability space X_0, which is not supported on a single atom. Assume that the Bernoulli action $G \curvearrowright \prod_{g\in G}(X_0, \mu_g)$ is non-singular. Let ν also be a probability measure on X_0. Then for every $t \in [0, 1]$ the Bernoulli action*

$$G \curvearrowright (X, \mu_t) = \prod_{g\in G} (X_0, (1-t)\nu + t\mu_g) \tag{3.2}$$

is non-singular. Assume, in addition, that one of the following conditions holds.

(1) $\nu \sim \mu_e$.

(2) $\nu \prec \mu_e$ *and* $\sup_{g\in G} |\log d\mu_g/d\mu_e(x)| < +\infty$ *for a.e* $x \in X_0$.

Then there exists a $t_1 \in [0, 1]$ such that $G \curvearrowright (X, \mu_t)$ is dissipative for every $t > t_1$ and weakly mixing for every $t < t_1$.

Remark 3.2. One might hope to prove a completely general phase transition result that only requires $\nu \prec \mu_e$, and not the additional assumption that $\sup_{g\in G} |\log d\mu_g/d\mu_e(x)| < +\infty$ for a.e. $x \in X_0$. However, the following example shows that this is not possible.

Let G be any countable infinite group and let $G \curvearrowright \prod_{g\in G}(C_0, \eta_g)$ be a conservative non-singular Bernoulli action. Note that Theorem 3.1 implies that

$$G \curvearrowright \prod_{g\in G} (C_0, (1-t)\eta_e + t\eta_g)$$

is conservative for every $t < 1$. Let C_1 be a standard Borel space and let $(\mu_g)_{g\in G}$ be a family of equivalent probability measures on $X_0 = C_0 \sqcup C_1$ such that $0 < \sum_{g\in G} \mu_g(C_1) < +\infty$ and such that $\mu_g|_{C_0} = \mu_g(C_0)\eta_g$. Then the Bernoulli action $G \curvearrowright (X, \mu) = \prod_{g\in G}(X_0, \mu_g)$ is non-singular with non-negligible conservative part $C_0^G \subset G$ and dissipative part $X \setminus C_0^G$. Taking $\nu = \eta_e \prec \mu_e$, for each $t < 1$ the Bernoulli action $G \curvearrowright (X, \mu_t) = \prod_{g\in G}(X_0, (1-t)\eta_e + t\mu_g)$ is constructed in the same way, by starting with the conservative Bernoulli action $G \curvearrowright \prod_{g\in G}(C_0, (1-t)\eta_e + t\eta_g)$. So for every $t \in (0, 1)$ the Bernoulli action $G \curvearrowright (X, \mu_t)$ has non-negligible conservative part and non-negligible dissipative part.

We can also prove a version of Theorem B in the more general setting of Theorem 3.1.

THEOREM 3.3. *Let G be a countable infinite non-amenable group. Make the same assumptions as in Theorem 3.1 and consider the non-singular Bernoulli actions $G \curvearrowright (X, \mu_t)$ given by (3.2). Assume, moreover, that:*

(1) $\nu \sim \mu_e$, *or*

(2) $\nu \prec \mu_e$ *and* $\sup_{g\in G} |\log d\mu_g/d\mu_e(x)| < +\infty$ *for a.e.* $x \in X_0$.

Then there exists a $t_0 > 0$ such that $G \curvearrowright (X, \mu_t)$ is strongly ergodic for every $t < t_0$.

Proof of Theorem 3.1. Assume that $G \curvearrowright (X, \mu_1) = \prod_{g\in G}(X_0, \mu_g)$ is non-singular. For every $t \in [0, 1]$ we have that

$$\sum_{h\in G} H^2(\mu_h^t, \mu_{gh}^t) \leq t \sum_{h\in G} H^2(\mu_h, \mu_{gh}) \quad \text{for every } g \in G,$$

so that $G \curvearrowright (X, \mu_t)$ is non-singular for every $t \in [0, 1]$. The rest of the proof we divide into two steps.

CLAIM 1. *If $G \curvearrowright (X, \mu_t)$ is conservative, then $G \curvearrowright (X, \mu_s)$ is weakly mixing for every $s < t$.*

Proof of Claim 1. Note that for every $g \in G$ we have that

$$(\mu_g^s)^r = (1-r)\nu + r\mu_g^s = (1-r)\nu + r(1-s)\nu + rs\mu_g = \mu_g^{sr},$$

so that $(\mu_s)_r = \mu_{sr}$. Therefore, it suffices to prove that $G \curvearrowright (X, \mu_s)$ is weakly mixing for every $s < 1$, assuming that $G \curvearrowright (X, \mu_1)$ is conservative.

The claim is trivially true for $s = 0$. So assume that $G \curvearrowright (X, \mu_1)$ is conservative and fix $s \in (0, 1)$. Let $G \curvearrowright (Y, \eta)$ be an ergodic pmp action. Define $Y_0 = X_0 \times X_0 \times \{0, 1\}$ and define the probability measures λ on $\{0, 1\}$ by $\lambda(0) = s$. Define the map $\theta\colon Y_0 \to X_0$ by

$$\theta(x, x', j) = \begin{cases} x & \text{if } j = 0, \\ x' & \text{if } j = 1. \end{cases} \tag{3.3}$$

Then for every $g \in G$ we have that $\theta_*(\mu_g \times \nu \times \lambda) = \mu_g^s$. Write $Z = \{0, 1\}^G$ and equip Z with the probability measure λ^G. We identify the Bernoulli action $G \curvearrowright Y_0^G$ with the diagonal action $G \curvearrowright X \times X \times Z$. By applying θ in each coordinate we obtain a G-equivariant factor map

$$\Psi\colon X \times X \times Z \to X : \quad \Psi(x, x', z)_h = \theta(x_h, x_h', z_h). \tag{3.4}$$

Then the map $\mathrm{id}_Y \times \Psi\colon Y \times X \times X \times Z \to Y \times X$ is G-equivariant and we have that $(\mathrm{id}_Y \times \Psi)_*(\eta \times \mu_1 \times \mu_0 \times \lambda^G) = \eta \times \mu_s$. The construction above is similar to [**KS20**, §4].

Take $F \in L^\infty(Y \times X, \eta \times \mu_s)^G$. Note that the diagonal action $G \curvearrowright (Y \times X, \eta \times \mu_1)$ is conservative, since $G \curvearrowright (Y, \eta)$ is pmp. The action $G \curvearrowright (X \times Z, \mu_0 \times \lambda^G)$ can be identified with a pmp Bernoulli action with base space $(X_0 \times \{0, 1\}, \nu \times \lambda)$, so that it is mixing. By [**SW81**, Theorem 2.3] we have that

$$L^\infty(Y \times X \times X \times Z, \eta \times \mu_1 \times \mu_0 \times \lambda^G)^G = L^\infty(Y \times X, \eta \times \mu_1)^G \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} 1,$$

which implies that the assignment $(y, x, x', z) \mapsto F(y, \Psi(x, x', z))$ is essentially independent of x' and z. Choosing a finite set of coordinates $\mathcal{F} \subset G$ and changing, for $g \in \mathcal{F}$, the value z_g between 0 and 1, we see that F is essentially independent of the x_g-coordinates for $g \in \mathcal{F}$. As this is true for any finite set $\mathcal{F} \subset G$, we have that $F \in L^\infty(Y)^G \mathbin{\overline{\otimes}} 1$. The action $G \curvearrowright (Y, \eta)$ is ergodic and therefore F is essentially constant. We conclude that $G \curvearrowright (X, \mu_s)$ is weakly mixing. □

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CLAIM 2. *If $\nu \sim \mu_e$ and if $G \curvearrowright (X, \mu_t)$ is not dissipative, then $G \curvearrowright (X, \mu_s)$ is conservative for every $s < t$.*

Proof of Claim 2. Again it suffices to assume that $G \curvearrowright (X, \mu_1)$ is not dissipative and to show that $G \curvearrowright (X, \mu_s)$ is conservative for every $s < 1$.

When $s = 0$, the statement is trivial, so assume that $G \curvearrowright (X, \mu_1)$ is not dissipative and fix $s \in (0, 1)$. Let $C \subset X$ denote the non-negligible conservative part of $G \curvearrowright (X, \mu_1)$. As in the proof of Claim 1, write $Z = \{0, 1\}^G$ and let λ be the probability measure on $\{0, 1\}$ given by $\lambda(0) = s$. Writing $\Psi\colon X \times X \times Z \to X$ for the G-equivariant map (3.4). We claim that $\Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C\times X\times Z}) \sim \mu_s$, so that $G \curvearrowright (X, \mu_s)$ is a factor of a conservative non-singular action, and therefore must be conservative itself.

As $\Psi_*(\mu_1 \times \mu_0 \times \lambda^G) = \mu_s$, we have that $\Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C\times X\times Z}) \prec \mu_s$. Let $\mathcal{U} \subset X$ be the Borel set, uniquely determined up to a set of measure zero, such that $\Psi_*((\mu_1 \times \mu_0 \times \lambda^G)|_{C\times X\times Z}) \sim \mu_s|_{\mathcal{U}}$. We have to show that $\mu_s(X \setminus \mathcal{U}) = 0$. Fix a finite subset $\mathcal{F} \subset G$. For every $t \in [0, 1]$ define

$$(X_1, \gamma_1^t) = \prod_{g\in\mathcal{F}} (X_0, (1-t)\nu + t\mu_g),$$

$$(X_2, \gamma_2^t) = \prod_{g\in G\setminus\mathcal{F}} (X_0, (1-t)\nu + t\mu_g).$$

We shall write $\gamma_1 = \gamma_1^1$, $\gamma_2 = \gamma_2^1$. Also define

$$(Y_1, \zeta_1) = \prod_{g\in\mathcal{F}} (X_0 \times X_0 \times \{0, 1\}, \mu_g \times \nu \times \lambda),$$

$$(Y_2, \zeta_2) = \prod_{g\in G\setminus\mathcal{F}} (X_0 \times X_0 \times \{0, 1\}, \mu_g \times \nu \times \lambda).$$

By applying the map (3.3) in every coordinate, we get factor maps $\Psi_j\colon Y_j \to X_j$ that satisfy $(\Psi_j)_*(\zeta_j) = \gamma_j^s$ for $j = 1, 2$. Identify $X_1 \times Y_2 \cong X \times (X_0 \times \{0, 1\})^{G\setminus\mathcal{F}}$ and define the subset $C' \subset X_1 \times Y_2$ by $C' = C \times (X_0 \times \{0, 1\})^{G\setminus\mathcal{F}}$. Let $\mathcal{U}' \subset X$ be Borel such that

$$(\mathrm{id}_{X_1} \times \Psi_2)_*((\gamma_1 \times \zeta_2)|_{C'}) \sim (\gamma_1 \times \gamma_2^s)|_{\mathcal{U}'}.$$

Identify $Y_1 \times X_2 \cong X \times (X_0 \times \{0, 1\})^{\mathcal{F}}$ and define $V \subset Y_1 \times X_2$ by $V = \mathcal{U}' \times (X_0 \times \{0, 1\})^{\mathcal{F}}$. Then we have that

$$\begin{aligned}(\Psi_1 \times \mathrm{id}_{X_2})_*((\zeta_1 \times \gamma_2^s)|_V) &\sim (\Psi_1 \times \mathrm{id}_{X_2})_*(\mathrm{id}_{Y_1} \times \Psi_2)_*((\gamma_1 \times \zeta_1)|_{C'} \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}}) \\ &= \Psi_*((\zeta_1 \times \zeta_2)|_{C\times X\times Z}) \sim \mu_s|_{\mathcal{U}}.\end{aligned}$$

Let $\pi\colon X_1 \times X_2 \to X_2$ and $\pi'\colon Y_1 \times X_2 \to X_2$ denote the coordinate projections. Note that by construction we have that

$$\pi'_*((\zeta_1 \times \gamma_2^s)|_V) \sim \pi_*((\gamma_1 \times \gamma_2^s)|_{\mathcal{U}'}) \sim \pi_*(\mu_s|_{\mathcal{U}}). \tag{3.5}$$

Let $W \subset X_2$ be Borel such that $\pi_*(\mu_s|_{\mathcal{U}}) \sim \gamma_2^s|_W$. For every $y \in X_2$ define the Borel sets

$$\mathcal{U}_y = \{x \in X_1 : (x, y) \in \mathcal{U}\} \quad \text{and} \quad \mathcal{U}'_y = \{x \in X_1 : (x, y) \in \mathcal{U}'\}.$$

As $\pi_*((\gamma_1 \times \gamma_2^s)|_{\mathcal{U}'}) \sim \gamma_2^s|_W$, we have that

$$\gamma_1(\mathcal{U}'_y) > 0 \quad \text{for } \gamma_2^s\text{-a.e. } y \in W.$$

The disintegration of $(\gamma_1 \times \gamma_2^s)|_{\mathcal{U}'}$ along π is given by $(\gamma_1|_{\mathcal{U}'_y})_{y\in W}$. Therefore, the disintegration of $(\zeta_1 \times \gamma_2^s)|_V$ along π' is given by $(\gamma_1|_{\mathcal{U}'_y} \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}})_{y\in W}$. We conclude that the disintegration of $(\Psi_1 \times \mathrm{id}_{X_2})_*((\zeta_1 \times \gamma_2^s)|_V)$ along π is given by $((\Psi_1)_*(\gamma_1|_{\mathcal{U}'_y} \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}}))_{y\in W}$. The disintegration of $\mu_s|_{\mathcal{U}}$ along π is given by $(\gamma_2^s|_{\mathcal{U}_y})_{y\in W}$. Since $\mu_s|_{\mathcal{U}} \sim (\Psi_1 \times \ \mathrm{id}_{X_2})_*((\zeta_1 \times \gamma_2^s)|_V)$, we conclude that

$$(\Psi_1)_*(\gamma_1|_{\mathcal{U}'_y} \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}}) \sim \gamma_1^s|_{\mathcal{U}_y} \quad \text{for } \gamma_2^s\text{-a.e. } y \in W.$$

As $\gamma_1(\mathcal{U}'_y) > 0$ for γ_2^s-a.e. $y \in W$, and using that $\nu \sim \mu_e$, we see that

$$\begin{aligned}\gamma_1^s \sim \nu^{\mathcal{F}} &\sim (\Psi_1)_*((\gamma_1 \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}})|_{\mathcal{U}'_y \times X_0^{\mathcal{F}} \times \{1\}^{\mathcal{F}}}) \\ &\prec (\Psi_1)_*(\gamma_1|_{\mathcal{U}'_y} \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}}).\end{aligned}$$

for γ_2^s-a.e. $y \in W$. It is clear that also $(\Psi_1)_*(\gamma_1|_{\mathcal{U}'_y} \times \nu^{\mathcal{F}} \times \lambda^{\mathcal{F}}) \prec \gamma_1^s$, so that $\gamma_1^s|_{\mathcal{U}_y} \sim \gamma_1^s$ for γ_2^s-a.e. $y \in W$. Therefore, we have that $\gamma_1^s(X_1 \setminus \mathcal{U}_y) = 0$ for γ_2^s-a.e. $y \in W$, so that

$$\mu_s(\mathcal{U} \triangle (X_0^{\mathcal{F}} \times W)) = 0.$$

Since this is true for every finite subset $\mathcal{F} \subset G$, we conclude that $\mu_s(X \setminus \mathcal{U}) = 0$. □

The conclusion of the proof now follows by combining both claims. Assume that $G \curvearrowright (X, \mu_t)$ is not dissipative and fix $s < t$. Choose r such that $s < r < t$.

$\nu \sim \mu_e$. By Claim 2 we have that $G \curvearrowright (X, \mu_r)$ is conservative. Then by Claim 1 we see that $G \curvearrowright (X, \mu_s)$ is weakly mixing.

$\nu \prec \mu_e$. As $\nu \prec \mu_e$, the measures μ_e^t and μ_e are equivalent. We have that

$$\frac{d\mu_g^t}{d\mu_e^t} = \bigg((1-t)\frac{d\nu}{d\mu_e} + t\frac{d\mu_g}{d\mu_e}\bigg)\frac{d\mu_e}{d\mu_e^t}.$$

So if $\sup_{g\in G} |\log d\mu_g/d\mu_e(x)| < +\infty$ for a.e $x \in X_0$, we also have that

$$\sup_{g\in G} |\log d\mu_g^t/d\mu_e^t(x)| < +\infty \quad \text{for a.e. } x \in X_0.$$

It follows from [BV20, Proposition 4.3] that $G \curvearrowright (X, \mu_t)$ is conservative. Then by Claim 1 we have that $G \curvearrowright (X, \mu_s)$ is weakly mixing. □

Remark 3.4. Let I be a countably infinite set and suppose that we are given a family of equivalent probability measures $(\mu_i)_{i\in I}$ on a standard Borel space X_0. Let ν be a probability measure on X_0 that is equivalent to all the μ_i. If G is an lcsc group that acts

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on I such that for each $i \in I$ the stabilizer subgroup $G_i = \{g \in G : g \cdot i = i\}$ is compact, then the pmp generalized Bernoulli action

$$G \curvearrowright \prod_{i \in I} (X_0, \nu), \quad (g \cdot x)_i = x_{g^{-1} \cdot i}$$

is mixing. For $t \in [0, 1]$ write

$$(X, \mu_t) = \prod_{i \in I} (X_0, (1-t)\nu + t\mu_i)$$

and assume that the generalized Bernoulli action $G \curvearrowright (X, \mu_1)$ is non-singular.

Since [**SW81**, Theorem 2.3] still applies to infinitely recurrent actions of lcsc groups (see [**AIM19**, Remark 7.4]), it is straightforward to adapt the proof of Claim 1 in the proof of Theorem 3.1 to prove that if $G \curvearrowright (X, \mu_t)$ is infinitely recurrent, then $G \curvearrowright (X, \mu_s)$ is weakly mixing for every $s < t$. Similarly, we can adapt the proof of Claim 2, using that a factor of an infinitely recurrent action is again infinitely recurrent. Together, this leads to the following phase transition result in the lcsc setting.

Assume that $G_i = \{g \in G : g \cdot i = i\}$ is compact for every $i \in I$ and that $\nu \sim \mu_e$. Then there exists a $t_1 \in [0, 1]$ such that $G \curvearrowright (X, \mu_t)$ is dissipative up to compact stabilizers for every $t > t_1$ and weakly mixing for every $t < t_1$.

Recall the following definition from [**BKV19**, Definition 4.2]. When G is a countable infinite group and $G \curvearrowright (X, \mu)$ is a non-singular action on a standard probability space, a sequence (η_n) of probability measures on G is called *strongly recurrent* for the action $G \curvearrowright (X, \mu)$ if

$$\sum_{h \in G} \eta_n^2(h) \int_X \frac{d\mu(x)}{\sum_{k \in G} \eta_n(hk^{-1}) dk^{-1}\mu/d\mu(x)} \xrightarrow{n \to +\infty} 0.$$

We say that $G \curvearrowright (X, \mu)$ is *strongly conservative* if there exists a sequence (η_n) of probability measures on G that is strongly recurrent for $G \curvearrowright (X, \mu)$.

LEMMA 3.5. *Let $G \curvearrowright (X, \mu)$ and $G \curvearrowright (Y, \nu)$ be non-singular actions of a countable infinite group G on standard probability spaces (X, μ) and (Y, ν). Suppose that $\psi \colon (X, \mu) \to (Y, \nu)$ is a measure-preserving G-equivariant factor map and that η_n is a sequence of probability measures on G that is strongly recurrent for the action $G \curvearrowright (X, \mu)$. Then η_n is strongly recurrent for the action $G \curvearrowright (Y, \nu)$.*

Proof. Let $E \colon L^0(X, [0, +\infty)) \to L^0(Y, [0, +\infty))$ denote the conditional expectation map that is uniquely determined by

$$\int_Y E(F) H \, d\nu = \int_X F(H \circ \psi) \, d\mu$$

for all positive measurable functions $F \colon X \to [0, +\infty)$ and $H \colon Y \to [0, +\infty)$. Since

$$\frac{dk^{-1}\nu}{d\nu} = \frac{d\psi_*(k^{-1}\mu)}{d\psi_*\mu} = E\bigg(\frac{dk^{-1}\mu}{d\mu}\bigg)$$

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for every $k \in G$, we have that

$$\sum_{k\in G} \eta_n(hk^{-1}) \frac{dk^{-1}\nu}{d\nu}(y) = E\bigg(\sum_{k\in G} \eta_n(hk^{-1}) \frac{dk^{-1}\mu}{d\mu}\bigg)(y) \quad \text{for a.e. } y \in Y. \tag{3.6}$$

By Jensen's inequality for conditional expectations, applied to the convex function $t \mapsto 1/t$, we also have that

$$\frac{1}{E(\sum_{k\in G} \eta_n(hk^{-1}) dk^{-1}\mu/d\mu)(y)} \le E\bigg(\frac{1}{\sum_{k\in G} \eta_n(hk^{-1}) dk^{-1}\mu/d\mu}\bigg)(y) \quad \text{for a.e. } y \in Y. \tag{3.7}$$

Combining (3.6) and (3.7), we see that

$$\begin{aligned} &\sum_{h\in G} \eta_n^2(h) \int_Y \frac{d\nu(y)}{\sum_{k\in G} \eta_n(hk^{-1}) dk^{-1}\nu/d\nu(y)} \\ &\quad\le \sum_{h\in G} \eta_n^2(h) \int_Y E\bigg(\frac{1}{\sum_{k\in G} \eta_n(hk^{-1}) dk^{-1}\mu/d\mu}\bigg)(y)\, d\nu(y) \\ &\quad= \sum_{h\in G} \eta_n^2(h) \int_X \frac{d\mu(x)}{\sum_{k\in G} \eta_n(hk^{-1}) dk^{-1}\mu/d\mu(x)}, \end{aligned}$$

which converges to 0 as η_n is strongly recurrent for $G \curvearrowright (X, \mu)$. □

We say that a non-singular group action $G \curvearrowright (X, \mu)$ has an *invariant mean* if there exists a G-invariant linear functional $\varphi \in L^\infty(X)^*$. We say that $G \curvearrowright (X, \mu)$ is *amenable (in the sense of Zimmer)* if there exists a G-equivariant conditional expectation $E\colon L^\infty(G \times X) \to L^\infty(X)$, where the action $G \curvearrowright G \times X$ is given by $g \cdot (h, x) = (gh, g \cdot x)$.

PROPOSITION 3.6. *Let G be a countable infinite group and let $(\mu_g)_{g\in G}$ be a family of equivalent probability measures on a standard Borel space X_0 that is not supported on a single atom. Let ν be a probability measure on X_0 and for each $t \in [0, 1]$ consider the Bernoulli action (3.2). Assume that $G \curvearrowright (X, \mu_1)$ is non-singular.*

(1) *If $G \curvearrowright (X, \mu_t)$ has an invariant mean, then $G \curvearrowright (X, \mu_s)$ has an invariant mean for every $s < t$.*
(2) *If $G \curvearrowright (X, \mu_t)$ is amenable, then $G \curvearrowright (X, \mu_s)$ is amenable for every $s > t$.*
(3) *If $G \curvearrowright (X, \mu_t)$ is strongly conservative, then $G \curvearrowright (X, \mu_s)$ is strongly conservative for every $s < t$.*

Proof. (1) We may assume that $t = 1$. So suppose that $G \curvearrowright (X, \mu_1)$ has an invariant mean and fix $s < 1$. Let λ be the probability measure on $\{0, 1\}$ that is given by $\lambda(0) = s$. Then by [**AIM19**, Proposition A.9] the diagonal action $G \curvearrowright (X \times X \times \{0, 1\}^G, \mu_1 \times \mu_0 \times \lambda^G)$ has an invariant mean. Since $G \curvearrowright (X, \mu_s)$ is a factor of this diagonal action, it admits a G-invariant mean as well.

(2) It suffices to show that $G \curvearrowright (X, \mu_1)$ is amenable whenever there exists a $t \in (0, 1)$ such that $G \curvearrowright (X, \mu_t)$ is amenable. Write λ for the probability measure on $\{0, 1\}$ given by $\lambda(0) = t$. Then $G \curvearrowright (X, \mu_t)$ is a factor of the diagonal action $G \curvearrowright (X \times X \times$

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$\{0,1\}^G, \mu_1 \times \mu_0 \times \lambda^G)$, so by [Zim78, Theorem 2.4] also the latter action is amenable. Since $G \curvearrowright (X \times \{0,1\}^G, \mu_0 \times \lambda^G)$ is pmp, we have that $G \curvearrowright (X, \mu_1)$ is amenable.

(3) We may again assume that $t = 1$. Suppose that (η_n) is a strongly recurrent sequence of probability measures on G for the action $G \curvearrowright (X, \mu_1)$. Fix $s < 1$ and let λ be the probability measure on $\{0,1\}$ defined by $\lambda(0) = s$. As the diagonal action $G \curvearrowright (X \times \{0,1\}^G, \mu_0 \times \lambda^G)$ is pmp, the sequence η_n is also strongly recurrent for the diagonal action $G \curvearrowright (X \times X \times \{0,1\}, \mu_1 \times \mu_0 \times \lambda^G)$. Since $G \curvearrowright (X, \mu_t)$ is a factor of $G \curvearrowright (X \times X \times \{0,1\}^G, \mu_1 \times \mu_0 \times \lambda^G)$, it follows from Lemma 3.5 that the sequence η_n is strongly recurrent for $G \curvearrowright (X, \mu_t)$. □

We finally prove Theorem 3.3. The proof relies heavily upon the techniques developed in [MV20, §5].

Proof of Theorem 3.3. For every $t \in (0,1]$ write ρ^t for the Koopman representation

$$\rho^t : G \curvearrowright L^2(X, \mu_t) : \quad (\rho^t_g(\xi))(x) = \left(\frac{dg\mu_t}{d\mu_t}(x)\right)^{1/2} \xi(g^{-1} \cdot x).$$

Fix $s \in (0,1)$ and let $C > 0$ be such that $\log(1-x) \geq -Cx$ for every $x \in [0, s)$. Then for every $t < s$ and every $g \in G$ we have that

$$\begin{aligned}\log(\langle \rho^t_g(1), 1\rangle) &= \sum_{h \in G} \log(1 - H^2(\mu^t_{gh}, \mu^t_h)) \\ &\geq \sum_{h \in G} \log(1 - tH^2(\mu_{gh}, \mu_h)) \\ &\geq -Ct \sum_{h \in G} H^2(\mu_{gh}, \mu_h).\end{aligned}$$

Because $G \curvearrowright (X, \mu_1)$ is non-singular we get that

$$\langle \rho^t_g(1), 1\rangle \to 1 \quad \text{as } t \to 0, \text{ for every } g \in G. \tag{3.8}$$

We claim that there exists a $t' > 0$ such that $G \curvearrowright (X, \mu_t)$ is non-amenable for every $t < t'$. Suppose, to the contrary, that t_n is a sequence that converges to zero such that $G \curvearrowright (X, \mu_{t_n})$ is amenable for every $n \in \mathbb{N}$. Then it follows from [Nev03, Theorem 3.7] that ρ^{t_n} is weakly contained in the left regular representation λ_G for every $n \in \mathbb{N}$. Write 1_G for the trivial representation of G. It follows from (3.8) that $\bigoplus_{n \in \mathbb{N}} \rho^{t_n}$ has almost invariant vectors, so that

$$1_G \prec \bigoplus_{n \in \mathbb{N}} \rho^{t_n} \prec \infty\lambda_G \prec \lambda_G,$$

which is in contradiction to the non-amenability of G. By Theorem 3.1 there exists a $t_1 \in [0,1]$ such that $G \curvearrowright (X, \mu_t)$ is weakly mixing for every $t < t_1$. Since every dissipative action is amenable (see, for example, [AIM19, Theorem A.29]) it follows that $t_1 \geq t' > 0$.

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Write $Z_0 = [0, 1)$ and let λ denote the Lebesgue probability measure on Z_0. Let ρ^0 denote the reduced Koopman representation

$$\rho^0 : G \curvearrowright L^2(X \times Z_0^G, \mu_0 \times \lambda^G) \ominus \mathbb{C}1 : \quad (\rho^0_g(\xi))(x) = \xi(g^{-1} \cdot x).$$

As G is non-amenable, ρ^0 has stable spectral gap. Suppose that for every $s > 0$ we can find $0 < s' < s$ such that $\rho^{s'}$ is weakly contained in $\rho^{s'} \otimes \rho^0$. Then there exists a sequence s_n that converges to zero, such that ρ^{s_n} is weakly contained in $\rho^{s_n} \otimes \rho^0$ for every $n \in \mathbb{N}$. This implies that $\bigoplus_{n\in\mathbb{N}} \rho^{s_n}$ is weakly contained in $(\bigoplus_{n\in\mathbb{N}} \rho^{s_n}) \otimes \rho^0$. But by (3.8), the representation $\bigoplus_{n\in\mathbb{N}} \rho^{s_n}$ has almost invariant vectors, so that $(\bigoplus_{n\in\mathbb{N}} \rho^{s_n}) \otimes \rho^0$ weakly contains the trivial representation. This is in contradiction to ρ^0 having stable spectral gap. We conclude that there exists an $s > 0$ such that ρ^t is not weakly contained in $\rho^t \otimes \rho^0$ for every $t < s$.

We prove that $G \curvearrowright (X, \mu_t)$ is strongly ergodic for every $t < \min\{t', s\}$, in which case we can apply [**MV20**, Lemma 5.2] to the non-singular action $G \curvearrowright (X, \mu_t)$ and the pmp action $G \curvearrowright (X \times Z_0^G, \mu_0 \times \lambda^G)$ by our choice of t' and s. After rescaling, we may assume that $G \curvearrowright (X, \mu_1)$ is ergodic and that ρ^t is not weakly contained in $\rho^t \otimes \rho^0$ for every $t \in (0, 1)$.

Let $t \in (0, 1)$ be arbitrary and define the map

$$\Psi : X \times X \times Z_0^G \to X : \quad \Psi(x, y, z)_h = \begin{cases} x_h & \text{if } z_h \le t, \\ y_h & \text{if } z_h > t. \end{cases}$$

Then Ψ is G-equivariant and we have that $\Psi(\mu_1 \times \mu_0 \times \lambda^G) = \mu_t$. Suppose that $G \curvearrowright (X, \mu_t)$ is not strongly ergodic. Then we can find a bounded almost invariant sequence $f_n \in L^\infty(X, \mu_t)$ such that $\|f_n\|_2 = 1$ and $\mu_t(f_n) = 0$ for every $n \in \mathbb{N}$. Therefore, $\Psi_*(f_n)$ is a bounded almost invariant sequence for $G \curvearrowright (X \times X \times Z_0^G, \mu_1 \times \mu_0 \times \lambda^G)$. Let $E : L^\infty(X \times X \times Z_0^G) \to L^\infty(X)$ be the conditional expectation that is uniquely determined by $\mu_1 \circ E = \mu_1 \times \mu_0 \times \lambda^G$. By [**MV20**, Lemma 5.2] we have that $\lim_{n\to\infty} \|(E \circ \Psi_*)(f_n) - \Psi_*(f_n)\|_2 = 0$. As Ψ is measure-preserving we get, in particular, that

$$\lim_{n\to\infty} \|(E \circ \Psi_*)(f_n)\|_2 = 1. \tag{3.9}$$

Note that if $\mu_t(f) = 0$ for some $f \in L^2(X, \mu_t)$, we have that $\mu_1((E \circ \Psi_*)(f)) = 0$. So we can view $E \circ \Psi_*$ as a bounded operator

$$E \circ \Psi_* : L^2(X, \mu_t) \ominus \mathbb{C}1 \to L^2(X, \mu_1) \ominus \mathbb{C}1.$$

CLAIM. *The bounded operator $E \circ \Psi_* : L^2(X, \mu_t) \ominus \mathbb{C}1 \to L^2(X, \mu_1) \ominus \mathbb{C}1$ has norm strictly less than* 1.

The claim is in direct contradiction to (3.9), so we conclude that $G \curvearrowright (X, \mu_t)$ is strongly ergodic.

Proof of claim. For every $g \in G$, let φ_g be the map

$$\varphi_g : L^2(X_0, \mu_g^t) \to L^2(X_0, \mu_g) : \quad \varphi_g(F) = tF + (1 - t)\nu(F) \cdot 1.$$

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Then $E \circ \Psi_*\colon L^2(X_0, \mu_t) \to L^2(X, \mu_1)$ is given by the infinite product $\bigotimes_{g\in G} \varphi_g$. For every $g \in G$ we have that

$$\|F\|_{2,\mu_g} = \|(d\mu_g^t/d\mu_g)^{-1/2}F\|_{2,\mu_g^t} \leq t^{-1/2}\|F\|_{2,\mu_g^t},$$

so that the inclusion map $\iota_g\colon L^2(X_0, \mu_g^t) \hookrightarrow L^2(X_0, \mu_g)$ satisfies $\|\iota_g\| \leq t^{-1/2}$ for every $g \in G$. We have that

$$\varphi_g(F) = t(F - \mu_g(F)\cdot 1) + \mu_t(F)\cdot 1 \quad \text{for every } F \in L^2(X_0, \mu_g^t).$$

So if we write P_g^t for the projection map onto $L^2(X_0, \mu_g^t) \ominus \mathbb{C}1$, and P_g for the projection map onto $L^2(X_0, \mu_g) \ominus \mathbb{C}1$, we have that

$$\varphi_g \circ P_g^t = t(P_g \circ \iota_g) \quad \text{for every } g \in G. \tag{3.10}$$

For a non-empty finite subset $\mathcal{F} \subset G$ let $V(\mathcal{F})$ be the linear subspace of $L^2(X, \mu_t) \ominus \mathbb{C}1$ spanned by

$$\bigg(\bigotimes_{g\in\mathcal{F}} L^2(X_0, \mu_g^t) \ominus \mathbb{C}1\bigg) \otimes \bigotimes_{g\in G\setminus\mathcal{F}} 1.$$

Then, using (3.10), we see that

$$\|(E \circ \Psi_*)(f)\|_2 \leq t^{|\mathcal{F}|/2}\|f\|_2 \quad \text{for every } f \in V(\mathcal{F}).$$

Since $\bigoplus_{\mathcal{F}\neq\emptyset} V(\mathcal{F})$ is dense inside $L^2(X, \mu_t) \ominus \mathbb{C}1$, we have that

$$\|(E \circ \Psi_*)|_{L^2(X,\mu_t)\ominus\mathbb{C}1}\| \leq t^{1/2} < 1. \qquad \square$$

This also concludes the proof of Theorem 3.3. $\square$

4. *Non-singular Bernoulli actions arising from groups acting on trees: proof of Theorem C*

Let T be a locally finite tree and choose a root $\rho \in T$. Let μ_0 and μ_1 be equivalent probability measures on a standard Borel space X_0. Following [**AIM19**, §10], we define a family of equivalent probability measures $(\mu_e)_{e\in E}$ by

$$\mu_e = \begin{cases} \mu_0 & \text{if } e \text{ is oriented towards } \rho, \\ \mu_1 & \text{if } e \text{ is oriented away from } \rho. \end{cases} \tag{4.1}$$

Let $G \subset \mathrm{Aut}(T)$ be a subgroup. When $g \in G$ and $e \in E$, the edges e and $g \cdot e$ are simultaneously oriented towards, or away from ρ, unless $e \in E([\rho, g\cdot\rho])$. As $E([\rho, g\cdot\rho])$ is finite for every $g \in G$, the generalized Bernoulli action

$$G \curvearrowright (X, \mu) = \prod_{e\in E}(X_0, \mu_e): \quad (g\cdot x)_e = x_{g^{-1}\cdot e} \tag{4.2}$$

is non-singular. If we start with a different root $\rho' \in T$, let $(\mu'_e)_{e\in E}$ denote the corresponding family of probability measures on X_0. Then we have that $\mu_e = \mu'_e$ for all but finitely many $e \in E$, so that the measures $\prod_{e\in E}\mu_e$ and $\prod_{e\in E}\mu'_e$ are equivalent. Therefore, up to conjugacy, the action (4.2) is independent of the choice of root $\rho \in T$.

LEMMA 4.1. *Let T be a locally finite tree such that each vertex $v \in V(T)$ has degree at least 2. Suppose that $G \subset \mathrm{Aut}(T)$ is a countable subgroup. Let μ_0 and μ_1 be equivalent probability measures on a standard Borel space X_0 and fix a root $\rho \in T$. Then the action $\alpha\colon G \curvearrowright (X, \mu)$ given by (4.2) is essentially free.*

Proof. Take $g \in G \setminus \{e\}$. It suffices to show that $\mu(\{x \in X : g \cdot x = x\}) = 0$. If g is elliptic, there exist disjoint infinite subtrees $T_1, T_2 \subset T$ such that $g \cdot T_1 = T_2$. Note that

$$(X_1, \mu_1) = \prod_{e \in E(T_1)} (X_0, \mu_e) \quad \text{and} \quad (X_2, \mu_2) = \prod_{e \in E(T_2)} (X_0, \mu_e)$$

are non-atomic and that g induces a non-singular isomorphism $\varphi\colon (X_1, \mu_1) \to (X_2, \mu_2)$: $\varphi(x)_e = x_{g^{-1} \cdot e}$. We get that

$$\mu_1 \times \mu_2(\{(x, \varphi(x)) : x \in X_1\}) = 0.$$

A fortiori $\mu(\{x \in X : g \cdot x = x\}) = 0$. If g is hyperbolic, let $L_g \subset T$ denote its axis on which it acts by non-trivial translation. Then $\prod_{e \in E(L_g)} (X_0, \mu_e)$ is non-atomic and by [**BKV19**, Lemma 2.2] the action $g^{\mathbb{Z}} \curvearrowright \prod_{e \in E(L_g)} (X_0, \mu_e)$ is essentially free. This implies that also $\mu(\{x \in X : g \cdot x = x\}) = 0$. □

We prove Theorem 4.2 below, which implies Theorem C and also describes the stable type when the action is weakly mixing.

THEOREM 4.2. *Let T be a locally finite tree with root $\rho \in T$. Let $G \subset \mathrm{Aut}(T)$ be a closed non-elementary subgroup with Poincaré exponent $\delta = \delta(G \curvearrowright T)$ given by (1.5). Let μ_0 and μ_1 be non-trivial equivalent probability measures on a standard Borel space X_0. Consider the generalized non-singular Bernoulli action $\alpha\colon G \curvearrowright (X, \mu)$ given by (4.2). Then α is:*

- *weakly mixing if $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$;*
- *dissipative up to compact stabilizers if $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$.*

Let $G \curvearrowright (Y, \nu)$ be an ergodic pmp action and let $\Lambda \subset \mathbb{R}$ be the smallest closed subgroup that contains the essential range of the map

$$X_0 \times X_0 \to \mathbb{R} : \quad (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x').$$

Let $\Delta\colon G \to \mathbb{R}_{>0}$ denote the modular function and let Σ be the smallest subgroup generated by Λ and $\log(\Delta(G))$.

Suppose that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$. Then the Krieger flow and the flow of weights of $\beta\colon G \curvearrowright X \times Y$ are determined by Λ and Σ as follows.

(1) *If Λ (respectively, Σ) is trivial, then the Krieger flow (respectively, flow of weights) is given by $\mathbb{R} \curvearrowright \mathbb{R}$.*
(2) *If Λ (respectively, Σ) is dense, then the Krieger flow (respectively, flow of weights) is trivial.*
(3) *If Λ (respectively, Σ) equals $a\mathbb{Z}$, with $a > 0$, then the Krieger flow (respectively, flow of weights) is given by $\mathbb{R} \curvearrowright \mathbb{R}/a\mathbb{Z}$.*

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In general, we do not know the behaviour of the action (4.2) in the critical situation $1 - H^2(\mu_0, \mu_1) = \exp(-\delta/2)$. However, if T is a regular tree and $G \curvearrowright T$ has full Poincaré exponent, we prove in Proposition 4.3 below that the action is dissipative up to compact stabilizers. This is similar to [**AIM19**, Theorems 8.4 and 9.10].

PROPOSITION 4.3. *Let T be a q-regular tree with root $\rho \in T$ and let $G \subset \mathrm{Aut}(T)$ be a closed subgroup with Poincaré exponent $\delta = \delta(G \curvearrowright T) = \log(q-1)$. Let μ_0 and μ_1 be equivalent probability measures on a standard Borel space X_0.*

If $1 - H^2(\mu_0, \mu_1) = (q-1)^{-1/2}$, then the action (4.2) is dissipative up to compact stabilizers.

Interesting examples of actions of the form (4.2) arise when $G \subset \mathrm{Aut}(T)$ is the free group on a finite set of generators acting on its Cayley tree. In that case, following [**AIM19**, §6] and [**MV20**, Remark 5.3], we can also give a sufficient criterion for strong ergodicity.

PROPOSITION 4.4. *Let the free group $\mathbb{F}_d$ on $d \geq 2$ generators act on its Cayley tree T. Let μ_0 and μ_1 be equivalent probability measures on a standard Borel space X_0. Then the action (4.2) dissipative if $1 - H^2(\mu_0, \mu_1) \leq (2d-1)^{-1/2}$ and weakly mixing and non-amenable if $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/2}$. Furthermore, the action (4.2) is strongly ergodic when $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/4}$.*

The proof of Theorem 4.2 below is similar to that of [**LP92**, Theorem 4] and [**AIM19**, Theorems 10.3 and 10.4]

Proof of Theorem 4.2. Define a family $(X_e)_{e \in E}$ of independent random variables on $(X, \mu) = \prod_{e \in E}(X_0, \mu_e)$ by

$$X_e(x) = \begin{cases} \log(d\mu_1/d\mu_0)(x_e) & \text{if } e \text{ is oriented towards } \rho, \\ \log(d\mu_0/d\mu_1)(x_e) & \text{if } e \text{ is oriented away from } \rho. \end{cases} \tag{4.3}$$

For $v \in T$ we write

$$S_v = \sum_{e \in E([\rho, v])} X_e.$$

Then we have that

$$\frac{dg\mu}{d\mu} = \exp(S_{g \cdot \rho}) \quad \text{for every } g \in G.$$

Since $G \subset \mathrm{Aut}(T)$ is a closed subgroup, for each $v \in T$ the stabilizer subgroup $G_v = \{g \in G : g \cdot v = v\}$ is a compact open subgroup of G.

Suppose that $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$. Then we have that

$$\int_X \sum_{v \in G \cdot \rho} \exp(S_v(x)/2)\, d\mu(x) = \sum_{v \in G \cdot \rho} (1 - H^2(\mu_0, \mu_1))^{2d(\rho, v)} < +\infty,$$

by definition of the Poincaré exponent. Therefore, we have that $\sum_{v \in G \cdot \rho} \exp(S_v(x)/2) < +\infty$ for a.e. $x \in X$. Let λ denote the left invariant Haar measure on G and define $L = \lambda(G_\rho)$, where $G_\rho = \{g \in G : g \cdot \rho = \rho\}$. Then we have that

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$$\int_G \frac{dg\mu}{d\mu}(x)\, d\lambda(g) = L \sum_{v\in G\cdot\rho} \exp(S_v(x)) < +\infty \quad \text{for a.e. } x \in X.$$

We conclude that $G \curvearrowright (X, \mu)$ is dissipative up to compact stabilizers.

Now assume that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$. We start by proving that $G \curvearrowright (X, \mu)$ is infinitely recurrent. By [**AIM19**, Theorem 8.17] we can find a non-elementary closed compactly generated subgroup $G' \subset G$ such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(G')/2)$. Let $T' \subset T$ be the unique minimal G'-invariant subtree. Then G' acts cocompactly on T' and we have that $\delta(G') = \dim_H \partial T'$. Let X and Y be independent random variables with distributions $(\log d\mu_1/d\mu_0)_*\mu_0$ and $(\log d\mu_0/d\mu_1)_*\mu_1$, respectively. Set $Z = X + Y$ and write

$$\varphi(t) = \mathbb{E}(\exp(tZ)).$$

The assignment $t \mapsto \varphi(t)$ is convex, $\varphi(t) = \varphi(1 - t)$ for every t and $\varphi(1/2) = (1 - H^2(\mu_0, \mu_1))^2$. We conclude that

$$\inf_{t\geq 0} \varphi(t) = (1 - H^2(\mu_0, \mu_1))^2.$$

Write R_k for the sum of k independent copies of Z. By the Chernoff–Cramér theorem, as stated in [**LP92**], there exists an $M \in \mathbb{N}$ such that

$$\mathbb{P}(R_M \geq 0) > \exp(-M\delta(G')). \tag{4.4}$$

Below we define a new *unoriented* tree S. This means that the edge set of S consists of subsets $\{v, w\} \subset V(S)$. Fix a vertex $\rho' \in T'$ and define the unoriented tree S as follows.

- S has vertices $v \in T'$ so that $d_{T'}(\rho', v)$ is divisible by M.
- There is an edge $\{v, w\} \in E(S)$ between two vertices $v, w \in S$ if $d_{T'}(v, w) = M$ and $[\rho', v]_{T'} \subset [\rho', w]_{T'}$.

Here the notation $[\rho', v]_{T'}$ means that we consider the line segment $[\rho', v]$ as a subtree of T'. We have that $\dim_H \partial S = M \dim_H \partial T' = M\delta(G')$. Form a random subgraph $S(x)$ of S by deleting those edges $\{v, w\} \in E(S)$ where

$$\sum_{e\in E([v,w]_{T'})} X_e(x_e) < 0.$$

This is an edge percolation on S, where each edge remains with probability $p = \mathbb{P}(R_M \geq 0)$. So by (4.4) we have that $p \exp(\dim_H S) > 1$. Furthermore, if $\{v, w\}$ and $\{v', w'\}$ are edges of S so that $E([v, w]_{T'}) \cap E([v', w']_{T'}) = \emptyset$, their presence in $S(x)$ constitutes independent events. So the percolation process is a quasi-Bernoulli percolation as introduced in [**Lyo89**]. Taking $w \in (1, p \exp(\dim_H S))$ and setting $w_n = w^{-n}$, it follows from [**Lyo89**, Theorem 3.1] that percolation occurs almost surely, that is, $S(x)$ contains an infinite connected component for a.e. $x \in X$. Writing

$$S'_v(x) = \sum_{e\in E([\rho',v]_{T'})} X_e(x_e),$$

this means that for a.e. $x \in (X, \mu)$ we can find a constant $a_x > -\infty$ such that $S'_v(x) > a_x$ for infinitely many $v \in T'$. As T'/G' is finite, there exists a vertex $w \in T'$ such that

$$\sum_{v \in G' \cdot w} \exp(S'_v(x)) = +\infty \quad \text{with positive probability.} \tag{4.5}$$

Therefore, by Kolmogorov's zero–one law, we have that $\sum_{v \in G' \cdot w} \exp(S'_v(x)) = +\infty$ almost surely. Since a change of root results in a conjugate action, we may assume that $\rho = w$. Then (4.5) implies that $\sum_{v \in G \cdot \rho} \exp(S_v(x)) = +\infty$ for a.e. $x \in X$. Writing again L for the Haar measure of the stabilizer subgroup $G_\rho = \{g \in G : g \cdot \rho = \rho\}$, we see that

$$\int_G \frac{dg\mu}{d\mu}\, d\lambda(g) = L \sum_{v \in G \cdot \rho} \exp(S_v) = +\infty \quad \text{almost surely.}$$

We conclude that $G \curvearrowright (X, \mu)$ is infinitely recurrent. We prove that $G \curvearrowright (X, \mu)$ is weakly mixing using a phase transition result from the previous section. Define the measurable map

$$\psi : X_0 \to (0, 1] : \quad \psi(x) = \min\{d\mu_1/d\mu_0(x), 1\}.$$

Let ν be the probability measure on X_0 determined by

$$\frac{d\nu}{d\mu_0}(x) = \rho^{-1}\psi(x) \quad \text{where } \rho = \int_{X_0} \psi(x)\, d\mu_0(x).$$

Then we have that $\nu \sim \mu_0$ and for every $s > 1 - \rho$ the probability measures

$$\eta_0^s = s^{-1}(\mu_0 - (1 - s)\nu),$$
$$\eta_1^s = s^{-1}(\mu_1 - (1 - s)\nu)$$

are well defined. We consider the non-singular actions $G \curvearrowright (X, \eta_s) = \prod_{e \in E}(X_0, \eta_e^s)$, where

$$\eta_e^s = \begin{cases} \eta_0^s & \text{if } e \text{ is oriented towards } \rho, \\ \eta_1^s & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

By the dominated convergence theorem we have that $H^2(\eta_0^s, \eta_1^s) \to H^2(\mu_0, \mu_1)$ as $s \to 1$. So we can choose s close enough to 1, but not equal to 1, such that $1 - H^2(\eta_0^s, \eta_1^s) > \exp(-\delta/2)$. By the first part of the proof we have that $G \curvearrowright (X, \eta_s)$ is infinitely recurrent. Note that

$$\mu_j = (1 - s)\nu + s\eta_j^s \quad \text{for } j = 0, 1.$$

Since we assumed that $G \subset \mathrm{Aut}(T)$ is closed, all the stabilizer subgroups $G_v = \{g \in G : g \cdot v = v\}$ are compact. By Remark 3.4 we conclude that $G \curvearrowright (X, \mu)$ is weakly mixing.

Let $G \curvearrowright (Y, \nu)$ be an ergodic pmp action. To determine the Krieger flow and the flow of weights of $\beta : G \curvearrowright X \times Y$ we use a similar approach to [**AIM19**, Theorem 10.4] and [**VW17**, Proposition 7.3]. First we determine the Krieger flow and then we deal with the flow of weights.

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As before, let $G' \subset G$ be a non-elementary compactly generated subgroup such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(G')/2)$. By [AIM19, Theorem 8.7] we may assume that G/G' is not compact. Let $T' \subset T$ be the minimal G'-invariant subtree. Let $v \in T'$ be as in Lemma 4.5 below so that

$$\bigcap_{g \in G} \Big(E(gT') \cup E([v, g^{-1} \cdot v]) \Big) = \emptyset. \tag{4.6}$$

Since changing the root yields a conjugate action, we may assume that $\rho = v$. Let (Z_0, ζ_0) be a standard probability space such that there exist measurable maps $\theta_0, \theta_1 \colon Z_0 \to X_0$ that satisfy $(\theta_0)_* \zeta_0 = \mu_0$ and $(\theta_1)_* \zeta_0 = \mu_1$. Write

$$(Z, \zeta) = \prod_{e \in E(T) \setminus E(T')} (Z_0, \zeta_0),$$
$$(X_1, \rho_1) = \prod_{e \in E(T) \setminus E(T')} (X_0, \mu_e),$$
$$(X_2, \rho_2) = \prod_{e \in E(T')} (X_0, \mu_e).$$

By the first part of the proof we have that $G' \curvearrowright (X_2, \rho_2)$ is infinitely recurrent. Define the pmp map

$$\Psi \colon (Z, \zeta) \to (X_1, \rho_1) : \quad (\Psi(z))_e = \begin{cases} \theta_0(z_e) & \text{if } e \text{ is oriented towards } \rho, \\ \theta_1(z_e) & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

Consider

$$U = \{e \in E(T) : e \text{ is oriented towards } \rho\}.$$

Since $gU \triangle U = E(T)([\rho, g \cdot \rho]) \subset E(T')$ for any $g \in G'$, the set $(E(T) \setminus E(T')) \cap U$ is G'-invariant. Therefore, Ψ is a G'-equivariant factor map. Consider the Maharam extensions

$$G' \curvearrowright Z \times X_2 \times Y \times \mathbb{R} \quad \text{and} \quad G \curvearrowright X \times Y \times \mathbb{R}$$

of the diagonal actions $G' \curvearrowright Z \times X_2 \times Y$ and $G' \curvearrowright X \times Y \times \mathbb{R}$, respectively. Identifying $(X, \mu) = (X_1, \rho_1) \times (X_2, \rho_2)$, we obtain a G'-equivariant factor map

$$\Phi \colon Z \times X_2 \times Y \times \mathbb{R} \to X_1 \times X_2 \times Y \times \mathbb{R} : \quad \Phi(z, x, y, t) = (\Psi(z), x, y, t).$$

Take $F \in L^\infty(X \times Y \times \mathbb{R})^G$. By [AIM19, Proposition A.33] the Maharam extension $G' \curvearrowright X_2 \times Y \times \mathbb{R}$ is infinitely recurrent. Since $G' \curvearrowright Z$ is a mixing pmp generalized Bernoulli action we have that $F \circ \Phi \in L^\infty(Z \times X_2 \times Y \times \mathbb{R})^G \subset 1 \mathbin{\overline{\otimes}} L^\infty(X_2 \times Y \times \mathbb{R})^G$ by [SW81, Theorem 2.3]. Therefore, F is essentially independent of the $E(T) \setminus E(T')$-coordinates. Thus, for any $g \in G$ the assignment

$$(x, y, t) \mapsto F(g \cdot x, y, t) = F(x, y, t - \log(dg^{-1}\mu/d\mu)(x))$$

is essentially independent of the $E(T) \setminus E(gT')$-coordinates. Since $\log(dg^{-1}\mu/d\mu)$ only depends on the $E([\rho, g^{-1} \cdot \rho])$-coordinates, we deduce that F is essentially independent of

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the $E(T) \setminus (E(gT') \cup E([\rho, g^{-1} \cdot \rho]))$-coordinates, for every $g \in G$. Therefore, by (4.6), we have that $F \in 1 \mathbin{\overline{\otimes}} L^\infty(Y \times \mathbb{R})$.

So we have proven that any G-invariant function $F \in L^\infty(X \times Y \times \mathbb{R})$ is of the form $F(x, y, t) = H(y, t)$, for some $H \in L^\infty(Y \times \mathbb{R})$ that satisfies

$$H(y, t) = H(g \cdot y, t + \log(dg^{-1}\mu/d\mu)(x)) \quad \text{for a.e. } (x, y, t) \in X \times Y \times \mathbb{R}.$$

Since 0 is in the essential range of the maps $\log(dg\mu/d\mu)$, for every $g \in G$, we see that $H(g \cdot y, t) = H(y, t)$ for a.e. $(y, t) \in Y \times \mathbb{R}$. By ergodicity of $G \curvearrowright Y$, we conclude that H is of the form $H(y, t) = P(t)$, for some $P \in L^\infty(\mathbb{R})$ that satisfies

$$P(t) = P(t + \log(dg^{-1}\mu/d\mu)(x)) \quad \text{for a.e. } (x, t) \in X \times \mathbb{R}, \text{ for every } g \in G. \tag{4.7}$$

Let $\Gamma \subset \mathbb{R}$ be the subgroup generated by the essential ranges of the maps $\log(dg\mu/d\mu)$, for $g \in G$. If $\Gamma = \{0\}$ we can identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R})$. If $\Gamma \subset \mathbb{R}$ is dense, then it follows that P is essentially constant so that the Maharam extension $G \curvearrowright X \times Y \times \mathbb{R}$ is ergodic, that is, the Krieger flow of $G \curvearrowright X \times Y$ is trivial. If $\Gamma = a\mathbb{Z}$, with $a > 0$, we conclude by (4.7) that we can identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R}/a\mathbb{Z})$, so that the Krieger flow of $G \curvearrowright X \times Y$ is given by $\mathbb{R} \curvearrowright \mathbb{R}/a\mathbb{Z}$. Finally, note that the closure of Γ equals the closure of the subgroup generated by the essential range of the map

$$X_0 \times X_0 \to \mathbb{R}: \quad (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x').$$

So we have calculated the Krieger flow in every case, concluding the proof of the theorem in the case where G is unimodular.

When G is not unimodular, let $G_0 = \ker \Delta$ be the kernel of the modular function. Let $G \curvearrowright X \times Y \times \mathbb{R}$ be the modular Maharam extension and let $\alpha : G_0 \curvearrowright X \times Y \times \mathbb{R}$ be its restriction to the subgroup G_0. Then we have that

$$L^\infty(X \times Y \times \mathbb{R})^G \subset L^\infty(X \times Y \times \mathbb{R})^\alpha.$$

By [**AIM19**, Theorem 8.16] we have that $\delta(G_0) = \delta$, and we can apply the argument above to conclude that $L^\infty(X \times Y \times \mathbb{R})^\alpha \subset 1 \mathbin{\overline{\otimes}} 1 \mathbin{\overline{\otimes}} L^\infty(\mathbb{R})$. So for every $F \in L^\infty(X \times Y \times \mathbb{R})^G$ there exists a $P \in L^\infty(\mathbb{R})$ such that

$$P(t) = P(t + \log(dg^{-1}\mu/d\mu)(x) + \log(\Delta(g))) \quad \text{for a.e. } (x, t) \in X \times \mathbb{R}, \text{ for every } g \in G. \tag{4.8}$$

Let Π be the subgroup of $\mathbb{R}$ generated by the essential range of the maps

$$x \mapsto \log(dg^{-1}\mu/d\mu)(x) + \log(\Delta(g)) \quad \text{with } g \in G.$$

As 0 is contained in the essential range of $\log(dg^{-1}\mu/d\mu)$, for every $g \in G$, we get that $\log(\Delta(G)) \subset \Pi$. Therefore, Π also contains the subgroup $\Gamma \subset \mathbb{R}$ defined above. Thus, the closure of Π equals the closure of Σ, where $\Sigma \subset \mathbb{R}$ is the subgroup as in the statement of the theorem. From (4.8) we conclude that we may identify $L^\infty(X \times Y \times \mathbb{R})^G \cong L^\infty(\mathbb{R})^\Sigma$, so that the flow of weights of $G \curvearrowright X \times Y$ is as stated in the theorem. □

LEMMA 4.5. *Let T be a locally finite tree and let $G \subset \mathrm{Aut}(T)$ be a closed subgroup. Suppose that $H \subset G$ is a closed compactly generated subgroup that contains a hyperbolic element and assume that G/H is not compact. Let $S \subset T$ be the unique minimal H-invariant subtree. Then there exists a vertex $v \in S$ such that*

$$\bigcap_{g \in G} \Big(gS \cup [v, g^{-1} \cdot v] \Big) = \{v\}. \tag{4.9}$$

Proof. Let $k \in H$ be a hyperbolic element and let $L \subset T$ be its axis, on which k acts by a non-trivial translation. Then $L \subset S$, as one can show for instance as in the proof of [**CM11**, Proposition 3.8]. Pick any vertex $v \in L$. We claim that this vertex will satisfy (4.9). Take any $w \in V(T) \setminus \{v\}$. As G/H is not compact, one can show as in [**AIM19**, Theorem 9.7] that there exists a $g \in G$ such that $g \cdot w \notin S$. Since k acts by translation on L, there exists an $n \in \mathbb{N}$ large enough such that

$$[v, k \cdot v] \subset [v, k^n g \cdot v] \quad \text{and} \quad [v, k^{-1} \cdot v] \subset [v, k^{-n} g \cdot v],$$

so that in particular we have that $w \notin [v, k^n g \cdot v] \cap [v, k^{-n} g \cdot v] = \{v\}$. Since S is H-invariant, we also have that $k^n g \cdot w \notin S$ and $k^{-n} g \cdot w \notin S$ and we conclude that

$$w \notin ((k^n g)^{-1} S \cup [v, k^n g \cdot v]) \cap ((k^{-n} g)^{-1} S \cup [v, k^{-n} g \cdot v]). \qquad \square$$

Proof of Proposition 4.3. Define the family $(X_e)_{e \in E}$ of independent random variables on (X, μ) by (4.3) and write

$$S_v = \sum_{e \in E([\rho, v])} X_e.$$

CLAIM. *There exists a $\delta > 0$ such that*

$$\mu(\{x \in X : S_v(x) \leq -\delta \quad \textit{for every } v \in T \setminus \{\rho\}\}) > 0.$$

Proof of claim. Note that $\mathbb{E}(\exp(X_e/2)) = 1 - H^2(\mu_0, \mu_1)$ for every $e \in E$. Define a family of random variables $(W_n)_{n \geq 0}$ on (X, μ) by

$$W_n = \sum_{\substack{v \in T \\ d(v, \rho) = n}} \exp(S_v/2).$$

Using that $1 - H^2(\mu_0, \mu_1) = (q-1)^{-1/2}$, one computes that

$$\mathbb{E}(W_{n+1} | \; S_v, \; d(v, \rho) \leq n) = W_n \quad \text{for every } n \geq 1.$$

So the sequence $(W_n)_{n \geq 0}$ is a martingale, and since it is positive it converges almost surely to a finite limit when $n \to +\infty$. Write $\Sigma_n = \{v \in T : d(v, \rho) = n\}$. As $W_n \geq \max_{v \in \Sigma_n} \exp(S_v/2)$ we conclude that there exists a positive constant $C < +\infty$ such that

$$\mathbb{P}(S_v \leq C \text{ for every } v \in T) > 0.$$

For any vertex $w \in T$, write $T_w = \{v \in T : [\rho, w] \subset [\rho, v]\}$: the set of children of w, including w itself. Using the symmetry of the tree and changing the root from ρ to $w \in T$, we also have that

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$$\mathbb{P}(S_v - S_w \leq C \text{ for every } v \in T_w) > 0 \quad \text{for every } w \in T. \tag{4.10}$$

Set $\nu_0 = (\log d\mu_1/d\mu_0)_*\mu_0$ and $\nu_1 = (\log d\mu_0/d\mu_1)_*\mu_1$. Because $1 - H^2(\mu_0, \mu_1) \neq 0$ we have that $\mu_0 \neq \mu_1$, so that there exists a $\delta > 0$ such that

$$\nu_0 * \nu_1((-\infty, -\delta)) > 0.$$

Here $\nu_0 * \nu_1$ denotes the convolution product of ν_0 with ν_1. Therefore, there exists $N \in \mathbb{N}$ large enough such that

$$\mathbb{P}(S_w \leq -C - \delta \text{ for every } w \in \Sigma_N \text{ and } S_{w'} \leq -\delta \text{ for every } w' \in \Sigma_n \text{ with } n \leq N) > 0. \tag{4.11}$$

Since for any $w \in \Sigma_N$ and $w' \in \Sigma_n$ with $n \leq N$, we have that $S_v - S_w$ is independent of $S_{w'}$ for every $v \in T_w$, and since Σ_N is a finite set, it follows from (4.10) and (4.11) that

$$\mathbb{P}(S_v \leq -\delta \quad \text{for every } v \in T \setminus \{\rho\}) > 0.$$

This concludes the proof of the claim. □

Let $\delta > 0$ be as in the claim and define

$$\mathcal{U} = \{x \in X : S_v(x) \leq -\delta \text{ for every } v \in T \setminus \{\rho\}\},$$

so that $\mu(\mathcal{U}) > 0$. Let G_ρ be the stabilizer subgroup of ρ. Note that for every $g, h \in G$ we have that $S_{hg\cdot\rho}(x) = S_{g\cdot\rho}(h^{-1} \cdot x) + S_{h\cdot\rho}(x)$ for a.e. $x \in X$, so that for $h \in G$ we have that

$$h \cdot \mathcal{U} \subset \{x \in X : S_{hg\cdot\rho}(x) \leq -\delta + S_{h\cdot\rho}(x) \text{ for every } g \notin G_\rho\}.$$

It follows that if $h \notin G_\rho$, we have that

$$\mathcal{U} \cap h \cdot \mathcal{U} \subset \{x \in X : S_{h\cdot\rho}(x) \leq -\delta \text{ and } S_{h\cdot\rho}(x) \geq \delta\} = \emptyset.$$

Since $G \subset \mathrm{Aut}(T)$ is closed, we have that G_ρ is compact. So the action $G \curvearrowright (X, \mu)$ is not infinitely recurrent. Let λ denote the left invariant Haar measure on G. By an adaptation of the proof of [**BV20**, Proposition 4.3], the set

$$D = \left\{x \in X : \int_G \frac{dg\mu}{d\mu}(x)\, d\lambda(g) < +\infty\right\} = \left\{x \in X : \int_G \exp(S_{g\cdot\rho}(x))\, d\lambda(g) < +\infty\right\}$$

satisfies $\mu(D) \in \{0, 1\}$. Since $G \curvearrowright (X, \mu)$ is not infinitely recurrent, it follows from [**AIM19**, Proposition A.28] that $\mu(D) > 0$, so that we must have that $\mu(D) = 1$. By [**AIM19**, Theorem A.29] the action $G \curvearrowright (X, \mu)$ is dissipative up to compact stabilizers. □

We use a similar approach to [**MV20**, §6] in the proof of Proposition 4.4.

Proof of Proposition 4.4. It follows from Theorem 4.2 and Proposition 4.3 that the action $G \curvearrowright (X, \mu)$, given by (4.2), is dissipative when $1 - H^2(\mu_0, \mu_1) \leq (2d - 1)^{-1/2}$ and weakly mixing when $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}$. So it remains to show that $G \curvearrowright (X, \mu)$ is non-amenable when $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/2}$ and strongly ergodic when $1 - H^2(\mu_0, \mu_1) > (2d - 1)^{-1/4}$.

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Assume first that $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/2}$. By taking the kernel of a surjective homomorphism $\mathbb{F}_d \to \mathbb{Z}$ we find a normal subgroup $H_1 \subset \mathbb{F}_d$ that is free on infinitely many generators. By [**RT13**, Théorème 0.1] we have that $\delta(H_1) = (2d-1)^{-1/2}$. Then, using [**Sul79**, Corollary 6], we can find a finitely generated free subgroup $H_2 \subset H_1$ such that $H_1 = H_2 * H_3$ for some free subgroup $H_3 \subset H_1$ and such that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(H_2)/2)$. Let $\psi : H_1 \to H_3$ be the surjective group homomorphism uniquely determined by

$$\psi(h) = \begin{cases} e & \text{if } h \in H_2, \\ h & \text{if } h \in H_3. \end{cases}$$

We set $N = \ker \psi$, so that $H_2 \subset N$ and we get that $1 - H^2(\mu_0, \mu_1) > \exp(-\delta(N)/2)$. Therefore, $N \curvearrowright (X, \mu)$ is ergodic by Theorem 4.2. Also we have that $H_1/N \cong H_3$, which is a free group on infinitely many generators. Therefore, $H_1 \curvearrowright (X, \mu)$ is non-amenable by [**MV20**, Lemma 6.4]. A posteriori also $\mathbb{F}_d \curvearrowright (X, \mu)$ is non-amenable.

Let π be the Koopman representation of the action $\mathbb{F}_d \curvearrowright (X, \mu)$:

$$\pi : G \curvearrowright L^2(X, \mu) : \quad (\pi_g(\xi))(x) = \left(\frac{dg\mu}{d\mu}(x)\right)^{1/2} \xi(g^{-1} \cdot x).$$

CLAIM. *If $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/4}$, then π is not weakly contained in the left regular representation.*

Proof of claim. Let η denote the canonical symmetric measure on the generator set of $\mathbb{F}_d$ and define

$$P = \sum_{g \in \mathbb{F}_d} \eta(g)\pi_g.$$

The η-spectral radius of $\alpha : \mathbb{F}_d \curvearrowright (X, \mu)$, which we denote by $\rho_\eta(\alpha)$, is by definition the norm of P, as a bounded operator on $L^2(X, \mu)$. By [**AIM19**, Proposition A.11] we have that

$$\begin{aligned} \rho_\eta(\alpha) &= \lim_{n\to\infty} \langle P^n(1), 1\rangle^{1/n} \\ &= \lim_{n\to\infty} \left(\sum_{g \in \mathbb{F}_d} \eta^{*n}(g)(1 - H^2(\mu_0, \mu_1))^{2|g|} \right)^{1/n}, \end{aligned}$$

where $|g|$ denotes the word length of a group element $g \in \mathbb{F}_d$. By [**AIM19**, Theorem 6.10] we then have that

$$\rho_\eta(\alpha) = \frac{(1 - H^2(\mu_0, \mu_1))^2}{2d} \left((2d-1) + (1 - H^2(\mu_0, \mu_1))^{-4}\right)$$

if $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/4}$, and

$$\rho_\eta(\alpha) = \frac{\sqrt{2d-1}}{d}$$

if $1 - H^2(\mu_0, \mu_1) \leq (2d-1)^{-1/4}$. Therefore, if $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/4}$, we have that $\rho_\eta(\alpha) > \rho_\eta(\mathbb{F}_d)$, where $\rho_\eta(\mathbb{F}_d)$ denotes the η-spectral radius of the left regular

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representation. This implies that α is not weakly contained in the left regular representation (see, for instance, [**AD03**, §3.2]). □

Now assume that $1 - H^2(\mu_0, \mu_1) > (2d-1)^{-1/4}$. As in the proof of Theorem 4.2 there exist probability measures ν, η_0 and η_1 on X_0 that are equivalent to μ_0 and a number $s \in (0, 1)$ such that

$$\mu_j = (1-s)\nu + s\eta_j \quad \text{for } j = 0, 1,$$

and such that $1 - H^2(\eta_0, \eta_1) > (2d-1)^{-1/4}$. Consider the non-singular action

$$\mathbb{F}_d \curvearrowright (X, \eta) = \prod_{e \in E(T)} (X_0, \eta_e) \quad \text{where } \eta_e = \begin{cases} \eta_0 & \text{if } e \text{ is oriented towards } \rho, \\ \eta_1 & \text{if } e \text{ is oriented away from } \rho. \end{cases}$$

By Theorem 4.2 the action $\mathbb{F}_d \curvearrowright (X, \eta)$ is ergodic. Write ρ for the Koopman representation associated to $\mathbb{F}_d \curvearrowright (X, \eta)$. By the claim, ρ is not weakly contained in the left regular representation. Let λ be the probability measure on $\{0, 1\}$ given by $\lambda(0) = s$. Let ρ^0 be the reduced Koopman representation of the pmp generalized Bernoulli action $\mathbb{F}_d \curvearrowright (X \times \{0, 1\}^{E(T)}, \nu^{E(T)} \times \lambda^{E(T)})$. Then ρ^0 is contained in a multiple of the left regular representation. Therefore, as ρ is not weakly contained in the left regular representation, ρ is not weakly contained in $\rho \otimes \rho^0$.

Define the map

$$\Psi \colon X \times X \times \{0, 1\}^{E(T)} \to X \colon \quad \Psi(x, y, z)_e = \begin{cases} x_e & \text{if } z_e = 0, \\ y_e & \text{if } z_e = 1. \end{cases}$$

Then Ψ is $\mathbb{F}_d$-equivariant and we have that $\Psi_*(\eta \times \nu^{E(T)} \times \lambda^{E(T)}) = \mu$. Suppose that $\mathbb{F}_d \curvearrowright (X, \mu)$ is not strongly ergodic. Then there exists a bounded almost invariant sequence $f_n \in L^\infty(X, \mu)$ such that $\|f_n\|_2 = 1$ and $\mu(f_n) = 0$ for every $n \in \mathbb{N}$. Therefore, $\Psi_*(f_n)$ is a bounded almost invariant sequence for the diagonal action $\mathbb{F}_d \curvearrowright (X \times X \times \{0, 1\}^{E(T)}, \eta \times \nu^{E(T)} \times \lambda^{E(T)})$. Let $E \colon L^\infty(X \times X \times \{0, 1\}^{E(T)}) \to L^\infty(X)$ be the conditional expectation that is uniquely determined by $\mu \circ E = \eta \times \nu^{E(T)} \times \lambda^{E(T)}$. By [**MV20**, Lemma 5.2] we have that $\lim_{n\to\infty} \|(E \circ \Psi_*)(f_n) - \Psi_*(f_n)\|_2 = 0$, and in particular we get that

$$\lim_{n\to\infty} \|(E \circ \Psi_*)(f_n)\|_2 = 1. \tag{4.12}$$

But just as in the proof of Theorem 3.3 we have that

$$\left\| (E \circ \Psi_*)\big|_{L^2(X,\mu)\ominus\mathbb{C}1} \right\| < 1,$$

which is in contradiction with (4.12). We conclude that $\mathbb{F}_d \curvearrowright (X, \mu)$ is strongly ergodic. □

Proposition 4.6 below complements Theorem 4.2 by considering groups $G \subset \mathrm{Aut}(T)$ that are not closed. This is similar to [**AIM19**, Theorem 10.5].

PROPOSITION 4.6. *Let T be a locally finite tree with root $\rho \in T$. Let $G \subset \mathrm{Aut}(T)$ be an lcsc group such that the inclusion map $G \to \mathrm{Aut}(T)$ is continuous and such that*

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$G \subset \mathrm{Aut}(T)$ is not closed. Write $\delta = \delta(G \curvearrowright T)$ for the Poincaré exponent given by (1.5). Let μ_0 and μ_1 be non-trivial equivalent probability measures on a standard Borel space X_0. Consider the generalized non-singular Bernoulli action $\alpha\colon G \curvearrowright (X, \mu)$ given by (4.2). Let $H \subset \mathrm{Aut}(T)$ be the closure of G. Then the following assertions hold.

- *If $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$, then α is ergodic and its Krieger flow is determined by the essential range of the map*
$$X_0 \times X_0 \to \mathbb{R}: \quad (x, x') \mapsto \log(d\mu_0/d\mu_1)(x) - \log(d\mu_0/d\mu_1)(x') \tag{4.13}$$
as in Theorem 4.2.
- *If $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$, then each ergodic component of α is of the form $G \curvearrowright H/K$, where K is a compact subgroup of H. In particular, there exists a G-invariant σ-finite measure on X that is equivalent to μ.*

Proof. Let $H \subset \mathrm{Aut}(T)$ be the closure of G. Then $\delta(H) = \delta$ and we can apply Theorem 4.2 to the non-singular action $H \curvearrowright (X, \mu)$.

If $1 - H^2(\mu_0, \mu_1) > \exp(-\delta/2)$, then $H \curvearrowright X$ is ergodic. As $G \subset H$ is dense, we have that
$$L^\infty(X)^G = L^\infty(X)^H = \mathbb{C}1,$$
so that $G \curvearrowright X$ is ergodic. Let $H \curvearrowright X \times \mathbb{R}$ be the Maharam extension associated to $H \curvearrowright X$. Again, as $G \subset H$ is dense, we have that
$$L^\infty(X \times \mathbb{R})^G = L^\infty(X \times \mathbb{R})^H.$$
Note that the subgroup generated by the essential ranges of the maps $\log(dg^{-1}\mu/d\mu)$, with $g \in G$, is the same as the subgroup generated by the essential ranges of the maps $\log(dh^{-1}\mu/d\mu)$, with $h \in H$. Then one determines the Krieger flow of $G \curvearrowright X$ as in the proof of Theorem 4.2.

If $1 - H^2(\mu_0, \mu_1) < \exp(-\delta/2)$, the action $H \curvearrowright (X, \mu)$ is dissipative up to compact stabilizers. By [**AIM19**, Theorem A.29] each ergodic component is of the form $H \curvearrowright H/K$ for a compact subgroup $K \subset H$. Therefore, each ergodic component of $G \curvearrowright (X, \mu)$ is of the form $G \curvearrowright H/K$, for some compact subgroup $K \subset H$. □

Acknowledgements. T.B. thanks Stefaan Vaes for his valuable feedback during the process of writing this paper. T.B. is supported by a PhD fellowship fundamental research of the Research Foundation Flanders.

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