INEQUALITIES FOR THE PERMANENTAL MINORS OF NON-NEGATIVE MATRICES

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1. Introduction. Let A be an $n \times n$ *non-negative matrix*, that is, a matrix whose entries are non-negative numbers. The *permanent* of A is the scalar-valued function of A defined by

$$\text{per}\,(A) = \sum a_{1i_1} \ldots a_{ni_n}$$

where the summation extends over all permutations $i_1, \ldots, i_n$ of the integers $1, \ldots, n$. The purpose of this paper is to prove several inequalities involving the permanent of A and the permanent of submatrices of A when suitable restrictions are placed on the row sums. One result, for instance, states that when each of the row sums of A does not exceed 1, then the sum of the permanents of all $r \times r$ submatrices of A does not exceed $\binom{n}{r}$. This improves a result of Marcus and Gordon **(1)**. For such matrices it is also shown that the permanent cannot be greater than the maximum permanent of an $r \times r$ submatrix of A.

If A is an $n \times n$ non-negative matrix with row sums $r_1, \ldots, r_n$ and column sums $s_1, \ldots, s_n$, then A is called *row substochastic* if $r_i \leqslant 1$, $i = 1, \ldots, n$; *row stochastic* if $r_i = 1$, $i = 1, \ldots, n$; and *doubly stochastic* if $r_i = s_i = 1$, $i = 1, \ldots, n$. Doubly stochastic matrices and their permanents have been studied extensively **(2**; **3**; **4)** and it is known that their permanents are always positive.

Let r and n be positive integers with $1 \leqslant r \leqslant n$. Following Marcus and Minc **(4)** we denote by $Q_{r,n}$ the totality of strictly increasing sequences of r integers chosen from $1, \ldots, n$. Thus the sequence $i_1, \ldots, i_r$ is in $Q_{r,n}$ if and only if $1 \leqslant i_1 < \ldots < i_r \leqslant n$. $Q_{r,n}$, of course, contains $\binom{n}{r}$ sequences. If $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ are two sequences in $Q_{r,n}$, then $A[i_1, \ldots, i_r|j_1, \ldots, j_r]$ denotes the $r \times r$ submatrix of A formed by rows $i_1, \ldots, i_r$ and columns $j_1, \ldots, j_r$ and $A(i_1, \ldots, i_r\,|\,j_1, \ldots, j_r)$ denotes the $(n - r) \times (n - r)$ submatrix of A formed by the rows complementary to $i_1, \ldots, i_r$ and the columns complementary to $j_1, \ldots, j_r$. The permanent of $A[i_1, \ldots, i_r|j_1, \ldots, j_r]$ is called a *permanental minor* of order r of A. In case $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$ are identical, we denote the corresponding submatrices more briefly by $A[i_1, \ldots, i_r]$ and $A(i_1, \ldots, i_r)$. In this case the permanent of $A[i_1, \ldots, i_r]$

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is a *principal permanental minor* of order r. Suppose the sequences in $Q_{r,n}$ have been ordered lexicographically. Then the rth *permanental compound* of A, denoted by $P^r(A)$, is the $\binom{n}{r} \times \binom{n}{r}$ matrix whose entries are

$$\text{per}\,(A[i_1, \ldots, i_r \mid j_1, \ldots, j_r])$$

arranged lexicographically in $i_1, \ldots, i_r$ and $j_1, \ldots, j_r$. Observe that $P^1(A) = A$ and that $P^n(A)$ is the 1×1 matrix whose single entry is per (A).

2. Results. We first observe the following: Let $A = (a_{ij})$ be an $n \times n$ non-negative row substochastic matrix. Then

$$(1) \qquad a_{i1} + \ldots + a_{in} \leqslant 1$$

for $i = 1, \ldots, n$. Hence for any sequence of integers $k_1, \ldots, k_r$ in $Q_{r,n}$

$$\prod (a_{i1} + \ldots + a_{in}) \leqslant 1,$$

where the product is taken over all $i = k_1, \ldots, k_r$; or

$$(2) \qquad \sum_\sigma a_{k_1\sigma(k_1)} \ldots a_{k_r\sigma(k_r)} \leqslant 1,$$

the summation extending over all n^r mappings σ of $k_1, \ldots, k_r$ into $1, \ldots, n$. Put $N = \binom{n}{r}$. Suppose the N sequences in $Q_{r,n}$ have been ordered lexicographically, say $\alpha_1, \ldots, \alpha_N$. For $i = 1, \ldots, N$, let σ_i run over all one-to-one mappings of $k_1, \ldots, k_r$ onto α_i. Then inequality (2) can be written as

$$(3) \qquad \sum_{\sigma_1} a_{k_1\sigma_1(k_1)} \ldots a_{k_r\sigma_1(k_r)} + \ldots + \sum_{\sigma_N} a_{k_1\sigma_N(k_1)} \ldots a_{k_r\sigma_N(k_r)} + \sum_\tau a_{k_1\tau(k_1)} \ldots a_{k_r\tau(k_r)} \leqslant 1,$$

where τ runs over all mappings of $k_1, \ldots, k_r$ into $1, \ldots, n$ such that

$$\tau(k_i) = \tau(k_j)$$

for at least one pair i, j with $1 \leqslant i < j \leqslant r$. We can now write inequality (3) as

$$(4) \qquad \sum \text{per}\,(A[k_1, \ldots, k_r \mid j_1, \ldots, j_r]) + \sum_\tau a_{k_1\tau(k_1)} \ldots a_{k_r\tau(k_r)} \leqslant 1,$$

the first summation extending over all sequences $j_1, \ldots, j_r$ in $Q_{r,n}$. Since A is also a non-negative matrix, we may conclude from inequality (4) that

$$(5) \qquad \sum \text{per}\,(A[k_1, \ldots, k_r \mid j_1, \ldots, j_r]) \leqslant 1.$$

In equality (5) $k_1, \ldots, k_r$ is an arbitrary but fixed sequence in $Q_{r,n}$ and the summation extends over all sequences $j_1, \ldots, j_r$ in $Q_{r,n}$. Equality occurs in (5) if and only if equality occurs in (1) for $i = k_1, \ldots, k_r$ and

$$a_{k_1\tau(k_1)} \ldots a_{k_r\tau(k_r)} = 0$$

for each τ. We can now state and prove the following two theorems.

THEOREM 1. *Let A be an $n \times n$ non-negative row substochastic matrix. Then for $r = 1, \ldots, n$ the* rth *permanental compound of A, $P^r(A)$, is an $\binom{n}{r} \times \binom{n}{r}$ non-negative row substochastic matrix. $P^1(A)$ is row stochastic if and only if A is row stochastic. For $r = 2, \ldots, n$, $P^r(A)$ is row stochastic if and only if A is a permutation matrix.*

Proof. Since A is non-negative, $P^r(A)$ is clearly non-negative. The fact that $P^r(A)$ is row substochastic is immediate from inequality (5). Since $P^1(A) = A$, $P^1(A)$ is row stochastic if and only if A is. Let r be a positive integer with $2 \leqslant r \leqslant n$. By the preceding remarks, a necessary condition for $P^r(A)$ to be row stochastic is that A be row stochastic. Hence assume that A is row stochastic. Then $P^r(A)$ is row stochastic if and only if equality occurs in (5) for each sequence $k_1, \ldots, k_r$ in $Q_{r,n}$, which in turn happens if and only if

$$a_{k_1\tau(k_1)} \ldots a_{k_r\tau(k_r)} = 0 \tag{6}$$

for all sequences $k_1, \ldots, k_r$ in $Q_{r,n}$ and all mappings τ of $k_1, \ldots, k_r$ into $1, \ldots, n$ such that $\tau(k_i) = \tau(k_j)$ for at least one pair i, j with $i \neq j$. Holding $k_1, \ldots, k_r$ fixed, we may allow the $\tau(k_p)$, $p \neq i, j$, to vary independently over $1, \ldots, n$. By repeated summation of (6), using the fact that A is row stochastic, we obtain

$$a_{k_i\tau(k_i)}a_{k_j\tau(k_i)} = 0$$

for all $i \neq j$ and for $\tau(k_i) = 1, \ldots, n$. Summarizing, we have shown that

$$a_{ik}\, a_{jk} = 0, \qquad i \neq j, k = 1, 2, \ldots, n.$$

This means that A has at most one non-zero element in each column. Since A is row stochastic, there must be at least n non-zero elements in A, one in each row. Hence by the pigeon-hole principle each column has precisely one non-zero element and A is a permutation matrix. This completes the proof of the theorem.

THEOREM 2. *Let A be an $n \times n$ non-negative row substochastic matrix. For $r = 1, \ldots, n$ let $p_r(A)$ be the sum of all the permanental minors of A of order r. Then*

$$p_r(A) \leqslant \binom{n}{r}. \tag{7}$$

For $r = 1$, equality occurs in (7) if and only if A is row stochastic. For $r = 2, \ldots, n$ equality occurs in (7) if and only if A is a permutation matrix.

Proof. Inequality (7) follows from Theorem 1 and the observation that $p_r(A)$ is the sum of all the elements of the rth compound of A, $P^r(A)$. For $r = 1$, $p_1(A)$ is the sum of the elements of A and equals n if and only if A is row stochastic. For $r = 2, \ldots, n$ equality occurs in (7) if and only if $P^r(A)$ is row stochastic, which, by Theorem 1, will happen if and only if A is a permutation matrix. This concludes the proof.

Inequality (7) improves a result of M. Marcus and W. R. Gordon who obtained in **(1)** by entirely different methods that for A an $n \times n$ non-negative doubly stochastic matrix

$$s_r(A) \leqslant \binom{n}{r}$$

where $s_r(A)$ is the sum of the squares of all permanental minors of order r. Their condition for equality is the same as ours, namely A a permutation matrix.

The next two theorems are concerned with the principal permanental minors of row stochastic matrices.

THEOREM 3. *Let $A = (a_{ij})$ be an $n \times n$ non-negative row stochastic matrix. Then for $r = 1, \ldots, n-1$*

$$(8)\ \sum \text{per}\,(A[i_1, \ldots, i_r])(1 - \text{per}\,(A(i_1, \ldots, i_r))) \leqslant \binom{n-1}{r}(1 - \text{per}\,(A))$$

where the summation extends over all sequences $i_1, \ldots, i_r$ in $Q_{r,n}$.

Proof. We first make the following observation. Since A is row stochastic,

$$\binom{n-1}{r} = \binom{n-1}{r}\prod_{i=1}^{n}(a_{i1} + \ldots + a_{in})$$

or

$$(9)\qquad \binom{n-1}{r} = \binom{n-1}{r}\text{per}\,(A) + \binom{n-1}{r}\sum\nolimits_\tau a_{1\tau(1)} \ldots a_{n\tau(n)}$$

where τ runs over all mappings of $1, \ldots, n$ into itself such that $\tau(i) = \tau(j)$ for at least one pair i, j with $i \neq j$.

Consider now the expression

$$(10)\qquad \text{per}\,(A[i_1, \ldots, i_r])(1 - \text{per}\,(A(i_1, \ldots, i_r)))$$

for a fixed sequence $i_1, \ldots, i_r$ in $Q_{r,n}$. Set $s = n - r$ and let $j_1, \ldots, j_s$ be the complementary sequence in $Q_{s,n}$. Then (10) may be written as

$$(11)\qquad \text{per}\,(A[i_1, \ldots, i_r])(1 - \text{per}\,(A[j_1, \ldots, j_s])).$$

Since A is row stochastic, we may replace the number 1 in (11) by

$$\prod(a_{j1} + \ldots + a_{jn})$$

where the product is taken over all $j = j_1, \ldots, j_s$. Hence (11) may be written as

$$(12)\qquad \sum\nolimits_\rho\sum\nolimits_\sigma a_{i_1\rho(i_1)} \ldots a_{i_r\rho(i_r)}a_{j_1\sigma(j_1)} \ldots a_{j_s\sigma(j_s)}$$

where ρ runs over all permutations of $i_1, \ldots, i_r$ and σ runs over all mappings of $j_1, \ldots, j_s$ into $1, \ldots, n$ such that σ is not a permutation of $j_1, \ldots, j_s$. All of the terms in (12) are formally distinct. Every term in (12) occurs as a term in

$$(13)\qquad \sum\nolimits_\tau a_{1\tau(1)} \ldots a_{n\tau(n)}$$

where τ runs over all mappings of $1, \ldots, n$ into itself such that $\tau(i) = \tau(j)$ for at least one pair i, j with $i \neq j$. A term $a_{1\tau(1)} \ldots a_{n\tau(n)}$ in the sum (13) may occur as a term in the double sum (12) for more than one sequence $i_1, \ldots, i_r$ in $Q_{r,n}$. We may write such a term as

$$a_{1l_1} \ldots a_{nl_n}, \tag{14}$$

where $l_p = l_q$ for some pair p, q with $p \neq q$. There is then an integer k such that $l_i \neq k$ for $i = 1, \ldots, n$. Define D to be that subset of $Q_{r,n}$ consisting of those sequences $i_1, \ldots, i_r$ for which (14) occurs as a term in the corresponding double sum (12). Then for all sequences $i_1, \ldots, i_r$ in D we have that $i_j \neq k$ for $j = 1, \ldots, r$. Hence the number of sequences in D cannot exceed the number of sequences in $Q_{r,n-1}$, which is $\binom{n-1}{r}$. Hence each term (14) occurs as a term in (12) for at most $\binom{n-1}{r}$ sequences in $Q_{r,n}$. Therefore

$$\sum \text{per}\,(A[i_1, \ldots, i_r])(1 - \text{per}\,(A(i_1, \ldots, i_r))) \leqslant \binom{n-1}{r} \sum_\tau a_{1\tau(1)} \ldots a_{n\tau(n)}$$
$$= \binom{n-1}{r} - \binom{n-1}{r} \text{per}\,(A),$$

the equality following from our initial observation (9). This proves the theorem.

LEMMA 1. *Let A be an $n \times n$ non-negative matrix. Let r be an integer with $1 \leqslant r \leqslant n - 1$. Suppose that* per $(A) > 0$ *and that for all sequences $i_1, \ldots, i_r$ in $Q_{r,n}$*

$$\text{per}\,(A) = \text{per}(A[i_1, \ldots, i_r])\text{per}\,(A(i_1, \ldots, i_r)). \tag{15}$$

Then there exists a permutation matrix P such that

$$P'AP = \begin{bmatrix} x_1 & & 0 \\ & \ddots & \\ * & & x_n \end{bmatrix}$$

where per $(A) = x_1 \ldots x_n$. *Here* 0 *denotes all* 0's *while* * *denotes arbitrary elements.*

Proof. The lemma is true for $n = 1$. Suppose we have shown it for all $m \times m$ non-negative matrices with $m < n$ and all integers r with $1 \leqslant r \leqslant m - 1$. We proceed by induction.

Partition the matrix A as

$$\begin{bmatrix} A_{rr} & A_{rs} \\ A_{sr} & A_{ss} \end{bmatrix}$$

where A_{rr} and A_{ss} are $r \times r$ and $s \times s$ matrices respectively. If A_{rs} is a zero matrix, then A has an $r \times s$ submatrix of 0's with $r + s = n$. Otherwise A_{rs}

contains a non-zero element. Since by hypothesis per $(A) = \text{per}\,(A_{rr})\,\text{per}\,(A_{ss})$, it follows that the $(n-1) \times (n-1)$ matrix obtained by crossing out the row and column of this non-zero element must have a zero permanent. Hence, by the Frobenius–König theorem, it contains a $p \times q$ submatrix of 0's with $p + q = (n-1) + 1$. Thus in either case A has a $p \times q$ submatrix of 0's with $p + q = n$.

Suppose

$$a_{1j_1} \ldots a_{nj_n} \neq 0$$

where $j_1, \ldots, j_n$ is a permutation of $1, \ldots, n$ other than the identical permutation. Then the permutation $j_1, \ldots, j_n$ contains a cycle $(k_1, \ldots, k_t)$ of length $t > 1$. Choose a sequence $i_1, \ldots, i_r$ in $Q_{r,n}$ such that at least one, but not all, of the integers $k_1, \ldots, k_t$ is included among the integers $i_1, \ldots, i_r$. For such a sequence $i_1, \ldots, i_r$ it is easily seen that relation (15) does not hold. This contradicts our hypothesis and so

$$a_{1j_1} \ldots a_{nj_n} = 0$$

for all permutations $j_1, \ldots, j_n$ of $1, \ldots, n$ other than the identical permutation. Since per $(A) \neq 0$ by assumption, it follows that per $(A) = a_{11} \ldots a_{nn} \neq 0$ and no diagonal element of A is zero.

Let the zeros of the $p \times q$ zero submatrix of A occur in positions (i_α, j_β), $1 \leqslant \alpha \leqslant p$, $1 \leqslant \beta \leqslant q$. Then $i_\alpha \neq j_\beta$ since no diagonal element is zero. Hence there exists a permutation matrix Q such that

$$Q'AQ = \begin{bmatrix} B & 0 \\ D & C \end{bmatrix}$$

where B is a $p \times p$ matrix and C a $q \times q$ matrix. If $p = 1$, then C must satisfy relation (15) with A replaced by C and for r replaced by $r - 1$ if $r > 1$ or for r unchanged if $r = 1$. The lemma then follows by applying the induction hypothesis to C. We argue similarly if $q = 1$. Otherwise $p > 1$ and $q > 1$ and B and C will both satisfy (15) for appropriate r. In this case the lemma follows by applying the induction hypothesis to both B and C.

THEOREM 4. *Let $A = (a_{ij})$ be an $n \times n$ non-negative row stochastic matrix. Then*

$$(16) \qquad e_r(A) \leqslant \binom{n-1}{r} + \binom{n-1}{r-1} \text{per}\,(A) \qquad \textit{for } r = 1, \ldots, n-1$$

where $e_r(A)$ is the sum of the principal permanental minors of order r of A. If A is doubly stochastic, equality occurs in (16) *if and only if A is the $n \times n$ identity matrix.*

Proof. The inequality (16) follows from inequality (8) and the obvious inequality

$$(17) \qquad \text{per}\,(A) \geqslant \text{per}\,(A[i_1, \ldots, i_r])\,\text{per}\,(A(i_1, \ldots, i_r))$$

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for each sequence $i_1, \ldots, i_r$ in $Q_{r,n}$. If equality occurs in (16), it must also occur in (17). If A is doubly stochastic, it follows by the preceding lemma that there is a permutation matrix P such that $P'AP = I_n$ or $A = I_n$ where I_n is the $n \times n$ identity matrix. This establishes the theorem.

Our last theorem is also concerned with relationships between the permanent and permanental minors of a matrix.

THEOREM 5. *Let $A = (a_{ij})$ be an $n \times n$ non-negative row substochastic matrix. For $r = 1, 2, \ldots, n$, let m_r be the maximum of the permanental minors of A of order r. Then*

$$\operatorname{per}(A) \leqslant m_r, \qquad r = 1, \ldots, n.$$

In particular the permanent of a non-negative row substochastic matrix does not exceed its maximum element.

Proof. By the Laplace expansion for permanents for any sequence $i_1, \ldots, i_r$ in $Q_{r,n}$,

$$\operatorname{per}(A) = \sum \operatorname{per}(A[i_1, \ldots, i_r \mid j_1, \ldots, j_r]) \operatorname{per}(A(i_1, \ldots, i_r|j_1, \ldots, j_r))$$

where the summation extends over all sequences $j_1, \ldots, j_r$ in $Q_{r,n}$. Hence

$$\operatorname{per}(A) \leqslant m_r(\sum \operatorname{per}(A(i_1, \ldots, i_r \mid j_1, \ldots, j_r))),$$

the summation again extending over all sequences $j_1, \ldots, j_r$ in $Q_{r,n}$. By Theorem 1, this sum does not exceed one and the inequality follows.

COROLLARY 1. *For A an $n \times n$ non-negative row substochastic matrix,*

$$\operatorname{per}(P^s(A)) \leqslant m_s, \qquad s = 1, 2, \ldots, n.$$

Proof. This follows by applying Theorem 5 to $P^s(A)$ for the case $r = 1$.

COROLLARY 2. *Let A be an $n \times n$ 0, 1 matrix with k 1's in each row. Then*

$$\operatorname{per}(A) \leqslant k^n(k!/k^k).$$

Proof. The matrix $k^{-1}A$ is a non-negative row stochastic matrix and we may apply Theorem 5 to it. For this matrix $m_k \leqslant k!/k^k$. Since

$$\operatorname{per}(A) = k^n(\operatorname{per} k^{-1}A),$$

the inequality follows.

A generalization of Theorem 4 to $n \times n$ non-negative matrices A with row sums $s_1, \ldots, s_n$ can be obtained using the same methods. The inequality analogous to (16) is

$$s_1 \ldots s_n \sum \frac{1}{s_{i_1} \ldots s_{i_r}} \operatorname{per}(A[i_1, \ldots, i_r]) \leqslant \binom{n-1}{r} s_1 \ldots s_n + \binom{n-1}{r-1} \operatorname{per}(A)$$

where the summation extends over all sequences $i_1, \ldots, i_r$ in $Q_{r,n}$.

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