J. Austral. Math. Soc. **20** (Series A) (1975), 431–448.

STIELTJES-TYPE INTEGRALS

A. M. RUSSELL

(Received 13 June 1974)

Communicated by E. Strzelecki

Introduction

Burkill (1957) introduced extended Riemann-Stieltjes integrals of the form $\int_a^b f(x)\,d^k g(x)/dx^{k-1}$ to provide an alternative approach to the theory of distributions. We will denote Burkill's integral by $(B)\int_a^b f(x)\,d^k g(x)/dx^{k-1}$ Burkill's paper (1957) partially motivated the study of the Riemann-Stieltjes integral $\int_a^b f(x)\,d^2 g(x)/du(x)$ in Russell (1970). He showed that the two integrals are not equivalent.

In order to define integrals of the form $\int_a^b f(x)\,d^k g(x)/dx^{k-1}$ when $k \geqq 2$, it was necessary to define and develop the concept of bounded k^{th} variation. This has been done in Russell (1973) using k^{th} divide differences, and, as indicated in that paper, we now proceed to define, and obtain properties of $\int_a^b f(x) d^k g(x)/dx^{k-1}$.

1. Notation and Preliminaries

Unless otherwise stated, undefined terms and notation will be found in Russell (1973).

It will be assumed throughout that k is a positive integer greater than 1.

In Russell (1973) we introduced the concept of total k^{th} variation of a function in a closed interval. If we call that variation inner variation, then we now define a related variation which we describe as outer variation.

DEFINITION 1. *Let k be any positive integer greater than 1. Then we will denote by $\Gamma(x_{-k+1},\cdots,x_{n+k-1})$ a subdivision of the closed interval $[a,b]$ of the form*

$$a' \leqq x_{-k+1} < \cdots < x_0 = a < x_1 < \cdots < x_n = b < \cdots x_{n+k-1} \leqq b'.$$

Throughout the paper it will be understood that $a' < a < b < b'$. In addition we point out at this stage that a Γ subdivision, as opposed to a π subdivision of

$[a, b]$ (see Russell (1973)), requires a fixed number $2k-2$ of points outside $[a, b]$.

DEFINITION 2. *The total outer k^{th} variation of g on $[a, b]$ is defined by*

$$W_k(g; a, b, a', b') = \sup_{\Gamma} \sum_{i=-k+1}^{n-1} (x_{i+k} - x_i) |Q_k(g; x_i, \cdots, x_{i+k})|.$$

We will usually use an abbreviated notation $W_k(g; a', b')$ for this variation. If $W_k(g; a', b') < \infty$ we say that g is of bounded k^{th} variation on $[a, b]$, and write $g \in BW_k[a', b']$. The summations over which the supremum is taken are called approximating sums of $W_k(g; a', b')$.

LEMMA 1. *If $f \in BV_k[a', b']$, then $f \in BW_k[a', b']$, and*

$$W_k(f; a', b') \leqq V_k(f; a', b').$$

The proof of this lemma is easy, and will be omitted.

LEMMA 2. *The inclusion, in (a, b), of extra points of subdivision to an existing Γ subdivision of $[a, b]$, does not decrease the approximating sums for $W_k(g; a', b')$.*

A similar proof to that of Russell (1973; Theorem 3) applies, so we omit details.

LEMMA 3. *If $f \in BV_k[a, b]$, and*

$$F(x) = \int_c^x f(t)dt, \quad a < c < b, \quad a \leqq x \leqq b,$$

then $F \in BV_{k+1}[a, b]$.

PROOF. Since $f \in BV_k[a, b]$, it follows from Russell (1973; Theorem 19) that $f = u - v$, where u and v are $0, 1, 2, \cdots, k$-convex, and have right and left $(k-1)^{\text{th}}$ Riemann * derivatives at a and b, respectively. It now follows from Russell (1973; Theorem 13) that F can be expressed in the form $F = U - V$, where U and V are $0, 1, 2, \cdots, (k+1)$-convex in $[a, b]$. Using an argument similar to that in the proof of Russell (1973; Theorem 19) establishes that U and V have right and left k^{th} Riemann * derivatives at a and b respectively. Consequently $F \in BV_{k+1}[a, b]$.

2. Definition and Linearity Properties of the Integral

DEFINITION 3. *Let $\Gamma(x_{-k+1}, \cdots, x_{n+k-1})$ be any subdivision of $[a, b]$. We call*

$$\max_{i=-k+2, \cdots, n+k-1} (x_i - x_{i-1})$$

the norm of the subdivision Γ, and denote it by $\|\Gamma\|$.

DEFINITION 4. *The integral* $\int_a^b f(x)\, d^k g(x)/dx^{k-1}$ *is the real number* I, *if it exists uniquely and if for each* $\varepsilon > 0$ *there is a real number* $\delta(\varepsilon)$ *such that when* $x_i \leqq \xi_i \leqq x_{i+k}$, $i = -k+1, \cdots, n-1$,

$$\left| I - \sum_{i=-k+1}^{n-1} f(\xi_i)\left[Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})\right]\right| < \varepsilon$$

whenever $\|\Gamma\| < \delta(\varepsilon)$.

If the integral exists we will write $(f, g) \in RS_k[a, b]$, and we will refer to the integral as an RS_k integral. We remark that when the integral exists, its value is independent of a' and b'.

The proofs of the following two theorems will not be given.

THEOREM 1. *If* $(f_i, g) \in RS_k[a, b]$, $i = 1, 2, \cdots, n$, *and* $\lambda_1, \lambda_2, \cdots, \lambda_n$ *are real numbers, then* $(\sum_{i=1}^n \lambda_i f_i, g) \in RS_k[a, b]$, *and*

$$\int_a^b \left(\sum_{i=1}^n \lambda_i f_i(x)\right)\frac{d^k g(x)}{dx^{k-1}} = \sum_{i=1}^n \lambda_i \int_a^b f_i(x)\,\frac{d^k g(x)}{dx^{k-1}}.$$

THEOREM 2. *If* $(f, g_i) \in RS_k[a, b]$, $i = 1, 2, \cdots, n$, *and* $\mu_1, \mu_2, \cdots, \mu_n$ *are real numbers, then* $(f, \sum_{i=1}^n \mu_i g_i) \in RS_k[a, b]$, *and*

$$\int_a^b f(x)\frac{d^k\left(\sum_{i=1}^n \mu_i g_i(x)\right)}{dx^{k-1}} = \sum_{i=1}^n \mu_i \int_a^b f(x)\frac{d^k g_i(x)}{dx^{k-1}}.$$

THEOREM 3. *Let* $(f, g) \in RS_k[a, c]$ *and* $RS_k[c, b]$, *where* $a < c < b$. *If* f *is continuous and* g *has right and left* $(k-1)^{\text{th}}$ *Riemann * derivatives at* c, *with* g *having bounded* $(k-1)^{\text{th}}$ *divided differences in some neighbourhood of* c, *then* $(f, g) \in RS_k[a, b]$, *and*

$$\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = \int_a^c f(x)\frac{d^k g(x)}{dx^{k-1}} - \frac{f(c)}{(k-1)!}\left[D_+^{k-1} g(c) - D_-^{k-1} g(c)\right] + \int_c^b f(x)\,\frac{d^k g(x)}{dx^{k-1}}.$$

PROOF. We consider a subdivision $\Gamma(x_{-k+1}, \cdots, x_{n+k-1})$ of $[a, b]$, and suppose that $x_{p-1} < c < x_p$, $1 \leqq p \leqq n$. Since g has bounded $(k-1)^{\text{th}}$ divided differences in some neighbourhood of c, we can choose a $\delta > 0$ such that g has bounded $(k-1)^{\text{th}}$ divided differences in the interval $[c-\delta, c+\delta]$. We now confine our discussion to Γ subdivisions for which $(x_{p+k-1} - x_{p-k}) < \delta$. Let $S(a, b)$, $S(a, c)$ and $S(c, b)$ denote the approximating sums of

$$\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}, \quad \int_a^c f(x)\frac{d^k g(x)}{dx^{k-1}} \quad \text{and} \quad \int_c^b f(x)\,\frac{d^k g(x)}{dx^{k-1}}$$

respectively.

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

We re-label the points $x_{-k+1}, \cdots, x_{p-1}, c, x_p, \cdots, x_{n+k-1}$ as

$$y_{-k+1}, \cdots, y_0 = a, \cdots, y_{n+1} = b, \cdots, y_{n+k}, \text{ where}$$

$$y_i = x_i, \qquad i = -k+1, \cdots, p-1$$

$$y_p = c \qquad\qquad , \text{and}$$

$$y_i = x_{i-1}, \qquad i = p+1, \cdots, n+k.$$

If we define $y_i \leqq \eta_i \leqq y_{i+k}$, $i = p-k, \cdots, p$, then

(1) $\quad S(a,b)$

$$= \left\{ \sum_{i=-k+1}^{p-k-1} f(\xi_i)\,[Q_{k-1}(g\,;\,x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g\,;\,x_i, \cdots, x_{i+k-1})] \right.$$

$$\left. + \sum_{i=p-k}^{p-1} f(\eta_i)\,[Q_{k-1}(g\,;\,y_{i+1}, \cdots, y_{i+k}) - Q_{k-1}(g\,;\,y_i, \cdots, y_{i+k-1})] \right\}$$

$$+ \left\{ \sum_{i=p-k+1}^{p} f(\eta_i)\,[Q_{k-1}(g\,;\,y_{i+1}, \cdots, y_{i+k}) - Q_{k-1}(g\,;\,y_i, \cdots, y_{i+k-1})] \right.$$

$$\left. + \sum_{i=p}^{n-1} f(\xi_i)\,[Q_{k-1}(g\,;\,x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g\,;\,x_i, \cdots, x_{i+k-1})] \right\}$$

$$+ \sum_{i=p-k}^{p-1} f(\xi_i)\,[Q_{k-1}(g\,;\,x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g\,;\,x_i, \cdots, x_{i+k-1})]$$

$$- \sum_{i=p-k}^{p-1} f(\eta_i)\,[Q_{k-1}(g\,;\,y_{i+1}, \cdots, y_{i+k}) - Q_{k-1}(g\,;\,y_i, \cdots, y_{k+k-1})]$$

$$- \sum_{i=p-k+1}^{p} f(\eta_i)\,[Q_{k-1}(g\,;\,y_{i+1}, \cdots, y_{i+k}) - Q_{k-1}(g\,;\,y_i, \cdots, y_{i+k-1})]$$

(2) $\quad = S(a,c) + S(c,b) + R,$

where R consists of the last three summation terms of (1).

Hence,

$$R = \sum_{i=p-k+1}^{p-1} [f(\xi_{i-1}) - f(\xi_i)]Q_{k-1}(g\,;x_i, \cdots, x_{i+k-1}) + f(\xi_{p-1})Q_{k-1}(g\,;x_p, \cdots, x_{p+k-1})$$

$$- f(\xi_{p-k})Q_{k-1}(g\,;x_{p-k}, \cdots, x_{p-1}) - \sum_{i=p-k+1}^{p-1} [f(\eta_{i-1}) - f(\eta_i)]Q_{k-1}(g\,;y_i, \cdots, y_{i+k-1})$$

$$- f(\eta_{p-1})Q_{k-1}(g\,;\,y_p, \cdots, y_{p+k-1}) + f(\eta_{p-k})Q_{k-1}(g\,;\,y_{p-k}, \cdots, y_{p-1})$$

$$- \sum_{i=p-k+2}^{p} [f(\eta_{i-1}) - f(\eta_i)]Q_{k-1}(g\,;y_i, \cdots, y_{i+k-1}) - f(\eta_p)Q_{k-1}(g\,;y_{p+1}, \cdots, y_{p+k})$$

$$+ f(\eta_{p-k+1})Q_{k-1}(g\,;\,y_{p-k+1}, \cdots, y_p),$$

and after further re-arrangement, R becomes

$$\sum_{i=p-k+1}^{p-1}[f(\xi_{i-1})-f(\xi_i)]Q_{k-1}(g;x_i,\cdots,x_{i+k-1}) \tag{3}$$

$$-2\sum_{i=p-k+2}^{p-1}[f(\eta_{i-1})-f(\eta_i)]Q_{k-1}(g;y_i,\cdots,y_{i+k-1})$$

$$+[f(\xi_{p-1})-f(\eta_p)]Q_{k-1}(g;y_{p+1},\cdots,y_{p+k})$$
$$+[f(\eta_{p-k})-f(\xi_{p-k})]Q_{k-1}(g;y_{p-k},\cdots,y_{p-1})$$

$$-[f(\eta_{p-k})-f(\eta_{p-k+1})]Q_{k-1}(g;y_{p-k+1},\cdots,y_p)$$
$$-[f(\eta_{p-1})-f(\eta_p)]Q_{k-1}(g;y_p,\cdots,y_{p+k-1})$$

$$+f(\eta_{p-k+1})Q_{k-1}(g;y_{p-k+1},\cdots,y_p)-f(\eta_{p-1})Q_{k-1}(g;y_p,\cdots,y_{p+k-1}).$$

Consequently, using the continuity of f at c, the boundedness of the $(k-1)^{\text{th}}$ divided differences in $[c-\delta, c+\delta]$, and the existence of the right and left $(k-1)^{\text{th}}$ Reimann * derivatives at c, we conclude from (3) that R has limit

$$\frac{f(c)}{(k-1)!}[D_-^{k-1}g(c)-D_+^{k-1}g(c)] \text{ as } \|\Gamma\|$$

approaches zero. The required result now follows.

Corollary. *Let $(f,g)\in RS_k\ [a,c]$ and $RS_k\ [c,b]$, where $a<c<b$. If f is continuous and g has a $(k-1)^{\text{th}}$ Riemann * derivative at c, then*

$$(f,g)\in RS_k[a,b],$$

and

$$\int_a^b f(x)\frac{d^kg(x)}{dx^{k-1}}=\int_a^c f(x)\frac{d^kg(x)}{dx^{k-1}}+\int_c^b f(x)\frac{d^kg(x)}{dx^{k-1}}.$$

3. Existence Theorems

We establish the first existence theorem for a slightly modified form of the RS_k integral, defined as follows:

Definition 5. *If in Definition 4 we impose the restrictions*

$$\xi_{-k+1}=\cdots=\xi_{-1}=a \text{ and } \xi_{n-k+1}=\cdots=\xi_{n-1}=b,$$

then we denote the corresponding integral by $(M)\ \int_a^b f(x)\,d^kg(x)/dx^{k-1}$, and speak of an MRS_k integral.

It is now necessary to point out that when $k=2$ there is a conflict of notation with an integral of Russell (1970). Since the present paper generalises that work the conflict can be conveniently resolved by replacing the "MRS_2" by "M^*RS_2".

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

It is immediately clear that if $(f, g) \in RS_k[a, b]$, then $(f, g) \in MRS_k[a, b]$, but we point out later that the converse is not necessarily true. The linearity properties of the RS_k integral are easily seen to apply to the modified integral, and we make use of them wherever necessary.

The following lemma is required for the first existence theorem.

LEMMA 4. *Let f be continuous and g k-convex in $[a', b']$. If $\Gamma(x_{-k+1}, \cdots, x_{n+k-1})$ is any subdivision of $[a, b]$, define for $i = -k+1, \cdots, n-1$*

$$M_i = \sup_{x_i \leqq x \leqq x_{i+k}} f(x) \text{ and } m_i = \inf_{x_i \leqq x \leqq x_{i+k}} f(x),$$

$$S = \sum_{i=-k+1}^{n-1} M_i [Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})],$$

and

$$s = \sum_{i=-k+1}^{n-1} m_i [Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})].$$

If the upper and lower approximating sums S and s become S' and s' respectively on the addition of an extra subdivision point to Γ, the extra point belonging to (a, b), then $S' \leqq S$ and $s' \geqq s$.

PROOF. We now suppose that the extra point of subdivision is inserted in the sub-interval (x_{s-1}, x_s), $s = 1, 2, \cdots, n$. Denote the extra point of subdivision by y_s, and denote the points of the new subdivision Γ' so formed by y_i, where

$$y_i = \begin{cases} x_i, & i = -k+1, \cdots, s-1 \\ x_{i+1}, & i = s+1, \cdots, n+k-1 \end{cases}. \tag{4}$$

We then define

$$M'_i = \sup_{y_i \leqq y \leqq y_{i+k}} f(y),$$

and

$$m'_i = \inf_{y_i \leqq y \leqq y_{i+k}} f(y) \quad \text{when} \quad i = -k+1, \cdots, n+k.$$

We now obtain the following inequalities:

$$M_i = M'_i, \quad i = -k+1, \cdots, s-k-1,$$

$$M_i \geqq \begin{cases} M'_i \\ M'_{i+1} \end{cases}, \quad i = s-k, \cdots, s-1,$$

$$M_i = M'_{i+1}, \quad i = s, \cdots, n-1.$$

We deduce the following inequalities:

$$
\begin{aligned}
M'_i &= M_i, \qquad i = -k+1, \quad \cdots, s-k-1,\\
M'_{s-k} &\leqq M_{s-k},\\
M'_i &\leqq \begin{cases} M_{i-1}, \\ M_i \quad , \end{cases} \quad i = s-k+1, \cdots, s-1,\\
M'_s &\leqq M_{s-1},\\
M'_i &= M_{i-1}, \quad i = s+1, \quad \cdots, n.
\end{aligned}
$$

Corresponding inequalities can be obtained for m_i and m'_i by replacing "M" by "m", and reversing the inequality signs. We now prove that $S' \leqq S$.

Writing $Q(x_{i+1}, \cdots, x_{i+k})$ for $Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k})$, we have

$$
\begin{aligned}
S - S' = &\sum_{i=-k+s}^{s-1} M_i[Q(x_{i+1}, \cdots, x_{i+k}) - Q(x_i, \cdots, x_{i+k-1})]\\
&- \sum_{i=-k+s}^{s} M'_i[Q(y_{i+1}, \cdots, y_{i+k}) - Q(y_i, \cdots, y_{i+k-1})]\\
&= M_{-k+s}[Q(x_{-k+s+1}, \cdots, x_s) - Q(x_{-k+s}, \cdots, x_{s-1})]\\
&+ \sum_{i=-k+s+1}^{s-2} M_i[Q(x_{i+1}, \cdots, x_{i+k}) - Q(x_i, \cdots, x_{i+k-1})]\\
&+ M_{s-1}[Q(x_s, \cdots, x_{s+k-1}) - Q(x_{s-1}, \cdots, x_{s+k-2})]\\
&- \sum_{i=-k+s}^{s} M'_i[Q(y_{i+1}, \cdots, y_{i+k}) - Q(y_i, \cdots, y_{i+k-1})].
\end{aligned} \tag{5}
$$

We now use (4) and apply Russell (1973; Theorem 1) to the first term of (5) to give

$$
\begin{aligned}
&M_{-k+s}[Q(x_{-k+s+1}, \cdots, x_s) - Q(x_{-k+s}, \cdots, x_{s-1})]\\
&= M_{-k+s}[\alpha_{-k+s}Q(y_{-k+s+1}, \cdots, y_s) + \beta_{-k+s}Q(y_{-k+s+2}, \cdots, y_{s+1})\\
&\qquad - Q(y_{-k+s}, \cdots, y_{s-1})]\\
&= M_{-k+s}[\{Q(y_{-k+s+1}, \cdots, y_s) - Q(y_{-k+s}, \cdots, y_{s-1})\}\\
&+ \beta_{-k+s}\{Q(y_{-k+s+2}, \cdots, y_{s+1}) - Q(y_{-k+s+1}, \cdots, y_s\}]
\end{aligned} \tag{6}
$$

since $\alpha_{-k+s} + \beta_{-k+s} = 1$.

In a similar way, the third term of (5) can be written in the form

$$
\begin{aligned}
&M_{s-1}[\{Q(y_{s+1}, \cdots, y_{s+k}) - Q(y_s, \cdots, y_{s+k-1})\}\\
&\qquad + \alpha_{s-2}\{Q(y_s, \cdots, y_{s+k-1}) - Q(y_{s-1}, \cdots, y_{s+k-2})\}],
\end{aligned} \tag{7}
$$

where $\alpha_{s-2} + \beta_{s-2} = 1$.

The general term of the summation

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

$$\sum_{i=-k+s+1}^{s-2} M_i[Q(x_{i+1},\cdots,x_{i+k})-Q(x_i,\cdots,x_{i+k-1})]$$

in (5) can be written in the form

$$(8)\quad \begin{aligned} &M_i[\alpha_{i-1}\{Q(y_{i+1},\cdots,y_{i+k})-Q(y_i,\cdots,y_{i+k-1})\} \\ &\quad +\beta_i\{Q(y_{i+2},\cdots,y_{i+k-1})-Q(y_{i+1},\cdots,y_{i+k}\}] \end{aligned}$$

since

$$\alpha_{i-1}+\beta_{i-1}=\alpha_i+\beta_i=1.$$

Substituting (6), (7) and (8) in (5), and re-arranging terms gives

$$(9)\quad \begin{aligned} S-S' &= (M_{-k+s}-M'_{-k+s})\{Q(y_{-k+s+1},\cdots,y_s)-Q(y_{-k+s},\cdots,y_{s-1})\} \\ &\quad +\sum_{i=-k+s}^{s-2}(\beta_i M_i+\alpha_i M_{i+1}-M'_{i+1})\{Q(y_{i+2},\cdots,y_{i+k+1}) \\ &\qquad -Q(y_{i+1},\cdots,y_{i+k})\} \\ &\quad +(M_{s-1}-M'_s)\{Q(y_{s+1},\cdots,y_{s+k})-Q(y_s,\cdots,y_{s+k-1})\}. \end{aligned}$$

Each term of (9) enclosed by curly brackets is non-negative because f is k-convex. Since $M_{-k+s}\geqq M'_{-k+s}$ and $M_{s-1}\geqq M'_s$, the first and last terms of (9) are non-negative. Furthermore, since $\alpha_i+\beta_i=1$, and $M_i\geqq M'_{i+1}$ and $M_{i+1}\geqq M'_{i+1}$ when $i=-k+s,\cdots,s-2$, it follows that $\beta_i M_i+\alpha_i M_{i+1}\geqq M'_{i+1}$. Consequently all terms of (9) are non-negative, and so we have shown that

$$(10)\qquad S'\leqq S.$$

If we replace "M" by "m" and reverse the inequality signs, a similar argument will prove that

$$(11)\qquad s'\geqq s.$$

This completes the proof.

Suppose now that F and G are defined only on $[a,b]$, and that G has right and left $(k-1)^{\text{th}}$ Riemann * derivatives $D_+^{k-1}G(a)$ and $D_-^{k-1}G(b)$. Then clearly there exists an extension f of F to $[a',b']$ such that f is continuous in $[a',b']$. For example, define $f(x)=F(a), x\leqq a$ and $f(x)=F(b), x\geqq b$. Furthermore, there exists a function g which agrees with G on $[a,b]$, is a polynomial when $x\leqq b$ and $x\geqq a$, and has $(k-1)^{\text{th}}$ Riemann * derivatives at a and b. We omit the proof. If extensions f and g are obtained in this way we say that they satisfy Condition A.

THEOREM 4. *Let F be continuous and G k-convex in $[a,b]$. Then there exist extensions f and g of F and G, respectively, to $[a',b']$ such that*

$$(f,g)\in MRS_k[a,b].$$

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

PROOF. Let f and g satisfy Condition A. We consider any $\Gamma(x_{-k+1}, \cdots, x_{n+k-1})$ subdivision of $[a, b]$, and make the following definitions:

$$M_{-1} = \sup_{x_{-k+1} \leqq x \leqq x_{k-1}} f(x),$$

$$m_{-1} = \inf_{x_{-k+1} \leqq x \leqq x_{k-1}} f(x),$$

$$M_i = \sup_{x_i \leqq x \leqq x_{i+k}} f(x),$$

$$m_i = \inf_{x_i \leqq x \leqq x_{i+k}} f(x), \qquad i = 0, 1, \cdots, n-k,$$

$$M_{n-k+1} = \sup_{x_{n-k+1} \leqq x \leqq x_{n+k-1}} f(x),$$

$$m_{n-k+1} = \inf_{x_{n-k+1} \leqq x \leqq x_{n+k-1}} f(x),$$

and

$$\begin{aligned} S = {} & M_{-1}\{Q_{k-1}(g; x_0, \cdots, x_{k-1}) - Q_{k-1}(g; x_{-k+1}, \cdots, x_0)\} \\ & + \sum_{i=0}^{n-k} M_i\{Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})\} \\ & + M_{n-k+1}\{Q_{k-1}(g; x_n, \cdots, x_{n+k-1}) - Q_{k-1}(g; x_{n-k+1}, \cdots, x_n)\}, \text{ and} \end{aligned}$$

let s be the expression obtained from S by replacing "M" by "m". Finally, we define

$$U = \inf_{\Gamma} S \quad \text{and} \quad L = \sup_{\Gamma} s.$$

Let S and s become S' and s' on the addition of an extra subdivision point to Γ. It follows immediately from Lemma 4 that if the extra subdivision point belongs to (a, b), then $S' \leqq S$ and $s' \geqq s$. If the extra subdivision point belongs to either $[a', a)$ or $(b, b']$, it follows readily that $S' = S$ and $s' = s$. Thus in all cases we have

$$S' \leqq S, \text{ and } s' \geqq s. \tag{12}$$

Let Γ_1 and Γ_2 be any two subdivisions of $[a, b]$, and let Γ_3 be a subdivision obtained by combining Γ_1 and Γ_2, all points of Γ_1 and Γ_2 in $[a, b]$ being included. If the upper and lower approximating sums corresponding to Γ_i are denoted, respectively, by S_i and s_i, $i = 1, 2, 3$, then it follows from (12) that

$$s_1 \leqq s_3 \leqq S_3 \leqq S_2.$$

Similarly $s_2 \leqq S_1$, and so we conclude that $L \leqq U$.

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

Since f is continuous in $[a', b']$ it is uniformly continuous there, so, given $\varepsilon > 0$, there exists $\delta(\varepsilon)$ such that for each i,

$$M_i - m_i < \frac{\varepsilon}{M} \text{ whenever } \|\Gamma\| < \delta(\varepsilon),$$

M being an upperbound for the non-negative sum

$$\sum_{i=-k+1}^{n-1} \{Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})\}$$
$$= Q_{k-1}(g; x_n, \cdots, x_{n+k-1}) - Q_{k-1}(g; x_{-k+1}, \cdots, x_0);$$

and the upper bound exists since $D_+^{k-1}g(x)$ and $D_-^{k-1}g(x)$ exist for all x in (a', b') by Bullen (1971; Theorem 7(b)). Hence

$$0 \leqq U - L \leqq S - s = \sum_{i=-k+1}^{n-1} (M_i - m_i)\,[Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})] < \varepsilon.$$

Therefore, since $\varepsilon > 0$ is arbitrary, $U = L = I$, say. Consequently if $\xi_{-k+1} = \xi_{-k+2} = \cdots = \xi_{-1} = a$, $x_i \leqq \xi_i \leqq x_{i+k}$, $i = 0, 1, \cdots, n-k$, and

$$\xi_{n-k+1} = \xi_{n-k+2} = \cdots = \xi_{n-1} = b,$$

it follows that

$$\left| I - \sum_{i=-k+1}^{n-1} f(\xi_i)\,[Q_{k-1}(g; x_{i+1}, \cdots, x_{i+k}) - Q_{k-1}(g; x_i, \cdots, x_{i+k-1})] \right| < \varepsilon$$

whenever $\|\Gamma\| < \delta(\varepsilon)$. This completes the proof.

REMARK. The necessity of introducing the interval $[a', b']$ in association with the interval $[a, b]$ arose from determinations of the behaviour of S and s on the addition of extra subdivision points. An example is given in Russell (1970) of the difficulties arising when π, and not Γ, subdivisions of $[a, b]$ are used.

We now relax the restrictions imposed upon f and g in the intervals $[a', a]$ and $[b, b']$, and also obtain existence theorems for the RS_k integral.

THEOREM 5. *Let f be continuous on $[a', b']$. If g is k-convex in $[a', b']$, then $(f, g) \in MRS_k[a, b]$.*

The proof is straightforward and will be omitted.

THEOREM 6. *If f is continuous on $[a', b']$, and $g \in BV_k[a', b']$, then $(f, g) \in MRS_k[a, b]$.*

PROOF. The proof follows from Theorem 18 of Russell (1973), Theorem 5, and Theorem 2.

THEOREM 7. *If f is continuous and $g \in BV_k[a',b']$, then $(f,g) \in RS_k[a,b]$, and*

$$\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = (M)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

The proof is straightforward and will be omitted.

We have obtained an existence theorem for the RS_k integral when f and g are defined on $[a',b']$. If F and G are only defined on $[a,b]$, F being continuous and G of bounded k^{th} variation, then it is clear from the previous discussions that extensions f and g of F and G, respectively, exist such that $(f,g) \in RS_k[a,b]$. We now state two theorems, without proof, which show effectively that the existence of the RS_k integral is determined by the behaviour of f and g in $[a,b]$ and at the end points a and b, and is otherwise independent of the extensions used.

THEOREM 8. *Let f_1 and f_2 be two functions that are continuous at a and b, and, in addition, are continuous and equal on $[a,b]$. If $g \in BV_k[a',b']$, then (f_1,g) and (f_2,g) belong to $RS_k[a,b]$, and*

$$\int_a^b f_1(x)\frac{d^k g(x)}{dx^{k-1}} = \int_a^b f_2(x)\frac{d^k g(x)}{dx^{k-1}}.$$

THEOREM 9. *Let f be continuous on $[a',b']$, and let g_1, and g_2 belong to $BV_k[a,b]$. If, in addition, $g_1(x) = g_2(x)$, $a \leqq x \leqq b$, and g_1 and g_2 have $(k-1)^{\text{th}}$ Riemann * derivatives at a and b, then (f,g_1) and (f,g_2) belong to $RS_k[a,b]$, and*

$$\int_a^b f(x)\frac{d^k g_1(x)}{dx^{k-1}} = \int_a^b f(x)\frac{d^k g_2(x)}{dx^{k-1}}.$$

If in Definition 4 we consider only π subdivisions of $[a,b]$, so that we necessarily consider only functions f and g defined on $[a,b]$, then we obtain an RS_k^* integral,

$$*\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

We now present existence theorems for this integral.

THEOREM 10. *If $(f,g) \in RS_k[a,b]$, and g has $(k-1)^{\text{th}}$ Riemann * derivatives at a and b, then $(f,g) \in RS_k^*[a,b]$, and*

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

$$*\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = \int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}.$$

The proof is straightforward and will be omitted.

THEOREM 11. *If f is continuous on $[a,b]$, and $g \in BV_k[a,b]$, then $(f,g) \in RS_k^*[a,b]$.*

PROOF. Since $g \in BV_k[a,b]$, it follows from Russell (1973; Theorem 19) that g has right and left $(k-1)^{\text{th}}$ Riemann * derivatives at a and b respectively, and so there extensions F and G satisfying Condition A and such that F is continuous and G is of bounded k^{th} variation in $[a',b']$. Thus $(F,G) \in RS_k[a,b]$, and the required result now follows from Theorem 10.

We conclude this section by giving an example which shows that the existence of the MRS_k integral does not imply the existence of the RS_k integral. Let $k=2$, $a=0$ and $b=1$. If $f(x)=0$ when $x \neq 0$, $f(0)=1$, and g is any function for which $g_-^{\prime(a)}$ and $g_+^{\prime(a)}$ exist and are unequal, then

$$\int_0^1 f(x)\frac{d^2g(x)}{dx}$$

does not exist, whereas

$$(M)\int_0^1 f(x)\frac{d^2g(x)}{dx}$$

does exist, and equals $g_+^{\prime(a)} - g_-^{\prime(a)}$.

4. Related Integrals.

We now discuss further useful modifications of the RS_k integral. These modifications will be obtained by giving ξ_i, in Definition 4, a specific value in the sub-interval $[x_i, x_{i+k}]$, and also by restricting our subdivisions so that all sub-intervals are of equal length. In Russell (1970) examples of modified integrals are given, one of which exhibits properties of Dirac's delta function.

If in Definition 4 we put $\xi_i = x_i$, we call the corresponding integral the M_0RS_k integral.

Again, if in Definition 4 we consider only Γ subdivisions in which all sub-intervals $(x_i - x_{i-1})$ are of equal length, and put $\xi_i = x_{i+k}$, then we call the corresponding integral the M_kRS_k integral. This integral will be useful in the context of integration by parts.

We conclude this section by showing that $(f,g) \in M_0RS_k[a,b]$ when f is quasi-continuous and g is of bounded k^{th} variation on $[a',b']$. We define step functions and quasi-continuous functions that are anchored at a as in Webb

(1967). We must now define the following extensions of the unit step functions L_c and R_c of Webb (1967):

$$r_c(x) = 0, a' \leqq x < c$$
$$= 1, c \leqq x \leqq b', \text{ and}$$
$$l_c(x) = 0, a' \leqq x \leqq c$$
$$= 1, c < x \leqq b'.$$

THEOREM 12. *If $k \geqq 2$ and $g \in BV_k[a', b']$, then*

$$(M_0)\int_a^b l_c(x)\frac{d^k g(x)}{dx^{k-1}}$$

and

$$\int_a^b r_c(x)\frac{d^k g(x)}{dx^{k-1}}$$

exist and equal

$$\frac{D_+^{k-1}g(b) - D_+^{k-1}g(c)}{(k-1)!}$$

The proof is straightforward and will be omitted.

THEOREM 13. *If F is a step function on $[a, b]$, and $g \in BV_k[a', b']$, then there exists an extension f of F such that $(f, g) \in M_0RS_k[a, b]$.*

Again the proof is straightforward and will be omitted.

THEOREM 14. *If F is a quasi-continuous function anchored at a, and $g \in BV_k[a', b']$, then there is an extension f of F such that*

$$(M_0)\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}}$$

exists.

PROOF. Since F is quasi-continuous on $[a, b]$, there exists a sequence $\{F_n\}$ of step functions which converges uniformly to F. We extend each step function F_n by defining

$$f_n(x) = \begin{cases} 0, x \leqq a \\ F_n(x), a \leqq x \leqq b \\ F_n(b), x > b. \end{cases}$$

Then the sequence $\{f_n\}$ converges uniformly to, say, f on $[a', b']$. Then, using Theorem 13 we obtain

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

$$\left|(M_0)\int_a^b f_m(x)\frac{d^k g(x)}{dx^{k-1}} - (M_0)\int_a^b f_n(x)\frac{d^k g(x)}{dx^{k-1}}\right|$$

$$= \left|(M_0)\int_a^b \{f_m(x) - f_n(x)\}\frac{d^k g(x)}{dx^{k-1}}\right|$$

$$\leqq \|f_m - f_n\| \, W_k(g\,;\,a', b'), \text{ where}$$

$$\|f_m - f_n\| = \sup_{a' \leqq x \leqq b'} |f_m(x) - f_n(x)|.$$

Hence
$$\left\{(M_0)\int_a^b f_n(x)\frac{d^k g(x)}{dx^{k-1}}\right\}$$

is a Cauchy sequence of real numbers, so let its limit be I.

Let $\varepsilon > 0$ be given. Then there is a positive integer n such that

$$\left|I - (M_0)\int_a^b f_n(x)\frac{d^k g(x)}{dx^{k-1}}\right| < \frac{\varepsilon}{3}, \tag{13}$$

and

$$\|f_n - f\| < \frac{\varepsilon}{3W_k(g\,;\,a', b')}. \tag{14}$$

Since $(f_n, g) \in M_0RS_k[a, b]$, there exists $\delta(\varepsilon)$ such that whenever $\|\Gamma\| < \delta(\varepsilon)$,

$$\left|(M_0)\int_a^b f_n(x)\frac{d^k g(x)}{dx^{k-1}} - S(\Gamma, f_n, g)\right| < \frac{\varepsilon}{3}, \tag{15}$$

where $S(\Gamma, f_n, g)$ is an approximating sum for

$$(M_0)\int_a^b f_n(x)\frac{d^k g(x)}{dx^{k-1}}.$$

Consequently, if $S(\Gamma, f, g)$ is an approximating sum for the M_0RS_k integral, and $\|\Gamma\| < \delta(\varepsilon)$, then using (13), (14) and (15) we obtain

$$|I - S(\Gamma, f, g)| < \varepsilon,$$

and this completes the proof.

5. Integration by Parts

In order to obtain an integration by parts result we will need to introduce Hellinger-type integrals. The simplest integration by parts result is given in Russell (1970) for the case $k = 2$. For larger values of k it appears to be convenient to work with sub-intervals of equal length.

In order to define our Hellinger-type integrals we make use of the operator Δ_h^k, where

https://doi.org/10.1017/S1446788700016153 Published online by Cambridge University Press

$$\Delta_h^1 f(x) = f(x+h) - f(x), \text{ and } \Delta_h^k f(x) = \Delta(\Delta_h^{k-1} f(x)).$$

DEFINITION 6. *Let $\varepsilon > 0$ be arbitrary and let s be a fixed positive integer such that $1 \leqq s \leqq k-1$. Then*

$$\int_a^b \frac{d^s f(x) d_{g(x)}^{k-s}(x)}{dx^{k-1}}$$

is the real number I, if it exists uniquely and there is a real number $\delta(\varepsilon)$ such that

$$\left| I - \frac{1}{(s-1)!\,(k-s)!} \sum_{i=-k+1}^{n-s-1} \frac{\Delta_h^s f(x_{i+k-s}) \Delta_h^{k-s} g(x_{i+s})}{h^{k-1}} \right| < \varepsilon$$

whenever $\|\Gamma_h\| < \delta(\varepsilon)$.

We retain the factorial terms in our definition as a consequence of the result.

$$Q_k(f; x_0, \cdots, x_k) = \frac{\Delta_h^k f(x_0)}{k!\, h^k}$$

when all sub-intervals are of equal length h.

THEOREM 15. *Let $(f,g) \in RS_k[a,b]$. If $D_-^{k-2} g(a)$, $D_+^{k-2} g(b)$, $D_-^{k-2} f(a)$ and $D^{k-2} f(b)$ exist, and $1 \leqq s \leqq k-1$, then*

$$\int_a^b \frac{d^s f(x) s^{k-s} g(x)}{dx^{k-1}} \text{ exists and}$$

(16)
$$\int_a^b f(x) \frac{d^k g(x)}{dx^{k-1}} = \frac{(-1)^s (s-1)!\,(k-s)!}{(k-1)!} \int_a^b \frac{d^s f(x) d^{k-s} g(x)}{dx^{k-1}}$$
$$+ \frac{1}{(k-1)!} \sum_{m=1}^{s} (-1)^{m-1} [D^{m-1} f(b) D_+^{k-m} g(b) - D_-^{m-1} f(a) D_-^{k-m} g(a)].$$

PROOF. We consider any Γ_h subdivision of $[a,b]$ in which all sub-intervals are of equal length h. Denote the corresponding approximating sum of the RS_k integral by $S(\Gamma_h, f, g)$. Since $(f,g) \in RS_k[a,b]$, we can choose $\xi_i = x_{i+k-s}$, $i = -k+1, \cdots, n-1$. Then, writing Δ^n instead of Δ_h^n, and using the property

$$\Delta_h^n f(x) = \Delta_h^{n-1} f(x+h) - \Delta_h^{n-1} f(x),$$

we are able to show that

$$h^{k-1}(k-1)!\, S(\Gamma_h, f, g)$$

(17)
$$= (-1)^s \sum_{i=-k+1}^{n-s-1} \Delta^s f(x_{i+k-s}) \Delta^{k-s} g(x_{i+s})$$
$$+ \sum_{m=1}^{s} (-1)^{m-1} [\Delta^{m-1} f(x_{n+k-s-m}) \Delta^{k-m} g(x_n) - \Delta^{m-1} f(x_{-s+1}) \Delta^{k-m} g(x_{-k+m})].$$

Dividing (17) by $h^{k-1}(k-1)!$, noting that the existence of a $(k-2)^{\text{th}}$ Riemann * derivative implies the existence of a $(k-s-2)^{\text{th}}$ Riemann * derivative when $s = 1, 2, \cdots, k-2$, and letting $\|\Gamma\|$ approach zero, establishes the required result.

THEOREM 16. *Let $(f, g) \in RS_k[a, b]$. If $D_-^{k-1}g(a)$, $D_+^{k-1}g(b)$, $D_-^{k-1}f(a)$ and $D^{k-1}f(b)$ exist, then*

$$(M_k)\int_a^b g(x)\frac{d^k f(x)}{dx^{k-1}}$$

exists, and

$$\int_a^b f(x)\frac{d^k g(x)}{dx^{k-1}} = (-1)^k (M_k)\int_a^b g(x)\frac{d^k f(x)}{dx^{k-1}}$$

(18)

$$+\frac{1}{(k-1)!}\sum_{m=1}^{k}(-1)^{m-1}[D_-^{m-1}f(b)D_+^{k-m}g(b) - D_-^{m-1}f(a)D_-^{k-m}g(a)].$$

PROOF. The proof is similar to that for Theorem 15, and will be omitted.

REMARK. If in addition to the hypothesis of Theorem 16 we know that $(g, f) \in RS_k[a, b]$, then the M_kRS_k integral in (18) can be replaced by the RS_k integral.

6. Reduction of the RS_k Integral

THEOREM 17. *Let f be a step-function on $[a, b]$, and let $g \in BV_{k-1}[a', b']$. If $G(x) = G(a') + \int_{a'}^{x} g(t)dt$, $a' \leqq x \leqq b'$, then there is an extension F of f to $[a', b']$ so that*

$$(k-1)(M_0)\int_a^b F(x)\frac{d^k G(x)}{dx^{k-1}} = (M_0)\int_a^b F(x)\frac{d^{k-1}g(x)}{dx^{k-2}}.$$

PROOF. The case $k = 2$ appears in Russell (1970), so we assume that $k \geqq 3$. We first note that since $g \in BV_{k-1}[a', b']$, it follows from Lemma 3 that $G \in BV_k[a', b']$. Then, as in Theorem 12, $(r_c, g) \in M_0RS_{k-1}[a, b]$, $(r_c, G) \in M_0RS_k[a, b]$, and

$$(M_0)\int_a^b r_c(x)\frac{d^{k-1}g(x)}{dx^{k-2}} = \frac{D_+^{k-2}g(b) - D_+^{k-2}g(c)}{(k-2)!}$$

and

$$(M_0)\int_a^b r_c(x)\frac{d^k G(x)}{dx^{k-1}} = \frac{D_+^{k-1}G(b) - D_+^{k-1}G(c)}{(k-1)!}.$$

Corresponding results apply for the extended unit step function l_c. It now follows from Russell (1973; Theorem 18) and Bullen (1971; Theorem 7), that

$$D_+^{k-1}G(b) = (G^{k-2})'_+(b) = (g^{k-3})'_+(b) = D_+^{k-2}g(b).$$

Similarly, $D_+^{k-1}G(c) = D_+^{k-2}g(c)$, and consequently

$$(k-1)(M_0)\int_a^b r_c(x)\frac{d^kG(x)}{dx^{k-1}} = (M_0)\int_a^b r_c(x)\frac{d^{k-1}g(x)}{dx^{k-2}}.$$

Similarly,

$$(k-1)(M_0)\int_a^b l_c(x)\frac{d^kG(x)}{dx^{k-1}} = (M_0)\int_a^b l_c(x)\frac{d^{k-1}g(x)}{dx^{k-2}}.$$

The required result now follows using the definition of a step-function.

THEOREM 18. *Let f be anchored at a, continuous in $[a',b']$ and let $g \in BV_{k-1}[a',b']$. If*

$$G(x) = G(a') + \int_{a'}^x g(t)dt,$$

then

$$(k-1)\int_a^b f(x)\frac{d^kG(x)}{dx^{k-1}} = \int_a^b f(x)\frac{d^{k-1}g(x)}{dx^{k-2}}.$$

PROOF. We first observe that (f,g) and (f,G) belong to $RS_{k-1}[a,b]$ and $RS_k[a,b]$ respectively. Let F be the restriction of f to $[a,b]$. It is a well known result that a sequence $\{F_n\}$ of step functions can be obtained which converges uniformly to F. As in Theorem 14 we define an extension f_n of F_n for each n. Then the sequence $\{f_n\}$ converges uniformly to a function h on $[a',b']$, and h will be continuous in $[a',b']$. From Theorem 13 we conclude that $(f_n,G) \in M_0RS_k\ [a,b]$, and $(f_n,g) \in M_0RS_{k-1}[a,b]$, and Theorem 17 tells us that

$$(k-1)(M_0)\int_a^b f_n(x)\frac{d^kG(x)}{dx^{k-1}} = (M_0)\int_a^b f_n(x)\frac{d^{k-1}g(x)}{dx^{k-2}}$$

for all values of n.

Since g and G belong to $BV_{k-1}[a',b']$ and $BV_k[a',b']$ respectively, we let n tend to infinity and obtain

$$(k-1)(M_0)\int_a^b h(x)\frac{d^kG(x)}{dx^{k-1}} = (M_0)\int_a^b h(x)\frac{d^{k-1}g(x)}{dx^{k-2}}.$$

Since (h,G) and $(h,g) \in RS_k[a,b]$ and $RS_{k-1}[a,b]$ respectively, it follows that

$$(k-1)\int_a^b h(x)\frac{d^kG(x)}{dx^{k-1}} = \int_a^b h(x)\frac{d^{k-1}g(x)}{dx^{k-2}},$$

and finally since h is continuous on $[a',b']$, agreeing with f on $[a,b]$, we employ Theorem 8 to show that

$$(k-1)\int_a^b f(x)\frac{d^kG(x)}{dx^{k-1}} = \int_a^b f(x)\frac{d^{k-1}g(x)}{dx^{k-2}}.$$

References

P. S. Bullen (1971), 'A criterion for n-convexity', *Pacific J. Math.* **36**, 81–98.

J. C. Burkill (1957), 'An integral for distributions', *Proc. Cambridge Philos. Soc.* **53**, 821–824.

A. M. Russell (1970), 'Functions of bounded second variation and Stieltjes-type integrals', *J. London Math. Soc.* (2) **2**, 193–208.

A. M. Russell (1973), 'Functions of bounded k^{th} variation', *Proc. London Math. Soc.* (3) **26**, 547–563.

J. R. Webb (1967), 'A Hellinger integral representation for bounded linear functionals', *Pacific J. Math.* **20**, 327–337.

Department of Mathematics
University of Melbourne
Parkville 3052
Australia