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CORRIGENDUM

Instability of axisymmetric flow in thermocapillary liquid bridges: Kinetic and thermal energy budgets for two-phase flow with temperature-dependent material properties – CORRIGENDUM

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1. Thermal energy

In [3] we have claimed to take into account *the temperature dependence of all thermophysical parameters*. However, in the temperature equation (3.1c) of [3] we have neglected the term describing the advection of c_p. Even though the advection of c_p has little effect (see below), we present the temperature equation which includes the advection of c_p, but still neglects the pressure variations. The correspondingly revised temperature equation reads

$$\partial_t \left(\rho c_p \hat{T}\right) + \nabla \cdot \left(\rho c_p \hat{\boldsymbol{U}} \hat{T}\right) = \nabla \cdot (\lambda \nabla \hat{T}) + \rho \hat{T} \frac{\mathrm{D} c_p}{\mathrm{D} t} \tag{3.1c}$$

or, equivalently [1],

$$\rho c_p \left(\partial_t + \hat{\boldsymbol{U}} \cdot \nabla\right) \hat{T} = \nabla \cdot (\lambda \nabla \hat{T}). \tag{3.1c}$$

As a result the equations (4.1)–(4.6) must be replaced by the following expressions, where we use the same equation numbering as in the original publication.

$$\rho c_p \left[\partial_t + (\boldsymbol{u}_0 + \boldsymbol{u}) \cdot \nabla\right] (T_0 + T) = \nabla \cdot (\lambda \nabla T_0) + \nabla \cdot (\lambda \nabla T). \tag{4.1}$$

$$\rho c_p \left(\partial_t T_0 + \partial_t T + \boldsymbol{u}_0 \cdot \nabla T_0 + \boldsymbol{u}_0 \cdot \nabla T + \boldsymbol{u} \cdot \nabla T_0\right) = \nabla \cdot (\lambda \nabla T_0) + \nabla \cdot (\lambda \nabla T). \tag{4.2}$$

$$\begin{aligned} \rho_0 c_{p0} \partial_t T_0 &+ \left(\rho_0 c_p' + \rho_0' c_{p0}\right) T \partial_t T_0 + \rho_0 c_{p0} \partial_t T + \left[\rho_0 c_{p0} + (\rho_0 c_{p0}' + \rho_0' c_{p0}) T\right] \boldsymbol{u}_0 \cdot \nabla T_0 \\ &+ \rho_0 c_{p0} \left(\boldsymbol{u}_0 \cdot \nabla T + \boldsymbol{u} \cdot \nabla T_0\right) = \nabla \cdot (\lambda_0 \nabla T_0) + \nabla \cdot (\lambda_0' T \nabla T_0) + \nabla \cdot (\lambda_0 \nabla T). \end{aligned} \tag{4.3}$$

$$\rho_0 c_{p0} \boldsymbol{u}_0 \cdot \nabla T_0 = \nabla \cdot (\lambda_0 \nabla T_0). \tag{4.4}$$

$$\begin{aligned} \rho_0 c_{p0} \partial_t T &+ (\rho_0 c_{p0}' + \rho_0' c_{p0}) T \boldsymbol{u}_0 \cdot \nabla T_0 + \rho_0 c_{p0} \left(\boldsymbol{u}_0 \cdot \nabla T + \boldsymbol{u} \cdot \nabla T_0\right) \\ &= \nabla \cdot (\lambda_0' T \nabla T_0) + \nabla \cdot (\lambda_0 \nabla T). \end{aligned} \tag{4.5}$$

$$\begin{aligned} \underbrace{\rho_0 c_{p0} T \partial_t T}_{\text{T1}} = - \underbrace{\ 0\ }_{\text{T2}} - \underbrace{\rho_0' c_{p0} T^2 \boldsymbol{u}_0 \cdot \nabla T_0}_{\text{T3}} - \underbrace{\rho_0 c_{p0}' T^2 \boldsymbol{u}_0 \cdot \nabla T_0}_{\text{T4}} \\ - \underbrace{\rho_0 c_{p0} T \boldsymbol{u}_0 \cdot \nabla T}_{\text{T5}} - \underbrace{\rho_0 c_{p0} T \boldsymbol{u} \cdot \nabla T_0}_{\text{T6}} + \underbrace{T \nabla \cdot (\lambda_0' T \nabla T_0)}_{\text{T7}} + \underbrace{T \nabla \cdot (\lambda_0 \nabla T)}_{\text{T8}}. \end{aligned} \tag{4.6}$$

As a result of the inclusions of the c_p-advection term, the term T2 in (4.6) of [3] vanishes and the terms T3 to T6 are modified. The rate of change of thermal energy (4.8) formally remains the same, but with the following meaning.

$$D_{\text{th}} = -\int_{V_i} (\text{part of T8}) \, \mathrm{d}V = \int_{V_i} \lambda_0 (\nabla T)^2 \, \mathrm{d}V, \tag{4.9a}$$

$$J = -\int_{V_i} \text{T6} \, \mathrm{d}V = -\int_{V_i} \rho_0 c_{p0} T (u \partial_r T_0 + w \partial_z T_0) \, \mathrm{d}V, \tag{4.9b}$$

$$H_{\text{fs}} = -\int_{V_i} (\text{part of T8}) \, \mathrm{d}V = \alpha_i \int_{A_{\text{fs}}} \lambda_0 T \nabla T \cdot \boldsymbol{n} \, \mathrm{d}S, \tag{4.9c}$$

$$K_{\text{G,th}} = -\int_{V_i} (\text{part of T5}) \, \mathrm{d}V = -\frac{1-\alpha_i}{4} \int_{A_{\text{out}}} \rho_0 c_{p0} T^2 w_0 \, \mathrm{d}S, \tag{4.9d}$$

$$\Pi_\rho = -\int_{V_i} \text{T3} \, \mathrm{d}V = -\int_{V_i} \rho_0' c_{p0} T^2 \boldsymbol{u}_0 \cdot \nabla T_0 \, \mathrm{d}V, \tag{4.9e}$$

$$\Pi_{c_p} = -\int_{V_i} (\text{T4} + \text{part of T5}) \, \mathrm{d}V = -\frac{1}{2} \int_{V_i} \rho_0 c_{p0}' T^2 \boldsymbol{u}_0 \cdot \nabla T_0 \, \mathrm{d}V, \tag{4.9f}$$

$$\Pi_\lambda = -\int_{V_i} \text{T7} \, \mathrm{d}V = \alpha_i \int_{A_{\text{fs}}} \lambda_0' T^2 \nabla T_0 \cdot \boldsymbol{n} \, \mathrm{d}S - \frac{1}{2} \int_{V_i} \lambda_0' \nabla T_0 \cdot \nabla T^2 \, \mathrm{d}V, \tag{4.9g}$$

$$\partial_t E_T' = -\int_{V_i} \text{T2} \, \mathrm{d}V = 0. \tag{4.9h}$$

For the sake of completeness we have specified all subequations of (4.9) of [3]. Note that equations (4.9a), (4.9c), (4.9d), (4.9g) and (4.9h) remain unchanged, while the subequations (4.9b), (4.9e), (4.9f) and (4.9h) are updated. The notation'part of T#' in (4.9), # ∈ [5, 8], should indicate that only part of the respective term T# from (4.6) enters the integral in (4.9).

We note that the integral over T5 yields

$$\begin{aligned}
\int_{V_i} \text{T5} \, \mathrm{d}V = \int_{V_i} \rho_0 c_{p0} T \boldsymbol{u}_0 \cdot \nabla T \, \mathrm{d}V &= \overbrace{\int_{A_{\text{out}}} \rho_0 c_{p0} T^2 \boldsymbol{u}_0 \cdot \boldsymbol{n} \, \mathrm{d}S}^{:=-2K_{\text{G,th}}} - \int_{V_i} T \nabla \cdot \left(\rho_0 c_{p0} \boldsymbol{u}_0 T \right) \, \mathrm{d}V \\
&= -2K_{\text{G,th}} - \int_{V_i} T \rho_0 \boldsymbol{u}_0 \cdot \left(c_{p0} \nabla T + T \nabla c_{p0} \right) \, \mathrm{d}V \\
&= -2K_{\text{G,th}} - \int_{V_i} \rho_0 c_{p0} T \boldsymbol{u}_0 \cdot \nabla T \, \mathrm{d}V - \int_{V_i} \rho_0 c_{p0}' T^2 \boldsymbol{u}_0 \cdot \nabla T_0 \, \mathrm{d}V.
\end{aligned}$$

From which one concludes

$$\int_{V_i} \rho_0 c_{p0} T \boldsymbol{u}_0 \cdot \nabla T \, \mathrm{d}V = -K_{\text{G,th}} - \frac{1}{2} \int_{V_i} \rho_0 c_{p0}' T^2 \boldsymbol{u}_0 \cdot \nabla T_0 \, \mathrm{d}V.$$

For a derivation of the remaining expressions in the revised equation (4.9) the Appendix A of [3] is not required anymore and should be dropped.

2. Effect of c_p advection on the linear stability boundary

To demonstrate the effect of c_p advection on the linear stability boundary, we supplement the linear stability boundaries from table 3 of [1] by critical Reynolds numbers (and temperature differences) obtained including the term $\rho \hat{T} \, \mathrm{D}c_p / \mathrm{D}t$, i.e. the effect of c_p advection (superscript 'adv'). Table 1 reveals that the advection of c_p reduces the critical Reynolds number in the full-temperature-dependence model (FTD) by 1.6%. Considering the deviations from the critical Reynolds number in the Oberbeck–Boussinesq (OB) model we find that the temperature dependence of c_p without c_p advection reduces the critical Reynolds number by 2.2%. From table 3 of [1] we note that the temperature dependence of ρ and λ (including advection of these quantities) individually reduce Re_c^{OB} by about 2%, while

Table 1. *Critical temperature difference ΔT_c and critical Reynolds number $Re_c = \gamma \bar{\rho}_L \Delta T_c d / \bar{\mu}_L^2$ for a slender liquid bridge with $\Gamma = 0.66$ and $\mathcal{V} = 0.9$ made of 2-cSt silicone oil as in table 3 of [1]. The superscript 'adv' indicates results when the advection of c_p is included in the governing equations. For all models, the critical wave number is $m_c = 3$. The relative deviations $\epsilon_c^{FTD} := (Re_c - Re_c^{FTD})/Re_c^{FTD}$ and $\epsilon_c^{OB} := (Re_c - Re_c^{OB})/Re_c^{OB}$ are given in percent. For the definition of $\Delta^{(i)} Re_c$, please see the text*

Approximation	ΔT_c [K]	Re_c	$\epsilon_c^{\mathrm{FTD}}$ [%]	ϵ_c^{OB} [%]	$\Delta^{(i)} Re_c$
FTD	44.49	1471			
FTD$^{\mathrm{adv}}$	43.78	1448	-1.6	–	
OB	55.50	1835			
OB + $\rho(\hat{T})$	54.63	1806	–	-1.6	-29
OB + $\lambda(\hat{T})$	54.33	1797	–	-2.1	-38
OB + $c_p(\hat{T})$	54.28	1795	–	-2.2	
OB + $c_p^{\mathrm{adv}}(\hat{T})$	53.11	1756	–	-4.3	-79
OB + $\mu(\hat{T})$	45.60	1509	–	-17.8	-326

individually taking into account $c_p(\hat{T})$ (including c_p advection) reduces Re_c^{OB} by to 4.3 percent, where the advection of c_p contributes about as much as the mere temperature dependence (without c_p advection) does. Therefore, among the three quantities ρ, λ and c_p, c_p has the largest share in reducing the critical Reynolds number relative to Re_c^{OB}. Of course, the temperature dependence of the viscosity μ alone dominates the reduction of the critical Reynolds number from Re_c^{OB}. The change of the critical Reynolds number $\Delta^{(i)} Re_c := Re_c^{(i)} - Re_c^{\mathrm{OB}}$ due to the temperature dependence of each individual thermophysical parameter, where i symbolizes $i \in [\mathrm{OB} + \rho,\ \mathrm{OB} + \mu,\ \mathrm{OB} + c_p^{\mathrm{adv}},\ \mathrm{OB} + \mu]$ (including advection in all cases), is almost additive: For the case considered, the sum of the Reynolds number reductions amounts to $\sum_i \Delta^{(i)} Re_c = -473$ with the deviation $Re_c^{\mathrm{FTD,\,adv}} - Re_c^{\mathrm{OB}} = 1448 - 1835 = -387$. An updated version of the code `MaranStable` which includes the effect of c_p advection is available under https://github.com/fromano88/MaranStable. The driver file for the old version is '`main_v3d1.m`', while the one for the new version is '`main_v3d2.m`'.

3. Kinetic energy

Equation (B25) in [1] contains a sign error. The corrected equation (B25) reads

$$
\begin{aligned}
\int_{V_i} \mathtt{K7a}\, \mathrm{d}V &= \int_{V_i} \mu_0 u_l \partial_m \partial_m u_l \, \mathrm{d}V \\
&= \underbrace{\alpha_i \int_{A_{\mathrm{fs}}} \mu_0 u_l n_m \partial_m u_l \, \mathrm{d}S}_{:=M} - \int_{V_i} \mu_0 (\partial_m u_l)^2 \, \mathrm{d}V - \int_{V_i} u_l (\partial_m \mu_0)(\partial_m u_l) \, \mathrm{d}V \\
&= - \int_{V_i} \mu_0 (\partial_m u_l)^2 \, \mathrm{d}V + M - \int_{V_i} u_l (\partial_m u_l)(\partial_m \mu_0) \, \mathrm{d}V \\
&= - \int_{V_i} \mu_0 (\partial_m u_l)^2 \, \mathrm{d}V + M_r + M_\varphi + M_z \\
&\quad - \alpha_i \int_{A_{\mathrm{fs}}} \mu_0 (h_0 w^2 h_{0zz} - v^2) \, \mathrm{d}\varphi \, \mathrm{d}z - \int_{V_i} \mu_0' \boldsymbol{u} \cdot (\nabla \boldsymbol{u})^{\mathrm{T}} \cdot \nabla T_0 \, \mathrm{d}V.
\end{aligned}
$$

As a consequence (B26) must read

$$
\int_{V_i} \mathtt{K7a}\, \mathrm{d}V = -D_{\mathrm{kin}} + M_r + M_\varphi + M_z - \frac{1}{2} \int_{V_i} \mu_0' \cdot (\nabla \boldsymbol{u}^2) \cdot \nabla T_0 \, \mathrm{d}V,
$$

Table 2. *Critical Reynolds numbers Re_c and critical temperature differences ΔT_c for a liquid bridge volume ratio $\mathcal{V} = 0.88$ and different approximations of the transport equations. Shown are the results of [4] for the FTD, LTD and OB models (all exclusive of c_p advection) in comparison with the present results for the FDT^{adv} and LTD^{adv} models (both including the effect of c_p advection). The relative deviation $\epsilon_c = (Re_c - Re_c^{FTD})/Re_c^{FTD}$ is given in percent. All other parameters are identical to those for table 5 of [4]: Shin-Etsu silicone oil with $\nu(\hat{T} = 25^\circ C) = 2$ cSt, $\Gamma = 0.66$, $\mathcal{V} = 1$, $Bd = 0.363$*

Approximation	Re_c	ΔT_c [K]	ϵ_c [%]
FTD	1679	50.79	0
FTD$^{\text{adv}}$	1659	50.17	−1.2
LTD	1572	47.53	−6.4
LTD$^{\text{adv}}$	1556	47.05	−7.3
OB	2263	68.45	34.8

and equation (5.8i) needs to be updated to

$$\Lambda_\mu = \int_{V_i} \mu_0' \boldsymbol{u} \cdot \nabla \boldsymbol{u} \cdot \nabla T_0 \, \mathrm{d}V + \int_{V_i} (\mu_0' + \mu_0'' T_0) \boldsymbol{u} \cdot [\mathcal{S}_0 + (\nabla \boldsymbol{u}_0)^{\mathrm{T}}] \cdot \nabla T \, \mathrm{d}V$$
$$- \int_{V_i} \mu_0' T (\nabla \boldsymbol{u}_0) : (\nabla \boldsymbol{u}) \, \mathrm{d}V + \alpha_i \int_{A_{\mathrm{fs}}} \mu_0' w T \left(N^2 \partial_r w_0 - N^2 h_{0z} \partial_z w_0 - h_{0z}^2 h_{0zz} w_0 \right) \mathrm{d}\varphi \, \mathrm{d}z.$$

We apologize for the errors made.

4. Related publications

Publications [2] and [4] also considered the stability of the thermocapillary flow in liquid bridges when the specific heat $c_p(T)$ is temperature dependent. Similar as in table 1 above, we provide in table 2 the equivalent of table 5 of [4], here supplemented by the result when the advection of c_p is taken into account (superscript 'adv'). Note that the deviations specified in the last column are taken relative to the full-temperature-dependence model (FTD) without c_p advection, as considered in [4].

Table 2 shows that the inclusion of c_p advection reduces the critical Reynolds number of the FTD model by 1.2%. In contrast, the linearization of the temperature dependence of all material properties yields a 6.4% reduction, mainly caused by the insufficient representation of $\mu(\hat{T})$ by a linear function of $\hat{T}$. The percentage change of the critical Reynolds number due to c_p advection is further corroborated by considering the critical data for the same liquid near the global maximum of Re_c from figures 5 (green square) and 9 (black square) of [4]. Table 3 shows the corresponding comparisons. In both cases considered the advection of c_p slightly decreases the critical Reynolds number by 1.9 % (case A) and 1.1 % (case B) which is compatible with the trend seen in table 2. In contrast, the critical frequency is slightly increased.

We have demonstrated that the advection of c_p affects the critical data presented in [2, 4] by reducing Re_c by 1 to 2 percent. This is much less than the effect a temperature dependent viscosity of 2 cSt silicone oil has on the critical Reynolds number. Nevertheless, the c_p advection influences the critical data about as much as the inclusion or neglect of the temperature dependence of the liquid's density, its thermal conductivity or of c_p itself has.

In particular, the stability curves obtained by including $\rho \hat{T} \, \mathrm{D}_t c_p$ in the governing equations are within a 2% tolerance level from the critical Reynolds numbers reported in [2, 4] for $\Delta T \leqslant 56$ K (see table 3). All conclusions drawn and discussions reported in [3, 4] still hold true when the c_p advection

Table 3. *Critical Reynolds numbers Re_c near their extrema for a liquid bridge from 2 cSt silicone oil and selected cases of [4]. Case A: $\Gamma = 0.93$, $\mathcal{V} = 1$, $Bd = 0.721$, $Re_g = 0$. Case B: $\Gamma = 0.66$, $\mathcal{V} = 1$, $Bd = 0.363$, $Re_g = -500$. Values are given for the full-temperature-dependence model without (FTD) and with inclusion of c_p advection (FTD,adv). The relative deviation $\epsilon_c = (Re_c^{FTD,adv} - Re_c^{FTD})/Re_c^{FTD}$ is given in percent; correspondingly for the critical frequency ω_c*

Case	Variable	FTD	FTD,adv	ϵ_c [%]
A	Re_c	1438	1411	−1.9
A	ΔT_c[K]	30.87	30.29	−1.9
A	ω_c	35.0	35.4	1.1
B	Re_c	1853	1833	−1.1
B	ΔT_c [K]	56.04	55.45	−1
B	ω_c	41.7	42.7	2.4

term $\rho \hat{T}\, \mathrm{D}_t c_p$ is consistently included in the energy equation, except for small quantitative deviations discussed herein.

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