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Some symmetric varieties of groups

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Let $\underline{\underline{V}}(n, \sigma, d)$ denote the variety of all groups defined by the left-normed commutator identity $[x_1, \ldots, x_n] = [x_{1\sigma}, \ldots, x_{n\sigma}]^d$, where σ is a non-identity permutation of $\{1, \ldots, n\}$, and d is an integer, possibly negative. It is shown that $\underline{\underline{V}}(n, \sigma, d)$ is nilpotent-by-nilpotent if $\sigma \neq (1, 2)$, abelian by nilpotent if $n > 2$, $n\sigma \neq n$, and nilpotent of class at most $n + 1$ if $\{1, 2\} \neq \{1\sigma, 2\sigma\}$. This improves on a result of E.B. Kikodze that $\underline{\underline{V}}(n, \sigma, 1)$ is locally soluble and if $\{1, 2\} \neq \{1\sigma, 2\sigma\}$ is locally nilpotent.

1. Introduction

In the present note we consider groups satisfying a left-normed commutator identity

$$[x_1, \ldots, x_n] = [x_{1\sigma}, \ldots, x_{n\sigma}]^d , \tag{1}$$

where σ is a non-identity permutation of $\{1, \ldots, n\}$ and d is an integer. E.B. Kikodze [2] has shown that any finitely generated group satisfying (1) with $d = 1$ is soluble and is nilpotent if $\{1, 2\} \neq \{1\sigma, 2\sigma\}$. The following theorems extend these results to arbitrary d without the assumption of finite generation.

THEOREM 1. *Let G be a group satisfying an identity* (1) *for some σ such that $\{1, 2\} \neq \{1\sigma, 2\sigma\}$. Then G is nilpotent of class at most $n + 1$.*

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THEOREM 2. *Let* G *satisfy* (1) *for some* $n \geq 3$ *and* $\sigma \neq (12)$. *Then for some integers* m, k , G *satisfies* $\left[\Gamma_m(G), \Gamma_2(G), x_1, \ldots, x_k\right] = 1$, *where* $\Gamma_m(G)$ *denotes the* m*-th term of the lower central series of* G . *In particular,* G *is nilpotent-by-nilpotent.*

Note. Theorem 2 extends to $\sigma = (12)$ for $d = 1$ by a result of I.D. Macdonald [5].

THEOREM 3. *Let* G *satisfy* (1) *for some* $n \geq 3$ *and* $n\sigma \neq n$. *Then for some* m , G *satisfies* $\left[\Gamma_m(G), \Gamma_2(G)\right] = 1$ *and, hence, is abelian-by-nilpotent.*

The proofs of these results are given in Section 4. Before closing this introduction we record some of the more pertinent known results concerning groups G satisfying (1) with $d = 1$.

(A) $[x_1, x_2] = [x_2, x_1]$ implies $[\Gamma_3(G), \Gamma_2(G)] = 1$ (Macdonald [5]).

(B) $[x_1, x_2, x_1] = 1$ implies nilpotence class ≤ 3 (Levi [3]).

(C) For $m \geq 3$, $\sigma = (n-1, n)$, (1) implies $\left[\Gamma_{n-2}(G), \Gamma_2(G)\right] = 1$ (Levin [4]).

(D) For $n \geq 4$, $\sigma = (123 \ldots n)$, (1) implies nilpotence class $\leq n-1$ (Macdonald [5] and Meir-Wunderli (*cf.* [5])).

2. Notation

For unexplained notation we refer the reader to Hanna Neumann [6]. In particular, for elements $a, b, c, \ldots$ in a group A we define $[a, b] = a^{-1}b^{-1}ab$, $[a, b, c] = \left[[a, b], c\right]$, $a^b = b^{-1}ab$. If H, K are subgroups of A , then $[H, K]$ denotes the subgroup of A generated by all $[h, k]$, $h \in H$, $k \in K$. Further, we define $[H, 0K] = H$ and for $t \geq 1$, $[H, tK] = \left[[H, (t-1)K], K\right]$. Finally, $\Gamma_m(A)$ $\left(\zeta_m(A)\right)$ denotes the m-th term of the lower (upper) central series of A .

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3. Two lemmas

LEMMA 3.1. *Let* u, v *be fixed elements of a group* G *such that for some* $r \geq 1$ *and all* $x_1, \ldots, x_r \in G$

(2) $$[u, x_1, \ldots, x_r, v] = 1 .$$

Then $[U, sG, V] = \{1\}$ *for any* $s \geq r$ *, where* U *and* V *are the normal closures of* u *and* v *, respectively.*

Proof. Replacing x_r by $x_r g^{-1}$ in (2) and expanding (using the identity $[a, bc] = [a, c][a, b]^c$) gives $\left[[u, x_1, \ldots, x_r]^{g^{-1}}, v\right] = 1$, which after conjugation with g , gives $\left[u, x_1, \ldots, x_r, v^g\right] = 1$. If we now replace g by gh^{-1} , x_i by $x_i^{h^{-1}}$ for $i = 1, \ldots, r$, and conjugate the resulting expression by h , we obtain $\left[u^h, x_1, \ldots, x_r, v^g\right] = 1$. Hence, $[U, rG, V] = \{1\}$. In particular $\left[\left[u^h, x_{r+1}, \ldots, x_s\right], x_1, \ldots, x_r, v^g\right] = 1$ for any $s > r$, so that $[U, sG, V] = \{1\}$, as required.

LEMMA 3.2. *Let* u, v *be fixed elements of a group* G *which satisfies an identity* (1) *with* $1\sigma = 1$, $2\sigma = 2$ *and* $n\sigma = n-i \neq n$. *If* G *satisfies*

(3) $$[u, x_2, \ldots, x_r, v, x_{r+2}, \ldots, x_k] = 1$$

for some $k \geq r+1$ *,* $r \geq 2$ *, and all* $x_i \in G$ *, then for some integer* t *,*

$$[U, tG, V] = \{1\} .$$

Proof. By Lemma 3.1, G satisfies

(4) $$[u_1, x_2, \ldots, x_{n-i-1}, v, x_{n-i+1}, \ldots, x_m] = 1 ,$$

with $u_1 = u$ if $r \leq n-i-1$ and sufficiently large m . Otherwise, G satisfies (4) with $u_1 = [u, y_2, \ldots, y_{r-(n-i-1)}]$. Choose m of the form $n + p(n-i)$. By (4), G satisfies, in particular,

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$\left[[u_1, x_2, \ldots, x_{n-i-1}, v, \ldots, x_n]^d, \ldots, x_m\right] = 1$ so that, by (1), G satisfies $[u_1, x_2, \ldots, x_{n-1}, v, \ldots, x_m] = 1$. Iterating this process p times yields $[u_1, \ldots, x_m, v] = 1$ which, by Lemma 3.1, proves the lemma.

Finally, we remark that (1) implies

$[x_1, \ldots, x_n] = [x_{1\rho}, \ldots, x_{n\rho}]^{d^{N-1}}$, where $\rho = \sigma^{-1}$ and N is the order of σ .

4. Proofs of the theorems

Proof of Theorem 1: Case 1. $\{1\sigma, 2\sigma\} = \{i, j\}$ and $\{i, j\} \cap \{1, 2\} \neq \emptyset$. In (1) replacing x_i by x_1 or x_2 shows that G satisfies the law $[x_1, x_2, \ldots, x_i, x_1, x_{i+1}, \ldots, x_n] = 1$ or $[x_2, x_1, \ldots, x_{i-1}, x_1, x_{i+1}, \ldots, x_n] = 1$. Thus, if $i = 3$, G is nilpotent of class $\leq n$ by the Levi result [3], while if $i > 3$, G is nilpotent of class $\leq n-1$ by a result of Heineken and Macdonald (*cf.* 34.33 of [6]).

Case 2. $\{1\sigma, 2\sigma\} = \{i, j\}$, $\{i, j\} \cap \{1, 2\} = \emptyset$, $i < j$. Replacing x_j by x_i in (1) shows that G satisfies the law

(5) $$[x_1, x_2, \ldots, x_i, \ldots, x_{j-1}, x_i, x_{j+1}, \ldots, x_n] = 1 .$$

If $j = i+1$, then, replacing $[x_1, x_2]$ by $[x_1, x_2, y]$, shows that G satisfies

$$[x_1, x_2, \ldots, x_i, x_{i+1}, x_{i+3}, \ldots, x_n, x_{n+1}] = 1$$

so that

(6) $$\left[[x_1, x_2, \ldots, x_i, x_{i+1}, x_{i+1}, x_{i+3}, \ldots, x_n]^{d^{N-1}}, x_{n+1}\right] = 1 ,$$

where N is the order of σ . By the remark at the end of Section 3 we may apply σ to the indexes of the first n x's in (6) to obtain an identity of the form

$$[x_1, x_{i+1}, \ldots, x_{i+1}, \ldots, x_n, x_{n+1}] = 1 ,$$

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which implies, as in Case 1, that G is nilpotent of class $\leq n+1$. Finally, if $j > i+1$ we may apply Lemma 3.1 to (5) in order to insert one extra component between the two x_i's to obtain an identity

$$[x_1, x_2, \ldots, x_i, \ldots, x_j, x_i, x_{j+1}, \ldots, x_n, x_{n+1}] = 1 ,$$

so that, as in (6),

$$\left[[x_1, x_2, \ldots, x_i, \ldots, x_j, x_i, x_{j+1}, \ldots, x_n]^{d^{N-1}}, x_{n+1}\right] = 1 ,$$

which, as above, implies that G is nilpotent of class $\leq n+1$. This completes the proof of Theorem 1.

Proof of Theorem 2: Case 1. $d = 1$, $1\sigma = 1$, $2\sigma = 2$. Here we may assume that G satisfies the law

(7) $$[u, x_1, \ldots, x_n] = [u, x_{1\sigma}, \ldots, x_{n\sigma}] ,$$

where $u = [y_1, \ldots, y_r]$ $(r \geq 2)$ and $1\sigma = i \neq 1$. Replacing x_i by u in (7) yields the law

(8) $$[u, x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_n] = 1 ,$$

which, by Lemma 3.1, implies that $[U, (i-1)G, U, (n-i)G] = \{1\}$, where U is the normal closure of u in G.

In particular, $G_1 = G/\zeta_{n-i}(G)$ satisfies the identity

(9) $$\left[u^g, x_1, \ldots, x_p, u^h\right] = 1$$

where $p \geq i-1$. In (7) we may replace u by $v = [y_1, \ldots, y_{r+i-1}]$ to obtain

(10) $$[v, x_1, \ldots, x_n] = [v, x_{1\sigma}, \ldots, x_{n\sigma}] .$$

If we now replace x_i by $z_i x_i$ in (10) and expand using (9), we find that G satisfies

(11) $$\begin{aligned}[][v, x_1, \ldots, x_{i-1}, z_i, x_i, x_{i+1}, \ldots, x_n] &= [v, z_i, x_i, x_{2\sigma}, \ldots, x_{n\sigma}] \\ &= [v, z_i, x_1, \ldots, x_n] ,\end{aligned}$$

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by applying (10) to the last n entries of the right hand commutator. Thus, $G_2 = G_1/\zeta_{n-i+1}(G_1)$ satisfies the law

$$(12) \qquad [v, x_1, \ldots, x_{i-1}, x_i] = [v, x_i, x_1, \ldots, x_{i-1}] ,$$

by Lemma 1 of [2] (which states that G satisfies $[w_1, x_1, \ldots, x_k] = [w_2, x_1, \ldots, x_k]$ if and only if $G/\zeta_k(G)$ satisfies $w_1 = w_2$) . In particular, commuting both sides of (12) by x_{i+1} gives

$$(13) \qquad [v, x_1, \ldots, x_i, x_{i+1}] = [v, x_i, x_1, \ldots, x_{i-1}, x_{i+1}] .$$

Two applications of (12) to the last i entries of the left side of (13) and one application of (12) to the last i entries of the right side of (13) now give the law

$$(14) \quad [v, x_1, x_i, x_{i+1}, x_2, \ldots, x_{i-1}] = [v, x_i, x_{i+1}, x_1, x_2, \ldots, x_{i-1}] ,$$

so that again, by Lemma 1 of [2], $G_3 = G_2/\zeta_{i-2}(G_2)$ satisfies the law

$$(15) \qquad [v, x_1, x_2, x_3] = [v, x_2, x_3, x_1] .$$

Replacing x_3 by x_3x_4 in (15) and expanding using (9) gives

$$\begin{aligned} [v, x_1, x_2, x_3, x_4] &= [v, x_2, x_3, x_4, x_1] \\ &= [[v, x_2], x_4, x_1, x_3] , \text{ by (15)} , \\ &= [v, x_2, x_1, x_3, x_4] , \text{ by (15)} . \end{aligned}$$

Thus, $G_4 = G_3/\zeta_2(G_3)$ satisfies the law

$$[v, x_1, x_2] = [v, x_2, x_1] ,$$

which by a result of Levin [4] implies that $[\Gamma_{r+i-1}(G_4), \Gamma_2(G_4)] = \{1\}$. Since $G_4 = G/\zeta_k(G)$ where $k = 2 + (i-2) + (n-i+1) + (n-i) = 2n - i + 1$, it follows that $[\Gamma_{r+i-1}(G), \Gamma_2(G)] \le \zeta_k(G)$, or, equivalently, that $[\Gamma_{r+i-1}(G), \Gamma_2(G), kG] = \{1\}$.

Case 2. $1\sigma = 1$, $2\sigma = 2$, d arbitrary. As in Case 1 we may assume that G satisfies the law

$$(16) \qquad [u, x_1, \ldots, x_n] = [u, x_{1\sigma}, \ldots, x_{n\sigma}]^d ,$$

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where $u = [y_1, \ldots, y_r]$, $r \geq 2$, $1\sigma = i \neq 1$, and as in Case 1 this implies that $G_1 = G/\zeta_{n-i}(G)$ satisfies (9) and

$$(17) \qquad [v, x_1, \ldots, x_n] = [v, x_{1\sigma}, \ldots, x_{n\sigma}]^d ,$$

where $v = [y_1, \ldots, y_{r+i-1}]$. Again replacing x_i by $z_i x_i$ in (17) and expanding, using (9), yields

$$\begin{aligned}[][v, x_1, \ldots, x_{i-1}, z_i, x_i, x_{i+1}, \ldots, x_n] & \\ &= [v, z_i, x_i, x_{2\sigma}, \ldots, x_{(n-1)\sigma}, x_{n\sigma}]^d \\ &= [v, z_i, x_1, \ldots, x_n] ,\end{aligned}$$

as above. From this point on the proof is identical with that in Case 1.

Case 3. $1\sigma = 2$, $2\sigma = 1$, $\sigma \neq (12)$. As above, we may assume that G satisfies the law

$$(18) \qquad [u, x_1, \ldots, x_n] = [v, x_{1\sigma}, \ldots, x_{n\sigma}]^d ,$$

where $u = [y_1, y_2, \ldots, y_r]$, $v = [y_2, y_1, y_3, \ldots, y_r]$, $r \geq 2$, and $1 \neq 1\sigma = i$. In (18) replacing x_i by v gives $\left[u^g, x_1, \ldots, x_{i-1}, v, x_{i+1}, \ldots, x_n\right] = 1$ so that $G_1 = G/\zeta_{n-i}(G)$ satisfies the law

$$\left[u^g, x_1, \ldots, x_n, v^h\right] = 1 ,$$

so that, in particular,

$$(19) \qquad \left[\left[u^g, x_1, \ldots, x_n\right]^d, v^h\right] = 1 .$$

Thus, by (18), G_1 satisfies

$$(20) \qquad \left[v^g, x_1, \ldots, x_n, v^h\right] = 1 .$$

Further, in (18) replacing y_2 by $y_2^* = [y_{2,1}, \ldots, y_{2,n-r}]$ shows that G_1 satisfies the law

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(21) $$[u^*, x_1, \ldots, x_n] = [v^*, x_{1\sigma}, \ldots, x_{n\sigma}]^d$$

where $u^* = [y_1, y_2^*, y_3, \ldots, y_r]$, $v^* = [y_2^*, y_1, y_3, \ldots, y_r]$. Let k be the largest integer in $\{1, \ldots, n\}$ such that $k\sigma \neq k$. Then in (21) replace $x_{k\sigma}$ by $x_{k\sigma}x_{n+1}$ and expand both sides (using (20)) to obtain

$$[u^*, x_1, \ldots, x_k, x_{n+1}, \ldots, x_n] = [v^*, \ldots, x_{k\sigma}, x_{n+1}, \ldots, x_{n\sigma}]^d$$
$$= \left[[v^*, \ldots, x_{k\sigma}, x_{n+1}, \ldots, x_{(n-1)\sigma}]^d, x_{n\sigma}\right] ,$$

which, after an application of (21) to the latter commutator, yields an identity of the type considered in the previous cases. This completes the proof of Theorem 2.

Proof of Theorem 3. Theorem 3 follows immediately from Theorem 2 and Lemma 3.2.

5. Generalization

Let $w(x_1, \ldots, x_n)$ be a word which reduces to the identity if $x_{i_1} = \ldots = x_{i_k} = x_1$ for some sequence $3 \leq i_1 < \ldots < i_k \leq n$ satisfying $i_j < i_{j+1} - 1$. If G is a group satisfying the law

(22) $$[x_1, \ldots, x_n] = w(x_1, \ldots, x_n) ,$$

then G satisfies the law $[x_1, \ldots, x_n] = 1$ with $x_{i_1} = \ldots = x_{i_k} = x_1$, and by Lemma 2 of Kikodze [2] it follows that, in particular, G satisfies the law $\left[[x_1, \ldots, x_d, y], \ldots, [x_1, \ldots, x_d, y_m]\right] = 1$ for some integers $m, d \leq n$. By Lemma 7 of N.D. Gupta [1] it then follows that G is locally nilpotent-by-nilpotent. Thus, every group of the variety defined by (22) is locally nilpotent-by-nilpotent.

References

[1] N.D. Gupta, "Certain locally metanilpotent varieties of groups", *Arch. Math.* 20 (1969), 481-484.

https://doi.org/10.1017/S0004972700045706 Published online by Cambridge University Press

[2] E.B. Kikodze, "Some identities in groups" (English), *Math. USSR - Izvestija* 1 (1967), 253-258.

[3] F.W. Levi, "Groups in which the commutator relation satisfies certain algebraic conditions", *J. Indian Math. Soc. (N.S.)* 6 (1942), 87-97.

[4] Frank Levin, "On some varieties of soluble groups. II", *Math. Z.* 103 (1968), 162-172.

[5] I.D. Macdonald, "On certain varieties of groups", *Math. Z.* 76 (1961), 270-282.

[6] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, Berlin, Heidelberg, New York, 1967).

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