ON STRONG LIFTINGS ON PROJECTIVE LIMITS

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Abstract. The main problem investigated in this paper is the following. Assume that we are given a convergent projective system of topological measure spaces ordered by ordinals. When does there exist a consistent system of liftings (densities, linear liftings) on the projective system converging to a lifting (density, linear lifting) on the limit space. We look mainly for strong or strong completion Baire liftings. We reduce the problem to the question about the existence of strong liftings being inverse images of other strong liftings under measure preserving mappings (Proposition 2.4) and then we adapt a condition applied earlier by A. and C. Ionescu Tulcea [**14**] to get a strong lifting for an arbitrary measure on a product space (Theorem 2.7). In this way we get some results (see Theorems 2.7, 5.3, 5.7, 6.4 and 6.5) extending the well known achievements of A. and C. Ionescu Tulcea [**14**] and Fremlin [**9**].

The application of projective limits allows us to carry over results obtained earlier only for product spaces (see e.g. [**23**], [**18**], [**19**], [**20**], [**21**]) to more general classes of topological probability spaces. In particular, we can extend the class of spaces for which there is a positive answer to a problem of J. Kupka [**17**] concerning the permanence of the strong lifting property under the formation of products (see Theorem 6.5).

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1. Preliminaries. Given a probability space (Ω, Σ, μ) the family of all μ-null subsets of Ω is denoted by Σ_0. For $A, B \in \Sigma$ we write $A = B$ a.e. (μ) iff $A \triangle B$, the symmetric difference of A and B, is a μ-null set. The (Carathéodory) completion of (Ω, Σ, μ) will be denoted by $(\Omega, \widehat{\Sigma}, \widehat{\mu})$. If $\Xi \subset \Sigma$, then $\widehat{\mu \vert \Xi}$ will be the completion of $\mu \vert \Xi$. $\mathcal{L}^\infty(\mu)$ denotes the family of all bounded real-valued μ-measurable functions on (Ω, Σ, μ). Equivalent functions are not identified. The space of equivalence classes of functions that are μ-integrable (or bounded) is denoted by $L^1(\mu)$ (resp. by $L^\infty(\mu)$). The

σ-algebra generated by a family $\mathcal{L}$ of sets is denoted by $\sigma(\mathcal{L})$. $\mathbf{N}$ and $\mathbf{R}$ stand for the natural numbers and the real numbers respectively. If $M \subseteq \Omega$, then $M^c := \Omega \setminus M$. We use the notion of **(lower) density, linear lifting, lifting** in the sense of [**14**, Chapter III] and for every probability space (Ω, Σ, μ) we denote by $\vartheta(\mu)$, $\mathcal{G}(\mu)$ and $\Lambda(\mu)$ the system of all (lower) densities, linear liftings, and liftings respectively. If $\psi \in \mathcal{G}(\mu)$ then, following [**14**] we define density $\underline{\psi} \in \vartheta(\mu)$ by setting

$$\underline{\psi}(E) := \{\psi(\chi_E) = 1\} \quad \text{for each} \quad E \in \Sigma,$$

where χ_E denotes the characteristic function of E.

Throughout a quadruple $(\Omega, \mathcal{T}, \Sigma, \mu)$ will be called a **topological measure space**, if $(\Omega, \mathcal{T})$ is a Haussdorff topological space and (Ω, Σ, μ) is a measure space such that $\mathcal{T} \subseteq \Sigma$. $\mathcal{B}(\Omega)$ is the Borel σ-algebra of $(\Omega, \mathcal{T})$. The Baire σ-algebra of $(\Omega, \mathcal{T})$, i.e. the σ-algebra generated by the system of all cozero subsets of Ω, is denoted by $\mathcal{B}_0(\Omega)$. Let (Ω, Σ, μ) be a probability space and $\mathcal{T}$ be a topology on Ω. A family $\mathcal{F} \subseteq \Sigma$ is called a **measurable network** for $(\Omega, \mathcal{T})$ if for each $G \in \mathcal{T}$ there exists a subfamily $\mathcal{G} \subseteq \mathcal{F}$ such that $G = \bigcup \mathcal{G}$. If $\mathcal{F}$ can be taken to be countable, then we say that $\mathcal{F}$ is a **countable network**. If $(\Omega, \mathcal{T})$ is a (Hausdorff) topological space we denote by $C_b(\Omega)$ the space of all bounded continuous functions on Ω. The measure μ is called τ-**additive**, if for every incrasing family $\langle G_i \rangle_{i \in I}$ of open subsets of Ω we have $\mu(\bigcup_{i \in I} G_i) = \sup_{i \in I} \mu(G_i)$.

A **Radon measure space** is a complete topological measure space $(\Omega, \mathcal{T}, \widehat{\mathcal{B}}(\Omega), \mu)$ such that the measure μ is **locally finite**, i.e. for every $\omega \in \Omega$ there exists an open set G containing ω such that $\mu(G) < +\infty$, and **inner regular** with respect to the compact sets, i.e. $\mu(E) = \sup\{\mu(K) : K \text{ compact}, K \subseteq E\}$ for all $E \in \Sigma$.

For a topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ a lifting $\rho \in \Lambda(\mu)$ is called **strong**, if $G \subseteq \rho(G)$ for all $G \in \mathcal{T}$, and ρ is called **almost strong**, if there exists a $N \in \Sigma_0$ such that $G \subseteq \rho(G) \cup N$ for all $G \in \mathcal{T}$. We say that a topological measure space $(\Omega, \mathcal{T}, \Sigma, \mu)$ (or just μ) has the **almost strong lifting property**, $ASLP$ for short, if there exists an almost strong lifting $\rho \in \Lambda(\mu)$, and it has the **universal strong lifting property** ($USLP$ for short), if each $\rho \in \Lambda(\mu)$ is almost strong. For a list of spaces having the $USLP$ the reader should compare [**26**]. This list of spaces comprises most spaces appearing in applications. For each $\rho \in \Lambda(\mu)$ there exists exactly one (multiplicative) lifting $\widetilde{\rho}$ (in the sense of [**14**, Chapter III, Section 1, Definition 2]) on $\mathcal{L}^\infty(\mu)$ such that $\widetilde{\rho}(\chi_A) = \chi_{\rho(A)}$ for all $A \in \Sigma$ (see [**14**]). For simplicity we write $\widetilde{\rho} = \rho$. If $(\Omega, \mathcal{T})$ is completely regular, then a $\rho \in \Lambda(\mu)$ is strong if and only if $\rho(f) = f$ for every $f \in C_b(\Omega)$. If μ admits a strong lifting then μ is τ-additive (see [**1**], Prop. 3).

$\rho \in \Lambda(\mu)$ is called **completion Baire**, if $\rho(A) \in \widehat{\mathcal{B}}_0(\Omega)$ for each $A \in \Sigma$, and ρ is called **strong completion Baire**, if it is strong and completion Baire.

A measure μ on $\mathcal{B}_o(\Omega)$ is called **completion regular** if for every Borel set B there exist $A_1, A_2 \in \mathcal{B}_o(\Omega)$ such that $A_1 \subseteq B \subseteq A_2$ and $\mu(A_2 \setminus A_1) = 0$. A measure μ on $\mathcal{B}(\Omega)$ is called **completion regular**, if its restriction to $\mathcal{B}_o(\Omega)$ is completion regular. If μ is completion regular, then each lifting is completion Baire.

For a complete probability space (Θ, Ξ, ν) and $\varphi \in \vartheta(\nu)$ put

$$\mathcal{T}_\varphi := \{A \in \Xi : A \subseteq \varphi(A)\}.$$

Then $\mathcal{T}_\varphi$ is a topology on Θ called the **density topology** associated with φ (see [**14**, Chapter V, Section 1]). The same definition holds true for liftings instead of densities and the corresponding topology is called the **lifting topology**.

If γ is an ordinal, then we will often identify it with the set $\{\alpha < \gamma\}$ of all ordinals less than γ. I will always be a nonempty set and if $\langle (\Omega_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ is a family of probability spaces then, for each $\emptyset \neq J \subseteq I$ we denote by $(\Omega_J, \Sigma_J, \mu_J)$ the product measure space $\otimes_{i \in J}(\Omega_i, \Sigma_i, \mu_i)$. $\widehat{\otimes}_{i \in J}(\Omega_i, \Sigma_i, \mu_i)$ is the completion of $\otimes_{i \in J}(\Omega_i, \Sigma_i, \mu_i)$. For $\emptyset \neq J \subseteq K \subseteq I$ we denote by f_{JK} the canonical projection from Ω_K onto Ω_J and for each $\emptyset \neq J \subseteq I$ we write $f_J := f_{JI}$. For $J := \{i\} \subseteq I$ we write $f_i := f_{\{i\}}$. If for each $i \in I$ a topology $\mathcal{T}_i$ is given, then $\mathcal{T}_J$ is the product topology on Ω_J, whenever $\emptyset \neq J \subseteq I$. The collection of all finite subsets of I is denoted by $\mathcal{F}(I)$ and $\kappa(I)$ is the first ordinal of the cardinality $card(I)$.

If $I = \{\alpha < \gamma\}$ and $0 < \beta \leq \gamma$, then instead of $(\prod_{\alpha < \beta} \Omega_\alpha, \otimes_{\alpha < \beta} \Sigma_\alpha, \otimes_{\alpha < \beta} \mu_\alpha)$ we write $(\Omega_{<\beta}, \Sigma_{<\beta}, \mu_{<\beta})$ and, we denote by $f_{(<\alpha)(<\beta)}$ (resp. $f_{<\alpha}$) the canonical projection from $\Omega_{<\beta}$ onto $\Omega_{<\alpha}$ (resp. from $\Omega_{<\gamma}$ onto $\Omega_{<\alpha}$) for all $\alpha < \beta < \gamma$. $[I]^{<\kappa}$ is the collection of all non-empty subsets J of I such that $card(J) < \kappa$.

If (Ω, Σ, μ) is a probability space we write μ^I for the product measure on Ω^I and Σ^I for its domain.

For the product $(\Omega, \Sigma, \mu) = \otimes_{i \in I}(\Omega_i, \Sigma_i, \mu_i)$ of a family $\langle (\Omega_i, \Sigma_i, \mu_i)\rangle_{i \in I}$ of complete probability spaces let be given a density φ for $\widehat{\mu}$. We recall that φ **respects coordinates**, if for each non-empty subset J of I the inclusion $\varphi(\widehat{\Sigma}_J \times \Omega_{J^c}) \subseteq \widehat{\Sigma}_J \times \Omega_{J^c}$ holds true (cf. [**9**]).

2. Inverse images of densities and liftings. Let (Ω, Σ, μ), (Θ, Ξ, ν) be complete probability spaces, $f : \Omega \longrightarrow \Theta$ a measurable function with $\mu(f^{-1}(B)) = \nu(B)$ for each $B \in \Xi$, and let $\psi \in \mathcal{G}(\nu)$. A linear lifting $\varphi \in \mathcal{G}(\mu)$ is called an **inverse image** of ψ under f, if

$$\varphi(g \circ f) = \psi(g) \circ f \quad \text{for each} \quad g \in \mathcal{L}^\infty(\nu).$$

Let $\tau \in \vartheta(\nu)$. A density $\varphi \in \vartheta(\mu)$ is called an **inverse image** of τ under f, if

$$\varphi(f^{-1}(E)) = f^{-1}(\tau(E)) \quad \text{for each} \quad E \in T.$$

The definition of an inverse image of a lifting is covered by both definitions.

LEMMA 2.1. *For complete probability spaces (Ω, Σ, μ), (Θ, Ξ, ν), densities $\delta \in \vartheta(\mu)$ and $\zeta \in \vartheta(\nu)$, and a $\Sigma - \Xi$-measurable, measure preserving map $f : \Omega \longrightarrow \Theta$ the following conditions are all equivalent.*
  (i) *The map f is $\mathcal{T}_\delta$-$\mathcal{T}_\zeta$-continuous.*
  (ii) *For all $B \in \Xi$, $f^{-1}(\zeta(B)) \subseteq \delta(f^{-1}(B)))$.*
If in addition $\delta \in \Lambda(\mu)$ and $\zeta \in \Lambda(\nu)$ then we may add the following equivalent condition.
  (iii) *The lifting δ is the inverse image of the lifting ζ.*

Proof. Note that for all $B, C \in \Xi$ with $B = C$ a.e.(ν) we have $f^{-1}(B) = f^{-1}(C)$ a.e.(μ) since $\nu = f(\mu)$.

(i) $\Longrightarrow$ (ii). $B \in \Xi$ implies $\zeta(B) \in \mathcal{T}_\zeta$, hence $f^{-1}(\zeta(B)) \in \mathcal{T}_\delta$. Consequently $f^{-1}(\zeta(B)) \subseteq \delta(f^{-1}(\zeta(B))) = \delta(f^{-1}(B))$.

(ii) $\Longrightarrow$ (i). $G \in \mathcal{T}_\zeta$ means $G \subseteq \zeta(G)$, hence $f^{-1}(G) \subseteq f^{-1}(\zeta(G)) \subseteq \delta(f^{-1}(G))$, i.e. $f^{-1}(G) \in \mathcal{T}_\delta$. $\square$

LEMMA 2.2. *Let (Ω, Σ, μ), (Θ, Ξ, ν) be complete probability spaces and $f : \Omega \longrightarrow \Theta$ a measurable, measure preserving map. Then for each $\psi \in \mathcal{G}(\nu)$ the collection of inverse images of ψ in non-empty.*

Proof. Let $\mathcal{B}$ be the σ-subalgebra of Σ generated by $f^{-1}(\Xi) \cup \Sigma_0$. Define $\varphi \in \mathcal{G}(\mu \mid \mathcal{B})$ by means of $\varphi(g) := \psi(h) \circ f$ for each $g \in \mathcal{L}^{\infty}(\mu \mid \mathcal{B})$ and $h \in \mathcal{L}^{\infty}(\nu)$ such that $h \circ f = g$ a.e. (μ). Then, it is enough to extend φ in an arbitrary way to Σ. $\square$

The corresponding result for liftings instead of linear liftings can be found in [**21**, Lemma 2.1].

The following problem however remains open:

QUESTION 2.3. *Let* $(\Omega, \mathcal{T}, \Sigma, \mu)$ *and* $(\Theta, \mathcal{S}, \Xi, \nu)$ *be complete topological probability spaces and* $f : \Omega \longrightarrow \Theta$ *a measure preserving map. Suppose that ν admits a strong lifting τ. When does there exist a strong inverse image $\rho \in \Lambda(\mu)$ of τ under f?*

In general a strong lifting may have empty set of strong inverse images (see Remark 2.5). We know two classes of spaces $(\Omega, \mathcal{T}, \Sigma, \mu)$ admitting strong inverse images of $\tau \in \Lambda(\nu)$ under $f : \Omega \to \Theta$ if $(\Theta, \mathcal{S}, \Xi, \nu)$ is fixed. These are the hyperstonian spaces and the extremally disconnected Baire spaces where each set of the first category is closed, endowed with a category measure.

The following proposition solves the problem of the existence of a strong inverse image of a strong lifting under a measure preserving mapping in case of liftings possessing almost strong inverse images.

PROPOSITION 2.4. *Let* $(\Omega, \mathcal{T}, \Sigma, \mu)$ *and* $(\Theta, \mathcal{S}, \Xi, \nu)$ *be complete topological probability spaces and* $f : \Omega \longrightarrow \Theta$ *be a measure preserving map. If ν admits a strong lifting τ which has an almost strong inverse image in $\Lambda(\mu)$, then τ possesses a strong inverse image $\rho \in \Lambda(\mu)$ under f if and only if the following condition is satisfied:*
If $B \in \Xi$ *and* $G \in \mathcal{T}$ *are arbitrary, then*

$$f^{-1}(\tau(B)) \cap G \neq \emptyset \Longrightarrow \mu(f^{-1}(B) \cap G) > 0. \tag{1}$$

If $\varphi \in \Lambda(\mu)$ *is an almost strong inverse image of τ and $N \in \Sigma_0$ is a set satisfying the inclusion $G \subseteq \varphi(G) \cup N$ for all $G \in \mathcal{T}$, then ρ can be taken to satisfy the equality $\varphi|N^c = \rho|N^c$.*

Proof. For each $\omega \in N$, let

$$\mathcal{A}(\omega) := \{A : A \in f^{-1}(\Xi), \omega \in \varphi(A)\}, \quad \mathcal{E}(\omega) := \{G : G \in \mathcal{T}, \omega \in G\},$$

$$\mathcal{H}(\omega) := \{A \cap G : A \in \mathcal{A}(\omega), G \in \mathcal{E}(\omega)\}$$

and $\mathcal{F}(\omega) \subseteq \Sigma$ be the filter defined by

$$\mathcal{F}(\omega) := \{E \in \Sigma : \exists F \in \mathcal{H}(\omega) \quad \text{with} \quad F \subseteq E \quad \text{a.e.}(\mu)\}.$$

Due to the assumption (1) all members of $\mathcal{H}(\omega)$ and, consequently of $\mathcal{F}(\omega)$ are of positive measure and every $\mathcal{F}(\omega)$ is measure stable.

Define a mapping $\overline{\rho} : \Sigma \to \Sigma$ by means of

$$\overline{\rho}(E) := [\varphi(E) \cap N^c] \cup \rho^*(E) \quad \text{for each} \quad E \in \Sigma,$$

where $\rho^*(E) := \{\omega \in N : E \in \mathcal{F}(\omega)\}$. It is straightforward to verify that $\overline{\rho} \in \vartheta(\mu)$.

Claim 1. For each $E \in T$ we have $f^{-1}(\tau(E)) \subseteq \overline{\rho}(f^{-1}(E))$.

Proof of claim. Let $E \in T$. If $\omega \in N$ we have

$$\omega \in f^{-1}(\tau(E)) \Longleftrightarrow \omega \in \varphi(f^{-1}(E)) \Longleftrightarrow f^{-1}(E) \in \mathcal{A}(\omega) \subseteq \mathcal{F}(\omega)$$
$$\Longrightarrow \omega \in \rho^*(f^{-1}(E)) \Longrightarrow \omega \in \overline{\rho}(f^{-1}(E)),$$

i.e.

$$f^{-1}(\tau(E)) \cap N \subseteq \overline{\rho}(f^{-1}(E)) \cap N \quad \text{for all} \quad E \in T. \tag{2}$$

If $\omega \notin N$ we have

$$\omega \in \overline{\rho}(f^{-1}(E)) \Longleftrightarrow \omega \in \varphi(f^{-1}(E)) \Longleftrightarrow \omega \in f^{-1}(\tau(E)),$$

i.e.

$$\overline{\rho}(f^{-1}(E)) \cap N^c = f^{-1}(\tau(E)) \cap N^c \quad \text{for all} \quad E \in T. \tag{3}$$

From (2) and (3) follows the desired result.

Claim 2. $\overline{\rho}$ *is strong.*

Proof of claim. Let $G \in \mathcal{T}$ and $\omega \in G$.

If $\omega \in N$ then $G \in \mathcal{E}(\omega) \subseteq \mathcal{F}(\omega)$ hence $\omega \in \rho^*(G) \subseteq \overline{\rho}(G)$.

If $\omega \notin N$, then $\omega \in \varphi(G)$ hence $\omega \in \varphi(G) \cap N^c \subseteq \overline{\rho}(G)$. Consequently $\overline{\rho}$ is strong. This completes the proof of Claim 2.

By von Neumann [30] there exists a $\rho \in \Lambda(\mu)$ such that

$$\overline{\rho}(E) \subseteq \rho(E) \quad \text{for each} \quad E \in \Sigma. \tag{4}$$

Since $\overline{\rho}$ is strong it follows by (4) that ρ is strong.

Claim 3. ρ *is an inverse image of τ under f.*

Proof of claim. For each $A \in T$ applying (4) and Claim 1 we get

$$f^{-1}(\tau(A)) \subseteq \overline{\rho}(f^{-1}(A)) \subseteq \rho(f^{-1}(A)).$$

Consequently, applying the above relation for A^c instead of A and the lifting properties we get easily the ρ is an inverse image of τ under f.

Assume now that there exists a strong inverse image $\rho \in \Lambda(\mu)$ of τ under f. Let $B \in \Xi$, $G \in \mathcal{S}$ and $\omega \in f^{-1}(\tau(B)) \cap G$. We have then

$$\omega \in f^{-1}(\tau(B)) \cap G \Leftrightarrow \omega \in \rho(f^{-1}(B)) \cap G$$
$$\Rightarrow \omega \in \rho(f^{-1}(B)) \cap \rho(G) = \rho(f^{-1}(B) \cap G),$$

hence $\mu(f^{-1}(B) \cap G) > 0$. Thus, condition (1) holds true. This completes the proof of the proposition. $\square$

REMARK 2.5. In general the condition (1) alone does not guarantee the existence of a strong inverse image of a strong lifting. To see it let $(\Omega, \mathcal{T}, \Sigma, \mu)$ be the space from Fremlin's simplification [8] of Losert's example and let $\Theta = \Omega$. Then, let Ξ be generated by μ-null sets, $\mathcal{S}$ be the trivial topology on Θ and ν be the restriction of μ to Ξ. Then the relation (1) is obviously satisfied if f is the identity map.

For the proof of the next theorem we need the following lemma, which is a generalization of Proposition 2 from [16]. Its proof is a modification of the proof of Proposition 2 from [16].

LEMMA 2.6. *Let $(\Theta_1, \mathcal{S}_1)$ and $(\Theta_2, \mathcal{S}_2)$ be topological spaces such that $(\Theta_1, \mathcal{S}_1)$ has a countable network $\mathcal{G}_1 \subseteq \mathcal{B}(\Theta_1)$. Then $\mathcal{B}_0(\Theta_1) \otimes \mathcal{B}_0(\Theta_2) = \mathcal{B}_0(\Theta_1 \times \Theta_2)$.*

Proof. Let f be a continuous function on $\Theta_1 \times \Theta_2$ and let $\{G_n : n \in \mathbf{N}\}$ be a countable basis of the natural topology of $\mathbf{R}$. Suppose that $\mathcal{G}_1 = \{G_1^m : m \in \mathbf{N}\}$. Then for each n we have $f^{-1}(G_n) \in \mathcal{S}_1 \times \mathcal{S}_2$ and

$$f^{-1}(G_n) = \bigcup_{m \in \mathbf{N}} \left(G_{1n}^m \times U_{2n}^m \right) \quad \text{where} \quad U_{2n}^m = \bigcup \{V : V \in \mathcal{S}_2 \quad \& \quad G_{1n}^m \times V \subseteq f^{-1}(G_n)\}$$

and $\{G_{1n}^m : m, n \in \mathbf{N}\}$ is a sub-collection of $\{G_n : n \in \mathbf{N}\}$.

Denote by $\mathcal{S}_2^*$ the topology in Θ_2 generated by the sets U_{2n}^m for $m, n \in \mathbf{N}$. For every $m, n \in \mathbf{N}$ put

$$U_{1n}^m := \bigcup \{U : U \in \mathcal{S}_1 \quad \text{and} \quad U \times U_{2n}^m \subseteq f^{-1}(G_n)\}.$$

Then we get

$$f^{-1}(G_n) = \bigcup_{m \in \mathbf{N}} \left(U_{1n}^m \times U_{2n}^m \right).$$

Denote by by $\mathcal{S}_1^*$ the topology in Θ_1 generated by the sets U_{1n}^m for $m, n \in \mathbf{N}$.

If G is an open subset of $\mathbf{R}$, then

$$f^{-1}(G) = \bigcup_{k \in \mathbf{N}} f^{-1}(G_{n_k}),$$

for some $n_1, n_2, \ldots$ and so

$$f^{-1}(G) = \bigcup_{k \in \mathbf{N}} f^{-1}(G_{n_k}) = \bigcup_{m, k \in \mathbf{N}} \left(U_{1n_k}^m \times U_{2n_k}^m \right). \tag{5}$$

It follows that f is $\mathcal{S}_1^* \times \mathcal{S}_2^*$-continuous.

Since the topologies $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ have countable bases and f is $\mathcal{S}_1^* \times \mathcal{S}_2^*$-continuous, it follows from Proposition 1 of [16] that f is $\mathcal{B}_0(\Theta_1, \mathcal{S}_1^*) \otimes \mathcal{B}_0(\Theta_2, \mathcal{S}_2^*)$-measurable, where $\mathcal{B}_0(\Theta_i, \mathcal{S}_i^*)$ denotes the Baire σ-algebra of Θ_i with respect to the weaker than $\mathcal{S}_i$ topology $\mathcal{S}_i^*$ for $i = 1, 2$.

Consequently, f is $\mathcal{B}_0(\Theta_1) \times \mathcal{B}_0(\Theta_2)$-measurable, hence $\mathcal{B}_0(\Theta_1 \times \Theta_2) \subseteq \mathcal{B}_0(\Theta_1) \otimes \mathcal{B}_0(\Theta_2)$. But, because of the continuity of the canonical projections $f_i : \Theta_1 \times \Theta_2 \longrightarrow \Theta_i$ for $i = 1, 2$ the inclusion $\mathcal{B}_0(\Theta_1) \otimes \mathcal{B}_0(\Theta_2) \subseteq \mathcal{B}_0(\Theta_1 \times \Theta_2)$ holds true in any case, hence the result follows. □

The hypothesis (IT) in the next theorem, which is a consequence of Proposition 2.4, traces back to A. and C. Ionescu Tulcea [14, page 115]. It is implied by independence and is equivalent itself to the relation (1) of Proposition 2.4.

THEOREM 2.7. *Let (Θ_i, Σ_i) be a measurable space, $\mathcal{T}_i$ a topology on Θ_i for $i = 1, 2$, $\Omega = \Theta_1 \times \Theta_2$, $\Sigma = \Sigma_1 \otimes \Sigma_2$ and let $\mathcal{T} := \mathcal{T}_1 \times \mathcal{T}_2$ be the product topology on Ω. If μ is a probability measure on Σ supporting Ω denote by μ_i the image measure of μ under the canonical projection f_i from Ω onto Θ_i. Assume that $\mathcal{T}_2 \subseteq \widehat{\Sigma}_2$ and $\widehat{\mu}_1$ admits a strong*

lifting τ (*notice that this yields* $\mathcal{T}_1 \subseteq \widehat{\Sigma}_1$). *Then the following conditions are equivalent:*

  (i) *If* $U \in \mathcal{T}_2$ *and* $A \in \widehat{\Sigma}_1$ *are arbitrary, then*

$$\widehat{\mu}(A \times U) = 0 \Longrightarrow \widehat{\mu}_1(A)\widehat{\mu}_2(U) = 0. \qquad (IT)$$

 (ii) *For each* $G_i \in \mathcal{T}_i$ *for* $i = 1, 2$, *and for each* $A \in \widehat{\Sigma}_1$ *we have*

$$f_1^{-1}(\tau(A)) \cap (G_1 \times G_2) \neq \emptyset \Longrightarrow \widehat{\mu}\big(f_1^{-1}(A) \cap (G_1 \times G_2)\big) > 0.$$

and each of them follows from

 (iii) *There exists a strong inverse image* $\rho \in \Lambda(\widehat{\mu})$ *of* τ *under* f_1.

   If the topological space $(\Theta_2, \mathcal{T}_2)$ *has a countable network* $\mathcal{G}_2 \subseteq \widehat{\Sigma}_2$, *then all the above conditions are equivalent.*

   In particular, if condition (iii) *holds true then*
   (a) $\widehat{\Sigma}_i = \widehat{\mathcal{B}}(\Theta_i)$ *for* $i = 1, 2$ *implies* $\widehat{\Sigma} = \widehat{\mathcal{B}}(\Omega)$.
   (b)   *If for each* $i = 1, 2$ *we have* $\Sigma_i = \mathcal{B}_0(\Theta_i)$ *and* $(\Theta_2, \mathcal{T}_2)$ *has a countable network* $\mathcal{G}_2 \subseteq \mathcal{B}(\Theta_2)$, *then* μ_i *is completion regular,* $\Sigma = \mathcal{B}_0(\Omega)$ *and* μ *is completion regular.*

   Proof. (i) $\Longrightarrow$ (ii) Let $G_i \in \mathcal{T}_i$ for $i = 1, 2$ and $A \in \Sigma_1$ be arbitrary sets satisfying condition

$$f_1^{-1}(\tau(A)) \cap (G_1 \times G_2) \neq \emptyset. \qquad (6)$$

Assume, if possible, that $\widehat{\mu}(f_1^{-1}(A) \cap (G_1 \times G_2)) = 0$, i.e. $\widehat{\mu}((A \times \Theta_2) \cap (G_1 \times G_2)) = 0$, or equivalently $\widehat{\mu}((A \cap G_1) \times G_2)) = 0$. So applying condition (i) we get

$$\widehat{\mu}_1(A \cap G_1)\mu_2(G_2) = 0.$$

Since $G_2 \neq \emptyset$, we have $\widehat{\mu}_2(G_2) > 0$ because $\widehat{\mu}_2$ supports Θ_2, hence $\widehat{\mu}_1(A \cap G_1) = 0$ implying $\tau(A) \cap \tau(G_1) = \emptyset$. Since τ is strong, the latter implies $\tau(A) \cap G_1 = \emptyset$ hence $f_1^{-1}(\tau(A)) \cap f_1^{-1}(G_1) = \emptyset$, or equivalently $f_1^{-1}(\tau(A)) \cap (G_1 \times \Theta_2) = \emptyset$ hence $f_1^{-1}(\tau(A)) \cap (G_1 \times G_2) = \emptyset$, which contradicts (6).

   Consequently, we deduce that $\widehat{\mu}(f_1^{-1}(A) \cap (G_1 \times G_2)) > 0$, i.e. condition (ii) holds true.

   (ii) $\Longrightarrow$ (i) Let $A \in \Sigma_1$ and $U \in \mathcal{T}_2$ be arbitrary sets such that $\widehat{\mu}(A \times U) = 0$ but $\widehat{\mu}_1(A)\widehat{\mu}_2(U) > 0$. Then $\tau(A) \times U \neq \emptyset$ hence $f_1^{-1}(\tau(A)) \cap f_2^{-1}(U) \neq \emptyset$ therefore applying condition (ii) we get $\widehat{\mu}(f_1^{-1}(A) \cap f_2^{-1}(U)) > 0$ or $\widehat{\mu}(A \times U) > 0$, a contradiction.

   (iii) $\Longrightarrow$ (ii) If there exists a strong inverse image $\rho \in \Lambda(\widehat{\mu})$ of τ under f_1 then for arbitrary $A \in \Sigma_1$, $G_i \in \mathcal{T}_i$ for $i = 1, 2$ and $(\theta_1, \theta_2) \in f_1^{-1}(\tau(A)) \cap (G_1 \times G_2)$ we have

$$
\begin{aligned}
(\theta_1, \theta_2) \in f_1^{-1}(\tau(A)) \cap (G_1 \times G_2) &\Leftrightarrow (\theta_1, \theta_2) \in \rho\big(f_1^{-1}(A)\big) \cap (G_1 \times G_2) \\
&\Rightarrow (\theta_1, \theta_2) \in \rho\big(f_1^{-1}(A)\big) \cap \rho(G_1 \times G_2) \\
&\Leftrightarrow (\theta_1, \theta_2) \in \rho\big(f_1^{-1}(A) \cap (G_1 \times G_2)\big),
\end{aligned}
$$

hence $\mu(f_1^{-1}(A) \cap (G_1 \times G_2)) > 0$. Thus, condition (ii) holds true.

   (ii) $\Longrightarrow$ (iii) Assume now that $(\Theta_2, \mathcal{T}_2)$ has a countable network $\mathcal{G}_2 \subseteq \Sigma_2$. Then, according to [21, Lemma 2.1] there exists a lifting $\varphi \in \Lambda(\widehat{\mu})$ being an inverse image of τ under f_1. Since τ is strong we have

$$f_1^{-1}(G) \subseteq \varphi\big(f_1^{-1}(G)\big) \quad \text{for each} \quad G \in \mathcal{T}_1. \qquad (7)$$

Set

$$N := \bigcup \big\{ f_2^{-1}(U) \backslash \varphi\big(f_2^{-1}(U)\big) : U \in \mathcal{G}_2 \big\}. \tag{8}$$

Then $N \in \widehat{\Sigma}_0$.

Let

$$\mathcal{G} := \{U_1 \times U_2 : U_1 \in \mathcal{T}_1, U_2 \in \mathcal{G}_2\}.$$

It follows that $\mathcal{G} \subseteq \widehat{\Sigma}$. For each $U_1 \times U_2 \in \mathcal{G}$ applying (7) and (8) we get

$$
\begin{aligned}
U_1 \times U_2 &= f_1^{-1}(U_1) \cap f_2^{-1}(U_2) \\
&\subseteq \varphi\big(f_1^{-1}(U_1)\big) \cap \big[\varphi\big(f_2^{-1}(U_2)\big) \cup N\big] \\
&= \big[\varphi\big(f_1^{-1}(U_1)\big) \cap \varphi\big(f_2^{-1}(U_2)\big)\big] \cup \big[\varphi\big(f_1^{-1}(U_1)\big) \cap N\big] \\
&\subseteq \varphi(U_1 \times U_2) \cup N,
\end{aligned}
$$

i.e.

$$U_1 \times U_2 \subseteq \varphi(U_1 \times U_2) \cup N \quad \text{for each} \quad U_1 \times U_2 \in \mathcal{G}. \tag{9}$$

Let $U \in \mathcal{T}$. Then $U = \bigcup_{n \in \mathbf{N}}(U_{1n} \times U_{2n})$ where $U_{2n} \in \mathcal{G}_2$ and $U_{1n} = \bigcup\{U_1 : U_1 \in \mathcal{T}_1$ and $U_1 \times U_{2n} \subseteq U\}$ for $n \in \mathbf{N}$.

Consequently, we have $U \in \widehat{\Sigma}$, hence applying (9) we get

$$U \subseteq \varphi(U) \cup N \quad \text{for each} \quad U \in \mathcal{T},$$

i.e. the lifting φ is an almost strong inverse image of τ under f_1. Therefore applying Proposition 2.4 we conclude condition (iii).

In particular, if condition (iii) holds true then:

(a) Since $\widehat{\Sigma}_i = \widehat{\mathcal{B}}(\Theta_i)$, the canonical projections f_i are $\widehat{\mathcal{B}}(\Omega) - \widehat{\mathcal{B}}(\Theta_i)$-measurable for $i = 1, 2$ hence $\widehat{\Sigma} \subseteq \widehat{\mathcal{B}}(\Omega)$, which combined with (iii) implies $\widehat{\mathcal{B}}(\Omega) = \widehat{\Sigma}$.

(b) If for each $i = 1, 2$ we have $\Sigma_i = \mathcal{B}_0(\Theta_i)$ then it follows by Lemma 2.6 that $\Sigma = \mathcal{B}_0(\Omega)$. On the other hand, it follows by the assumptions that $\mathcal{T}_i \subseteq \widehat{\mathcal{B}}_0(\Theta_i)$ therefore the equality $\widehat{\mathcal{B}}_0(\Theta_i) = \widehat{\mathcal{B}}(\Theta_i)$ holds true hence μ_i is completion regular for each $i = 1, 2$. Consequently, applying (a) we get $\widehat{\Sigma} = \widehat{\mathcal{B}}(\Omega) = \widehat{\mathcal{B}}_0(\Omega)$ and so μ is completion regular. This completes the proof of the theorem. $\qquad \square$

REMARK 2.8. The above theorem remains true (with the same proof) under the weaker assumption that μ is **strictly localizable**, i.e. if there exists a disjoint family $\langle\Omega_i\rangle_{i\in I}$ of measurable sets of finite measure such that $\Omega = \bigcup_{i\in I} \Omega_i$ and

$$
\begin{aligned}
\Sigma &= \{E : E \subseteq \Omega, E \cap \Omega_i \in \Sigma \quad \forall i \in I\}, \\
\mu(E) &= \sum_{i\in I} \mu(E \cap \Omega_i) \quad \text{for each} \quad E \in \Sigma.
\end{aligned}
$$

Radon measures are strictly localizable, since they have a concassage (see e.g. [**32**]).

Theorem 2.7 improves Theorem 4 from [**14**, Chapter VIII, Section 2], where Θ_2 is assumed to be a metrizable space.

REMARKS 2.9. (a) From Theorem 2.7 it follows that the condition (IT) implies the existence of an inverse strong lifting for $\widehat{\mu}$ if $(\Theta_2, \mathcal{T}_2)$ has a countable network. It is not

in general true that the existence of a (not nesseccarily inverse image) strong lifting implies condition (IT) (see Remark 5.10 (b)).

(b) If $\langle (X_i, \mathcal{S}_i, \Sigma_i, \mu_i) \rangle_{i \in I}$ is a family of complete topological probability spaces such that each μ_i has the $USLP$ and supports X_i, and if $(\Theta, \mathcal{S}, \Xi, \nu)$ is a complete topological probability space admitting a strong lifting τ, then the product space

$$(\Omega, \mathcal{T}, \Sigma, \mu) = (\Theta \times X_I, \mathcal{S} \times \mathcal{S}_I, \Xi \,\widehat{\otimes}\, \Sigma_I, \nu \,\widehat{\otimes}\, \mu_I)$$

is a complete topological probability space admitting a strong inverse image $\rho \in \Lambda(\mu)$ of τ under the canonical projection f from Ω onto Θ. The above assertion follows from [**27**, Theorem 3.9].

(c) We don't know whether Theorem 2.7 remains true under the assumption that $\widehat{\mu}_2$ has the $USLP$ (instead of the stronger assumption that $(\Theta_2, \mathcal{T}_2)$ has a countable network $\mathcal{G}_2 \subseteq \Sigma_2$). Notice that the assumption of Theorem 2.7 for $(\Theta_2, \mathcal{T}_2)$ implies that $\widehat{\mu}_2$ has the $USLP$ but according to the Remark 5.10 (c) the inverse implication is not in general true.

If μ is the product of the probability measures μ_1 and μ_2 then according to [**22**, Theorem 5.5] there is a positive answer to the above question.

3. Projective limits of liftings. We use the notions of projective systems and projective limits of probability spaces in the sense of [**29**]. For a projective system $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I \rangle$ of probability spaces a family $\langle \delta_\alpha \rangle_{\alpha \in I}$ of densities $\delta_\alpha \in \vartheta(\mu_\alpha)$ is called **consistent**, if

$$\delta_\beta \circ f_{\alpha\beta}^{-1} = f_{\alpha\beta}^{-1} \circ \delta_\alpha \tag{10}$$

for all $\alpha, \beta \in I$ with $\alpha \leq \beta$.

If $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha \in I})$ is the projective limit of the above projective system and μ_p is an extension of μ to a σ-algebra Σ_p containing Σ, then a density $\delta \in \vartheta(\mu_p)$ is called a **projective limit** or a **p-density** of the consistent family $\langle \delta_\alpha \rangle_{\alpha \in I}$ if

$$\delta \circ f_\alpha^{-1} = f_\alpha^{-1} \circ \delta_\alpha \tag{11}$$

for all $\alpha \in I$. We write in that case $\delta \in proj_{\alpha \in I} lim \delta_\alpha$ (notice that projective limits of densities are not uniquely defined). We will say also that $\langle \delta_\alpha \rangle_{\alpha \in I}$ is **convergent** to δ. The same definitions hold true for liftings. We assume throughout that all the canonical projections f_α are surjections. We will use the notation **p-lifting** for a lifting being a projective limit of other liftings.

A family $\langle \psi_\alpha \rangle_{\alpha \in I}$ of linear liftings $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$ is called **consistent**, if for all $\alpha, \beta \in I$ with $\alpha \leq \beta$ we have

$$\psi_\beta(g \circ f_{\alpha\beta}) = \psi_\alpha(g) \circ f_{\alpha\beta} \quad \text{for all} \quad g \in \mathcal{L}^\infty(\mu_\alpha). \tag{12}$$

A linear lifting $\psi \in \mathcal{G}(\mu)$ is called a **projective limit** or a **linear p-lifting** of a consistent family $\langle \psi_\alpha \rangle_{\alpha \in I}$, where $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$, if for each $\alpha \in I$ we have

$$\psi(g \circ f_\alpha) = \psi_\alpha(g) \circ f_\alpha \quad \text{for all} \quad g \in \mathcal{L}^\infty(\mu_\alpha). \tag{13}$$

For liftings on complete probability spaces definitions (10) and (12) are equivalent and the same is true for (11) and (13).

If $I = \kappa$, where κ is an infinite ordinal, we say that a projective system $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ of probability spaces is **continuous**, if for every limit ordinal $\xi < \kappa$ the projective limit of the system $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \xi \rangle$ exists and equals to $(\Omega_\xi, \Sigma_\xi, \mu_\xi, \langle f_\alpha \rangle_{\alpha < \xi})$. For projective systems of topological probability spaces a similar definition can be formulated, but then the mappings $f_{\alpha\beta}$ are assumed to be in addition continuous.

Let $\langle (\Omega_i, \Sigma_i) \rangle_{i \in I}$ be a family of measurable spaces. If μ is a measure on Σ_I, then $(\Omega_I, \Sigma_I, \mu)$ can be represented as a projective limit of a continuous projective system of probability spaces in the following way.

Fix an index $i_0 \in I$ and assume that $I = \kappa := \kappa(I)$ and $i_0 = 0$. Then it is easily seen that $\langle (\Omega_{<\alpha}, \Sigma_{<\alpha}, \mu_\alpha, f_{(<\alpha)(<\beta)}, \kappa) \rangle$ is a continuous projective system of measure spaces and that $(\Omega_I, \Sigma_I, \mu, \langle f_{<\alpha} \rangle_{\alpha < \kappa})$ is its projective limit.

Consequently, we may consider the notion of a consistent family $\langle \rho_\alpha \rangle_{\alpha < \kappa}$ of liftings $\rho_\alpha \in \Lambda(\widehat{\mu}_\alpha)$ as well as the notion of a projective limit lifting (p-lifting, in short) ρ for the measure $\widehat{\mu}$, i.e. of a lifting $\rho \in \Lambda(\widehat{\mu}) \cap proj_{\alpha < \kappa} lim \rho_\alpha$.

PROPOSITION 3.1. *Let κ be an infinite ordinal and $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ a projective system of complete probability spaces. Suppose that $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha < \kappa})$ is the projective limit of the above projective system and $\langle \psi_\alpha \rangle_{\alpha < \kappa}$ is a consistent family of linear liftings $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$. Then there exists a linear lifting $\psi \in \mathcal{G}(\widehat{\mu})$, which is a projective limit of the system $\langle \psi_\alpha \rangle_{\alpha < \kappa}$. If all ψ_α are liftings then ψ can be chosen to be a lifting.*

Proof. The only interesting case is when κ is a limit ordinal. For every $\alpha < \kappa$ we denote by Σ_α^* the σ-subalgebra of Σ generated by $f_\alpha^{-1}(\Sigma_\alpha)$ and by the set of all μ-null subsets of Ω, we set $\mu_\alpha^* := \widehat{\mu} \mid \Sigma_\alpha^*$, and define a linear lifting $\psi_\alpha^* \in \mathcal{G}(\mu_\alpha^*)$ by setting

$$\psi_\alpha^*(h) := \psi_\alpha(g) \circ f_\alpha$$

for every $h \in \mathcal{L}^\infty(\mu_\alpha^*)$ with $h = g \circ f_\alpha \quad a.e.(\mu)$ for some $g \in \mathcal{L}^\infty(\mu_\alpha)$.

For all $\alpha, \beta < \kappa$ with $\alpha \le \beta$ we have $\Sigma_\alpha^* \subseteq \Sigma_\beta^*$ hence $\mathcal{L}^\infty(\mu_\alpha^*) \subseteq \mathcal{L}^\infty(\mu_\beta^*)$. A direct calculation shows also that $\psi_\beta^* \mid \mathcal{L}^\infty(\mu_\alpha^*) = \psi_\alpha^*$.

If κ is a limit ordinal of countable cofinality, then there is a strictly increasing sequence $\langle \alpha_n \rangle_{n \in \mathbf{N}}$ with supremum κ. Put for simplicity $\psi_n^* := \psi_{\alpha_n}$ and $\Sigma_n^* := \Sigma_{\alpha_n}^*$ for all $n \in \mathbf{N}$ and let $\mathcal{U}$ be a free ultrafilter on $\mathbf{N}$. Then $\Sigma_\kappa^* = \sigma(\bigcup_{n \in \mathbf{N}} \Sigma_n^*)$ and we can define ψ by setting

$$\psi(h) := \lim_{n \in \mathcal{U}} \psi_n^*[E_{\Sigma_n^*}(h)] \quad \text{for} \quad h \in \mathcal{L}^\infty(\mu),$$

where $E_{\Sigma_n^*}(h)$ denotes the conditional expectation of h with respect to Σ_n^*. Using the arguments of the proof of Theorem 2 in [**14**, Chapter IV, Section 1], we get $\psi \in \mathcal{G}(\mu_\kappa)$. For each $\alpha < \kappa$ there exists an $n \in \mathbf{N}$ such that $\alpha < \alpha_n$, $\psi_{\alpha_n}^*$ extends ψ_α^* and ψ extends ψ_α^*. Thus ψ is a projective limit of $\langle \psi_\alpha \rangle_{\alpha < \kappa}$.

If κ is a limit ordinal of uncountable cofinality, then we proceed in the standard way. If all ψ_α are liftings then define for each $\alpha < \kappa$ a lifting $\widetilde{\psi}_\alpha \in \Lambda(\mu_\alpha^*)$ by setting

$$\widetilde{\psi}_\alpha(A) := f_\alpha^{-1}(\psi_\alpha(B))$$

for every $A \in \Sigma_\alpha^*$ with $A = f_\alpha^{-1}(B)$ a.e.$(\widehat{\mu})$ for some $A \in \Sigma_\alpha$. It can be easily seen that for any $\alpha \le \beta < \kappa$ the lifting $\widetilde{\psi}_\beta$ extends $\widetilde{\psi}_\alpha$.

Assume now that all ψ_α are liftings. If κ is of countable cofinality, then there exists a density $\varphi \in \vartheta(\widehat{\mu})$ such that φ is an inverse image of ψ_α under f_α for each $\alpha < \kappa$ (see [**10**, Lemma 1]). By von Neumann [**30**] there exists a lifting $\psi \in \Lambda(\widehat{\mu})$ such that $\varphi(E) \subseteq \psi(E)$ for each $E \in \widehat{\Sigma}$. It then follows that

$$\psi\left(f_\alpha^{-1}(A)\right) = f^{-1}(\psi_\alpha(A)) \quad \text{for each} \quad A \in \Sigma_\alpha.$$

Consequently, ψ is a projective limit of $\langle \psi_\alpha \rangle_{\alpha < \kappa}$.

If κ is of uncountable cofinality, then we define $\psi \in \Lambda(\widehat{\mu})$ by

$$\psi(E) := \widetilde{\psi}_\alpha(E) \quad \text{for each} \quad E \in \Sigma_\alpha^* \quad \text{for some} \quad \alpha < \kappa.$$

Again ψ is a projective limit of $\langle \psi_\alpha \rangle_{\alpha < \kappa}$. $\square$

It is known that not every convergent projective system $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I \rangle$ of complete probability spaces admits a consistent family $\langle \rho_\alpha \rangle_{\alpha \in I}$ of (linear) liftings ρ_α (see [**27**, Example 3.5]). In the next theorem we present a condition guaranteeing the existence of such a consistent family.

THEOREM 3.2. *Let κ be an infinite ordinal and $\langle \Omega_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ a continuous projective system of complete probability spaces. Suppose that $(\Omega, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha < \kappa})$ is the projective limit of the above system. Then:*

(1) There exist a consistent family $\langle \psi_\alpha \rangle_{\alpha < \kappa}$ of linear liftings $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$ and a linear lifting $\psi \in \mathcal{G}(\widehat{\mu})$ such that ψ is a projective limit of the system $\langle \psi_\alpha \rangle_{\alpha < \kappa}$. If all ψ_α are liftings then ψ can be chosen to be a lifting.

(2) There exist a consistent family $\langle \varphi_\alpha \rangle_{\alpha < \kappa}$ of densities $\varphi_\alpha \in \vartheta(\mu_\alpha)$ and a density $\varphi \in \vartheta(\widehat{\mu})$ such that φ is a projective limit of the system $\langle \varphi_\alpha \rangle_{\alpha < \kappa}$.

Proof. (1) (A) We construct a consistent family $\langle \psi_\alpha \rangle_{\alpha < \kappa}$ of $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$ by induction on $\alpha \leq \kappa$. The inductive hypothesis will be that ψ_γ extends ψ_β whenever $\beta \leq \gamma \leq \alpha$. To start the induction for $\alpha = 0$ we choose an arbitrary $\psi_0 \in \mathcal{G}(\mu_0)$.

(B) *Inductive step to a successor ordinal α.* If $\alpha = \beta + 1$ where $\beta < \kappa$ and $\psi_\beta \in \mathcal{G}(\mu_\beta)$ is known we choose by Lemma 2.2 a $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$ such that

$$\psi_\alpha(g \circ f_{\beta\alpha}) = \psi_\beta(g) \circ f_{\beta\alpha} \quad \text{for each} \quad g \in \mathcal{L}^\infty(\mu_\beta). \tag{14}$$

Since $\psi_\beta(g \circ f_{\delta\beta}) = \psi_\delta(g) \circ f_{\delta\beta}$ for all $\delta \leq \beta$ and all $g \in \mathcal{L}^\infty(\mu_\delta)$, we get $\psi_\alpha(g \circ f_{\delta\alpha}) = \psi_\delta(g) \circ f_{\delta\alpha}$ from (14), i.e. the family $\langle \psi_\delta \rangle_{\delta \leq \alpha}$ is consistent.

(C) *Inductive step to a limit ordinal α.* In this case we find by Proposition 3.1 a $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$ such that ψ_α is a projective limit of $\langle \psi_\beta \rangle_{\beta < \alpha}$, i.e. the family $\langle \psi_\beta \rangle_{\beta \leq \alpha}$ is consistent.

(D) Thus the induction can be pushed through to obtain a consistent family $\langle \psi_\alpha \rangle_{\alpha \leq \kappa}$ such that ψ is a projective limit of $\langle \psi_\alpha \rangle_{\alpha < \kappa}$.

The corresponding result for liftings instead of linear liftings can be found in [**25**, Proposition 2.2].

(2) It follows from (1) that there exists a consistent family $\langle \psi_\alpha \rangle_{\alpha < \kappa}$ of linear liftings $\psi_\alpha \in \mathcal{G}(\mu_\alpha)$ and a $\psi \in \mathcal{G}(\widehat{\mu})$ such that ψ is a projective limit of $\langle \psi_\alpha \rangle_{\alpha < \kappa}$. For each $\alpha \leq \kappa$ put $\varphi_\alpha := \underline{\psi}_\alpha$. Since $\langle \psi_\alpha \rangle_{\alpha \leq \kappa}$ is consistent, for each $\alpha \leq \beta \leq \kappa$ and each $E \in \Sigma_\alpha$ we

have

$$\varphi_\beta\big(f_{\alpha\beta}^{-1}(E)\big) = \big\{\psi_\beta\big(\chi_{f_{\alpha\beta}^{-1}(E)}\big) = 1\big\} = \big\{\psi_\beta\big(\chi_E \circ f_{\alpha\beta}^{-1}\big) = 1\big\}$$
$$= \big\{\psi_\alpha(\chi_E) \circ f_{\alpha\beta}^{-1} = 1\big\} = f_{\alpha\beta}^{-1}(\{\psi_\alpha(\chi_E) = 1\})$$
$$= f_{\alpha\beta}^{-1}(\varphi_\alpha(E)),$$

i.e. $\langle\varphi_\alpha\rangle_{\alpha\leq\kappa}$ is consistent, which means that $\langle\varphi_\alpha\rangle_{\alpha<\kappa}$ is consistent and $\varphi := \varphi_\kappa$ is a projective limit of $\langle\varphi_\alpha\rangle_{\alpha<\kappa}$.   □

The result (2) of the above theorem has been proven for arbitrary (possibly incomplete) probability spaces and for densities in [**24**, Section 2, Proposition 1] by a different way.

4. Convergence of systems of strong liftings.   In case of arbitrary directed sets we have the following result:

LEMMA 4.1. *Let be given a projective system $\langle\Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, I\rangle$ of complete topological probability spaces converging to $(\Omega, \mathcal{T}, \Sigma, \mu)$ and a consistent family $\langle\rho_\alpha\rangle_{\alpha\in I}$ of liftings $\rho_\alpha \in \Lambda(\mu_\alpha)$ converging to $\rho \in \Lambda(\mu)$. Then $\rho \in \Lambda(\mu)$ is strong if and only if each ρ_α is strong. The same for densities instead of liftings. In both cases we have*

$$\mathcal{T} \subseteq proj_{\alpha\in I}lim\mathcal{T}_{\rho_\alpha} \subseteq \mathcal{T}_\rho \subseteq \mathcal{B}(\Omega, \mathcal{T}_\rho) = \widehat{\Sigma}.$$

If for each $\alpha \in I$ the topological space $(\Omega_\alpha, \mathcal{T}_\alpha)$ is completely regular, then the same is true for linear liftings.

Proof. Since $\langle\rho_\alpha\rangle_{\alpha\in I}$ is a consistent family of densities it follows from Lemma 2.1 that for each $\alpha \leq \beta$ the map $f_{\alpha\beta}$ is $\mathcal{T}_{\rho_\beta} - \mathcal{T}_{\rho_\alpha}$-continuous and since ρ is a projective limit of $\langle\rho_\alpha\rangle_{\alpha\in I}$ we have again by Lemma 2.1 that for each $\alpha \in I$ the map f_α is $\mathcal{T}_\rho - \mathcal{T}_{\rho_\alpha}$-continuous.

But for each $\alpha \in I$ the density ρ_α is strong hence $\mathcal{T}_\alpha \subseteq \mathcal{T}_{\rho_\alpha}$ and therefore we get

$$\mathcal{T} \subseteq proj_{\alpha<\kappa}lim\mathcal{T}_{\rho_\alpha} \subseteq \mathcal{T}_\rho \subseteq \mathcal{B}(\Omega, \mathcal{T}_\rho) = \widehat{\Sigma}.$$

Consequently ρ is strong for $(\Omega, \mathcal{T}, \widehat{\Sigma}, \widehat{\mu})$.

Assume now that ρ is strong. Then for each $\alpha \in I$ and for each $U \in \mathcal{T}_\alpha$ we have

$$f_\alpha^{-1}(U) \subseteq \rho\big(f_\alpha^{-1}(U)\big) = f_\alpha^{-1}(\rho_\alpha(U)).$$

Since all f_α are surjections we then get

$$U \subseteq f_\alpha\big(f_\alpha^{-1}(\rho_\alpha(U))\big) = \rho_\alpha(U),$$

that is ρ_α is strong.

Suppose that for each $\alpha \in I$ the topological space $(\Omega_\alpha, \mathcal{T}_\alpha)$ is completely regular and that $\langle\rho_\alpha\rangle_{\alpha\in I}$ is a consistent family of strong linear liftings. It follows in the same way as in the proof of Theorem 3.2 that $\langle\underline{\rho}_\alpha\rangle_{\alpha\in I}$ is a consistent family of densities $\underline{\rho}_\alpha \in \vartheta(\mu_\alpha)$. To show that for each $\alpha \in I$ the density $\underline{\rho}_\alpha$ is strong, let $U \in \mathcal{T}_\alpha$. Since each ρ_α is strong, it follows in the same way as in [**14**, page 105], that $\chi_U \leq \rho_\alpha(\chi_U)$,

which implies

$$\underline{\rho_\alpha}(U) = \{\omega \in \Omega_\alpha : \rho_\alpha(\chi_U)(\omega) = 1\} \supseteq \{\omega \in \Omega_\alpha : \chi_U(\omega) = 1\} = U.$$

Consequently, $\underline{\rho} \in \vartheta(\mu)$ is strong hence $\mathcal{T} \subseteq \widehat{\Sigma}$ as above.

To show that ρ is strong, let $U \in \mathcal{T}$. Since I is directed, the family

$$\mathcal{U} := \left\{ f_\alpha^{-1}(U_\alpha) : \alpha \in I, \ U_\alpha \in \mathcal{T}_\alpha \right\}$$

is a basis for the topology $\mathcal{T}$ (cf. [5]), it follows that $U = \cup_{\alpha \in J} f_\alpha^{-1}(U_\alpha)$ for some $U_\alpha \in \mathcal{T}_\alpha$ and $J \subseteq I$. So, we get

$$\rho(\chi_U) \geq \rho\big(\chi_{f_\alpha^{-1}(U_\alpha)}\big) = \rho(\chi_{U_\alpha} \circ f_\alpha)$$
$$= \rho_\alpha(\chi_{U_\alpha}) \circ f_\alpha \geq \chi_{U_\alpha} \circ f_\alpha = \chi_{f_\alpha^{-1}(U_\alpha)}$$

for each $\alpha \in J$. Hence $\rho(\chi_U) \geq \chi_U$, i.e. ρ is strong. $\qquad\square$

PROPOSITION 4.2. *Let κ be an infinite ordinal and $\langle \Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ be a continuous projective system of topological probability spaces. Assume that $(\Omega, \mathcal{T}, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha<\kappa})$ is the projective limit of the above projective system and $\langle \varphi_\alpha \rangle_{\alpha<\kappa}$ is a consistent family of strong densities $\varphi_\alpha \in \vartheta(\widehat{\mu}_\alpha)$. Then there exists a strong density $\varphi \in \vartheta(\widehat{\mu})$ such that the system $\langle \varphi_\alpha \rangle_{\alpha<\kappa}$ is convergent to φ.*

If all densities φ_α are liftings, then φ can be taken to be a strong lifting. Moreover,

(a) if $\widehat{\Sigma}_\alpha = \widehat{\mathcal{B}}(\Omega_\alpha)$ for all $\alpha < \kappa$, then $\widehat{\Sigma} = \widehat{\mathcal{B}}(\Omega)$;

(b) if for each $\alpha < \kappa$ we have that Ω_α is compact, $\Sigma_\alpha = \mathcal{B}_0(\Omega_\alpha)$ and that μ_α is completion regular, then $\Sigma = \mathcal{B}_0(\Omega)$ and μ is completion regular (consequently, φ is completion Baire).

If for each $\alpha < \kappa$ the topological space $(\Omega_\alpha, \mathcal{T}_\alpha)$ is completely regular and φ_α is a strong linear lifting, then φ can be taken to be a strong linear lifting.

Proof. The proof for densites and lifting can be found in [27, Theorem 3.1].

If for each $\alpha < \kappa$ the space $(\Omega_\alpha, \mathcal{T}_\alpha)$ is completely regular, then it follows by Proposition 3.1 that there exists a $\varphi \in \mathcal{G}(\widehat{\mu}) \cap proj_{\alpha<\kappa} lim\varphi_\alpha$ which is strong according to Lemma 4.1.

In particular,

(a) if $\widehat{\Sigma}_\alpha = \widehat{\mathcal{B}}(\Omega_\alpha)$ for each $\alpha < \kappa$, then relation $\mathcal{T} \subseteq \widehat{\Sigma}$ and the obvious one $\widehat{\Sigma} \subseteq \widehat{\mathcal{B}}(\Omega)$ imply that $\widehat{\Sigma} = \widehat{\mathcal{B}}(\Omega)$.

(b) If for each $\alpha < \kappa$ the measure μ_α is completion regular, $\Sigma_\alpha = \mathcal{B}_0(\Omega_\alpha)$ and Ω_α is compact, then it follows by [3, Theorem 2.3] that $\Sigma = \mathcal{B}_0(\Omega)$ hence $\widehat{\Sigma} = \widehat{\mathcal{B}}_0(\Omega)$. Consequently, applying (a) we get $\widehat{\Sigma} = \widehat{\mathcal{B}}(\Omega) = \widehat{\mathcal{B}}_0(\Omega)$, which means that μ is completion regular and φ is completion Baire. $\qquad\square$

In Theorem 3.2 and in Proposition 4.2 it is crucial to have a well ordering of the index set (see [27, Example 3.2]). Below is still another example.

EXAMPLE 4.3. Let $\Omega := [0,1]^I$, where I is an index set with the cardinality of the continuum. Gryllakis and Grekas have constructed in [11] a Radon probability measure μ on Ω, which supports Ω and is not completion regular. Denote again by μ the restriction of μ to $\mathcal{B}_0(\Omega)$. Clearly we have that $(\Omega, \mathcal{B}_0(\Omega), \mu, \langle f_\alpha \rangle_{\alpha \in \mathcal{F}(I)})$ is the projective limit of the system $\langle \Omega_\alpha, \mathcal{B}_0(\Omega_\alpha), \mu_\alpha, f_{\alpha\beta}, \mathcal{F}(I) \rangle$, where $\Omega_\alpha := [0,1]^\alpha$ and $\mu_\alpha := \mu \mid \widehat{\mathcal{B}}_0(\Omega_\alpha)$. In particular, we set $\Omega_1 := \Omega_{\{i\}} = [0,1]$ and $\mu_1 := \mu_{\{i\}}$ for each $i \in I$.

It follows from [**11**, Remark 2.9], that for each $\alpha \in \mathcal{F}(I)$ the measure $\widehat{\mu}_\alpha$ is equivalent to the Lebesgue measure λ^α.

According to [**20**, Theorem 2.1], the Lebesgue measure λ on $[0,1]$ admits an admissible strong density φ_1, and since λ is equivalent to $\widehat{\mu}_1$, we get $\varphi_1 \in A\vartheta(\widehat{\mu}_1)$. For the definition of an admissible (strong) density or linear lifting we refer to [**18**] (and [**20**]) or [**19**], respectively.

Applying now [**18**, Theorem 2.6] we find a density $\widetilde{\varphi}$ for the completion of μ_1^I such that $\widetilde{\varphi}$ respects coordinates and

$$\widetilde{\varphi}\big(f_{\{i\}}^{-1}(A)\big) = f_{\{i\}}^{-1}(\varphi_1(A)) \quad \text{for each} \quad i \in I \quad \text{and} \quad A \in \widehat{\mathcal{B}}_0(\Omega_1). \tag{15}$$

Since $\widetilde{\varphi}$ respects coordinates, for each $\alpha \in \mathcal{F}(I)$ there exists exactly one $\varphi_\alpha \in \vartheta(\mu_\alpha)$ such that

$$\widetilde{\varphi}\big(f_\alpha^{-1}(E)\big) = f_\alpha^{-1}(\varphi_\alpha(E)) \quad \text{for each} \quad E \in \widehat{\mathcal{B}}_0(\Omega_\alpha), \tag{16}$$

which implies that $\langle \varphi_\alpha \rangle_{\alpha \in \mathcal{F}(I)}$ is a consistent family of densities.

Since φ_1 is strong, it follows by (15) that $\widetilde{\varphi}$ is strong. So applying (16) we get that φ_α is strong.

Suppose that Proposition 4.2 is true for the directed index set $\mathcal{F}(I)$ instead of κ. Then there exists a strong $\widehat{\varphi} \in \vartheta(\mu)$ such that $\widehat{\varphi}$ is a projective limit of $\langle \varphi_\alpha \rangle_{\alpha \in \mathcal{F}(I)}$, and consequently, $\widehat{\mathcal{B}}_0(\Omega) = \widehat{\mathcal{B}}(\Omega)$ should be true, which means that μ should be completion regular, contradicting to [**11**].

The following Claim has been proven also in [**11**] without using strong densities or liftings.

Claim. There is a countable subset α of I such that the completion of the image measure μ_α of μ under the canonical projection f_α from $[0,1]^I$ onto $[0,1]^\alpha$ is not equivalent to the Lebesgue measure λ_α on $[0,1]^\alpha$.

Proof of claim. Assume that the assertion is false. Then applying the same arguments as above for $I = \omega_1$ we would find a consistent family $\langle \rho_\alpha \rangle_{\alpha < \omega_1}$ of strong densities $\rho_\alpha \in \vartheta(\widehat{\mu}_\alpha)$. Applying Proposition 4.2 we would get then the completion regularity of μ, contradicting its properties. $\square$

Taking into account Lemma 4.1 and Proposition 4.2 one can pose the following question (see also [**27**, Question 3.4]).

QUESTION 4.4. For which projective systems of topological probability spaces with well-ordered index sets do there exist consistent families of strong liftings?

The next result gives a partial answer to Question 4.4.

THEOREM 4.5. *Let κ be an infinite ordinal, $\langle \Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa \rangle$ be a continuous projective system of topological probability spaces and $(\Omega, \mathcal{T}, \Sigma, \mu, \langle f_\alpha \rangle_{\alpha < \kappa})$ be its projective limit. Assume that for all α with $1 \le \alpha < \kappa$ the measure $\widehat{\mu}_\alpha$ supports Ω_α, $(\Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha)$ has USLP, and $\widehat{\mu}_0$ has a strong lifting ρ_0. If for each successor ordinal $\delta = \alpha + 1 < \kappa$ ($\alpha \ge 0$) and each strong lifting $\rho_\alpha \in \Lambda(\widehat{\mu}_\alpha)$ we have*

$$f_{\alpha\delta}^{-1}(\rho_\alpha(B)) \cap G \ne \emptyset \Longrightarrow \mu_\delta\left(f_{\alpha\delta}^{-1}(B) \cap G\right) > 0, \tag{17}$$

when $B \in \Sigma_\alpha^+$ and $G \in \mathcal{T}_\delta$ are arbitrary, then the system admits a consistent family of strong liftings $\rho_\alpha \in \Lambda(\widehat{\mu}_\alpha)$, $\alpha \geq 0$, converging to a strong $\rho \in \Lambda(\widehat{\mu}) \cap proj_{\alpha<\kappa} lim\rho_\alpha$ which is strong w.r.t. $\mathcal{T}^* = proj_{\alpha<\kappa} lim\mathcal{T}_{\rho_\alpha}$.

Proof. We construct by induction on $\delta < \kappa$ a consistent family $\langle\rho_\delta\rangle_{\delta<\kappa}$ of strong liftings $\rho_\delta \in \Lambda(\widehat{\mu}_\delta)$. We start the induction with $\rho_0 \in \Lambda(\widehat{\mu}_0)$.

(a) *Inductive step to a successor ordinal $\delta = \alpha + 1$.* Assume that $\rho_\alpha \in \Lambda(\widehat{\mu}_\alpha)$ has been defined. It then follows by Proposition 2.4 that there exists a strong lifting $\rho_\delta \in \Lambda(\widehat{\mu}_\delta)$ being an inverse image of ρ_α under $f_{\alpha\delta}$.

(b) *Inductive step to a limit ordinal δ.* It follows from Proposition 4.2 that there exists a strong p-lifting $\rho_\delta \in \Lambda(\widehat{\mu}_\delta) \cap proj_{\alpha<\delta} lim\rho_\alpha$. When $\delta = \kappa$, we get the result.

(c) Since $\langle\rho_\alpha\rangle_{\alpha<\kappa}$ is a consistent family of strong liftings, it follows by Proposition 4.2 that there exists a strong lifting $\rho \in \Lambda(\widehat{\mu})$ being a projective limit of $\langle\rho_\alpha\rangle_{\alpha<\kappa}$.

(d) It follows from Lemma 2.1 that for each $\alpha \leq \beta < \kappa$ the map $f_{\alpha\beta}$ is $\mathcal{T}_{\rho_\beta} - \mathcal{T}_{\rho_\alpha}$-continuous hence $\langle\Omega_\alpha, \mathcal{T}_{\rho_\alpha}, \widehat{\Sigma}_\alpha, \widehat{\mu}_\alpha, f_{\alpha\beta}, \kappa\rangle$ is a projective system of complete topological probability spaces. So, applying Lemma 4.1 we get that ρ is strong with respect to $\mathcal{T}^*$ and that $\mathcal{T} \subseteq \mathcal{T}^* \subseteq \mathcal{T}_\rho \subseteq \widehat{\Sigma}$. $\quad\square$

QUESTION 4.6. Assume that $\langle\Omega_\alpha, \mathcal{T}_\alpha, \Sigma_\alpha, \mu_\alpha, f_{\alpha\beta}, \kappa\rangle$ is a continuous projective system of arbitrary not necessarily complete topological probability spaces such that μ_0 has a strong density and for each $1 \leq \alpha < \kappa$ the measure μ_α supports Ω_α and has the *USDP*, i.e. each density for μ_α is almost strong. Does there exist a consistent family $\langle\psi_\alpha\rangle_{\alpha<\kappa}$ of strong densities $\psi_\alpha \in \vartheta(\mu_\alpha)$?

REMARK 4.7. In Theorem 4.5 we have in general that $\mathcal{T} \neq \mathcal{T}^* = proj_{\alpha<\kappa} lim\mathcal{T}_\alpha \neq \mathcal{T}_\rho$. In fact, for each $\delta < \kappa$ let $\mathcal{S}_{<\delta}, f_{<\delta}$ and ρ_δ be as in Theorem 5.2, and let $A \in \mathcal{S}_{\rho_\delta}\backslash\mathcal{S}_{<\delta}$. Then $f_{<\delta}^{-1}(A) \in \mathcal{S}^*$, where $\mathcal{S}^*$ denotes the projective limit of the projective system $\langle\mathcal{S}_{\rho_\delta}\rangle_{\delta<\kappa}$, but $f_{<\delta}^{-1}(A) \notin \mathcal{S}_I$, since if we assume that $f_{<\delta}^{-1}(A) \in \mathcal{S}_I$ we should have $f_{<\delta}(f_{<\delta}^{-1}(A)) = A \in \mathcal{S}_{<\delta}$ because $f_{<\delta}$ is surjective and open, a contradiction. Consequently, $\mathcal{T} \neq \mathcal{T}^*$.

To show that in general $\mathcal{T}^* \neq \mathcal{T}_\rho$, consider the continuous projective system $\langle D_n, \mathcal{T}_n, \mathcal{P}(D_n), \mu_n, f_{mn}, \mathbf{N}\rangle$, where $D_n := \{0, 1\}^n$ endowed with the discrete topology $\mathcal{T}_n$, $\mu_1(\{0\}) = \mu_1(\{1\}) = 1/2$, $\mu_n := \mu_1^n$ and f_{mn} is the canonical projection from D_n onto D_m for each $m \leq n \in \mathbf{N}$. Denote by $(\Omega, \mathcal{T}, \Sigma, \mu, \langle f_n\rangle_{n\in\mathbf{N}})$ its projective limit. Then μ is the usual product measure $\mu_1^\mathbf{N}$ on $\Omega := \{0, 1\}^\mathbf{N}$.

For each $n \in \mathbf{N}$ put $\rho_n := id_{D_n} \in \Lambda(\mu_n)$. Then $\langle\rho_n\rangle_{n\in\mathbf{N}}$ is a consistent sequence of strong liftings, and so by Theorem 4.2 there exists a strong lifting $\rho \in \Lambda(\widehat{\mu})$ being a projective limit of $\langle\rho_n\rangle_{n\in\mathbf{N}}$. Clearly for all $n \in \mathbf{N}$ we have $\mathcal{T}_n = \mathcal{T}_{\rho_n}$ hence $\mathcal{T}^* = \mathcal{T}$. Suppose that $\mathcal{T}^* = \mathcal{T}_\rho$. Then the topological space $(\Omega, \mathcal{T})$ should be extremally disconnected (cf. [**14**, Chapter V, Section 3]), a contradiction, because Ω is homeomorphic to the Cantor set $\Delta \subseteq \mathbf{R}$ endowed with the subspace topology (see e.g. [**5**, Chapter IV, Section 4]) which is not extremaly disconnected.

5. Measures on product spaces. Throughout this section I is an arbitrary index set, $\kappa = \kappa(I)$, $\langle(X_i, \mathcal{S}_i)\rangle_{i\in I}$ and $\langle\Xi_i\rangle_{i\in I}$ are families of topological spaces $(X_i, \mathcal{S}_i)$ and σ-algebras Ξ_i on X_i, respectively, ν is a probability measure on Ξ_I and ν_J the image measure of ν under f_J for each $J \in [I]^{<\kappa}$. To prove the next result, we need first the following lemma.

LEMMA 5.1. *Let $\langle (X_i, \mathcal{S}_i) \rangle_{i \in I}$ be a family of topological spaces and μ a probability measure on $\mathcal{B}(X_I)$ supporting X_I and being τ-additive. Assume that*

$$\mathcal{B}_0(X_J) = \otimes_{i \in J} \mathcal{B}_0(X_i) \quad \text{for each finite} \quad J \subseteq I. \tag{18}$$

Then $\mathcal{B}_0(X_I) = \otimes_{i \in I} \mathcal{B}_0(X_i)$.

Proof. (a) Since μ has full support the space X_I satisfies the *ccc*, i.e. every pairwise disjoint family of open sets in X_I is countable, and so every element of $\mathcal{B}_0(X_I)$ is determined by countably many coordinates; in fact the elements of $\mathcal{B}_0(X_I)$ have the form $f_J^{-1}(C)$, where $C \in \mathcal{B}_0(X_J)$ and J is countable (see [**31**]). Thus $I = \mathbf{N}$ may be assumed without loss of generality.

(b)   In case $I = \mathbf{N}$, let $f : X_{\mathbf{N}} \longrightarrow \mathbf{R}$ be a continuous function. Fix $y = \langle y_n \rangle_{n \in \mathbf{N}}$ in $X_{\mathbf{N}}$ and define $f_n : X_{\mathbf{N}} \longrightarrow \mathbf{R}$ by means of

$$f_n(\langle x_n \rangle_{n \in \mathbf{N}}) := f(\langle x_1, \ldots, x_n, y_{n+1}, y_{n+2}, \ldots \rangle) \quad \text{for} \quad n \in \mathbf{N}.$$

Then each f_n is a continuous function and $\lim_{n \to \infty} f_n(x) = f(x)$. Since the function f_n depends only on the first n coordinates, without loss of generality the result can be reduced to the case where I is finite. So the result follows.   □

The condition (18) in the above lemma is satisfied if for each $i \in I$ the space $(X_i, \mathcal{S}_i)$ is locally compact (see [**3**, lemma on page 326]), or it has a countable network consisting of elements of $\mathcal{B}(X_i)$ (see Lemma 2.6), or it is second countable or a Suslin space (see [**16**]).

The condition (IT) of the next theorem corresponds to (IT) from Theorem 2.7 but now for infinite products.

THEOREM 5.2. *Let I be an arbitrary infinite index set and let $\kappa = \kappa(I)$. For each $i \in I$ let $(X_i, \mathcal{S}_i)$ be a topological space and let Ξ_i be a σ-algebra on X_i. If ν is a probability measure on Ξ_I, then for every $J \in [I]^{<\kappa}$ denote by ν_J the image measure on Ξ_J of μ under the canonical projection f_J. For every $J \in [I]^{<\kappa}$ assume that $\mathcal{S}_J \subseteq \widehat{\Xi}_J$ and $\widehat{\nu}_J$ supports X_J. Fix an index $i_0 \in I$ and assume that ν_{i_0} admits a strong lifting and that for every $J \in [I]^{<\kappa} \backslash \{i_0\}$ the measure $\widehat{\nu}_J$ has the USLP. Then the following conditions are equivalent:*

(i) *If $J \in [I]^{<\kappa}$, $i \in I \backslash J$, $B \in \widehat{\Xi}_J$ and $U \in \mathcal{S}_i$ then*

$$\widehat{\nu}_{J \cup \{i\}}(B \times U) = 0 \implies \widehat{\nu}_J(B) \widehat{\nu}_{\{i\}}(U) = 0. \tag{IT}$$

(ii) *For every $J \in [I]^{<\kappa}$, $i \in I \backslash J$, $B \in \widehat{\Xi}_J$, for every elementary open set $U \in \mathcal{S}_{J \cup \{i\}}$ and for every strong lifting $\rho_J \in \Lambda(\widehat{\nu}_J)$ we have*

$$f_{J,J \cup \{i\}}^{-1}(\rho_J(B)) \cap U \neq \emptyset \implies \widehat{\nu}_{J \cup \{i\}}\left(f_{J,J \cup \{i\}}^{-1}(B) \cap U \right) > 0. \tag{GIT}$$

If one of the above is satisfied, then there exists a strong lifting $\rho \in \Lambda(\widehat{\nu})$. Moreover,

(a) *if $\widehat{\Xi}_J = \widehat{\mathcal{B}}(X_J)$ for every $J \in [I]^{<\kappa}$, then $\widehat{\Xi}_I = \widehat{\mathcal{B}}(X_I)$;*

(b) *if $\Xi_K = \mathcal{B}_0(X_K)$ for every finite $K \subseteq I$ and ν_J is completion regular for every $J \in [I]^{<\kappa}$, then $\Xi_J = \mathcal{B}_0(X_J)$ for every $J \subseteq I$ and ν is completion regular (hence ρ is completion Baire).*

The whole thesis remains true when $\mathrm{card}(I) = \omega_1$ and all spaces $(X_i, \mathcal{T})_{i \neq i_0}$ satisfy the second axiom of countability.

Proof. The equivalence of (*i*) and (*ii*) follows in the same way as in Theorem 2.7. Fix an index $i_0 \in I$ and a strong lifting $\rho_{i_0} \in \Lambda(\widehat{v_{i_0}})$, where $\widehat{v_{i_0}} := \widehat{v}_{\{i_0\}}$. Assume that $I = \kappa := \kappa(I)$ and $i_0 = 0$. Then $\langle X_{<\alpha}, \mathcal{S}_{<\alpha}, \Xi_{<\alpha}, v_\alpha, f_{(<\alpha)(<\beta)}, \kappa \rangle$ is a continuous projective system of topological probability spaces, $(X_I, \mathcal{S}_I, \Xi_I, v, \langle f_{<\alpha} \rangle_{\alpha < \kappa})$ is its projective limit and the existence of a strong $\rho \in \Lambda(\widehat{\mu})$ follows exactly as in Theorem 4.5.

The assertion (*a*) follows in the same way as in the proof of Theorem 2.7.

(*b*) Since $\Xi_K = \mathcal{B}_0(X_K)$ for every finite $K \subseteq I$, it follows by Lemma 5.1 that $\Xi_J = \mathcal{B}_0(X_J)$ for every $J \subseteq I$. On the other hand, the completion regularity of v_J for every $J \in [I]^\kappa$ and assertion (*a*) imply that $\widehat{\Xi}_I = \widehat{\mathcal{B}}_0(X_I) = \widehat{\mathcal{B}}(X_I)$ hence v is completion regular and ρ is completion Baire. □

It remains an open question if assertion (*b*) in the above corollary remains true under the assumption "v_K is completion regular for every finite $K \subseteq I$" instead of "v_J is completion regular for every $J \in [I]^{<\kappa}$".

THEOREM 5.3. *Fix an index $i_0 \in I$ and assume that the inclusion $\mathcal{S}_{i_0} \subseteq \widehat{\Xi}_{i_0}$ holds true, the measure $v_{i_0} := v_{\{i_0\}}$ admits a strong lifting ρ_{i_0}, and that for each $i \in I \backslash \{i_0\}$ the topological space $(X_i, \mathcal{S}_i)$ has a countable network consisting of sets in $\widehat{\Xi}_i$ and the measure v_i supports X_i. Assume also that for every $J \in [I]^{<\kappa}$ we have $\mathcal{S}_J \subseteq \widehat{\Xi}_J$. If v is a probability measure on Ξ_I, then the following conditions are equivalent:*

(i) *Condition (IT) for $\widehat{v}$.*
(ii) *Condition (GIT) for $\widehat{v}$.*
Each of them yields
(ii) *There exists a strong lifting $\rho \in \Lambda(\widehat{v})$.*
If (iii) is satisfied then:
(a) *$\widehat{\Xi}_J = \widehat{\mathcal{B}}(X_J)$ for every $J \in [I]^{<\kappa}$ implies $\widehat{\Xi}_I = \widehat{\mathcal{B}}(X_I)$;*
(b) *If $\Xi_i = \mathcal{B}_0(X_i)$ for every $i \in I$ and $(X_i, \mathcal{S}_i)$ has a countable network consisting of sets in $\mathcal{B}(X_i)$, then v_i is completion regular for every $i \in I$, $\Xi_J = \mathcal{B}_0(X_J)$ for every $J \subseteq I$ and v is completion regular (hence ρ is completion Baire).*

Proof. If I is finite then the result follows by Theorem 2.7 and a finite induction. So we may assume without loss of generality that I is infinite. Moreover we may assume that $I = \kappa := \kappa(I)$ and $i_0 = 0$. Then $\langle X_{<\alpha}, \mathcal{S}_{<\alpha}, \Xi_{<\alpha}, v_\alpha, f_{(<\alpha)(<\beta)}, \kappa \rangle$ is a continuous projective system of topological probability spaces and $(X_I, \mathcal{S}_I, \Xi_I, v, \langle f_{<\alpha} \rangle_{\alpha < \kappa})$ is its projective limit.

The equivalence (i) $\Longleftrightarrow$ (ii) follows in the same way as in the proof of Theorem 5.2. It remains to show that (ii) $\Longrightarrow$ (iii). To this aim we construct by induction on $\delta < \kappa$ a consistent family $\langle \rho_\delta \rangle_{\delta < \kappa}$ of strong liftings $\rho_\delta \in \Lambda(\widehat{v}_\delta)$. The induction starts with $\rho_{i_0} \in \Lambda(\widehat{v}_{i_0})$.

(A) *Inductive step to a limit ordinal ordinal δ*. It follows by Proposition 4.2 that there exists a strong lifting $\rho_\delta \in \Lambda(\widehat{v}_\delta) \cap proj_{\alpha < \delta} \lim \rho_\alpha$.

(B) *Inductive step to a successor ordinal $\delta = \alpha + 1 < \kappa$ ($\alpha \geq 0$)*. Assume that $\rho_\alpha \in \Lambda(\widehat{v}_\alpha)$ has been constructed. It then follows by Theorem 2.7 that there exists a strong inverse image $\rho_\delta \in \Lambda(\widehat{v}_\delta)$ of ρ_α under $f_{\alpha\delta}$.

(C) Since $\langle \rho_\delta \rangle_{\delta < \kappa}$ is a consistent family of strong liftings, it follows by Proposition 4.2 that there exists a strong lifting $\rho \in \Lambda(\widehat{v}) \cap proj_{\delta < \kappa} \lim \rho_\delta$. This completes the proof of the implication (ii) $\Longrightarrow$ (iii).

If the condition (iii) holds true, then the assertion (*a*) follows in the same way as in the proof of Theorem 5.2.

If $\varXi_i = \mathcal{B}_0(X_i)$ for every $i \in I$ then this fact combined with the assumption $\mathcal{S}_i \subseteq \widehat{\varXi}_i$ implies that ν_i is completion regular for every $i \in I$. From Lemma 2.6 and a finite induction we get that $\mathcal{B}_0(X_F) = \otimes_{i \in F} \mathcal{B}_0(X_i)$ for each finite subset F of I. So we may apply Lemma 5.1 to get $\mathcal{B}_0(X_J) = \varXi_J = \otimes_{i \in J} \mathcal{B}_0(X_i)$ for each $J \subseteq I$, hence $\widehat{\mathcal{B}}_0(X_I) = \widehat{\varXi}_I$, which combined with (*a*) implies that $\widehat{\mathcal{B}}_0(X_I) = \widehat{\mathcal{B}}(X_I)$ and therefore ν is completion regular and ρ is completion Baire.   □

The following immediate consequence of Theorem 5.3 is a result of A. and C. Ionescu Tulcea [**14**, Chapter VIII, Section 2, Theorem 5 and its corollary].

COROLLARY 5.4. *Let $\langle X_i \rangle_{i \in I}$ be a family of compact topological spaces and let ν be a positive Radon measure on X_I supporting X_I. Suppose that*
(i) *X_i is metrizable for each $i \in I$,*
(ii) *ν satisfies condition (IT).*
Then ν has a strong lifting.

The next immediate consequence of Theorem 5.3 is a result of Fremlin [**9**, Proposition 453I].

COROLLARY 5.5. *Let $\langle (X_i, \mathcal{S}_i, \varXi_i, \nu_i) \rangle_{i \in I}$ be a family of complete topological probability spaces such that every $(X_i, \mathcal{S}_i)$ has a countable network consisting of measurable sets and every ν_i has full support. Let ν_I be the product measure on X_I. Then $\widehat{\nu}_I$ is a τ-additive measure and has a strong lifting.*

COROLLARY 5.6. *Let $(\Omega, \mathcal{T}, \Sigma, \mu)$ be a topological probability space such that $\widehat{\mu}$ admits a strong lifting ρ, and assume that each $(X_i, \mathcal{S}_i)$ has a countable network consisting of sets in $\widehat{\varXi}_i$ and each ν_i supports X_i. Assume also that $\mu \widehat{\otimes} \nu$, as a measure on the product of the σ-algebras Σ and all $\varXi_i$, $i \in I$, satisfies condition (IT). Then $\mu \widehat{\otimes} \nu$ admits a strong lifting φ being an inverse image of ρ under the canonical projection from $\Omega \times X_I$ onto Ω.*

In particular, if $\Sigma = \mathcal{B}_0(\Omega)$, for each $i \in I$ the equality $\varXi_i = \mathcal{B}_0(X_i)$ holds true and $(X_i, \mathcal{S}_i)$ has a countable network consisting of sets from $\mathcal{B}(X_i)$, then $\Sigma \otimes \varXi_I = \mathcal{B}_0(\Omega \times X_I)$, $\mu \otimes \nu$ is completion regular and φ is in addition completion Baire.

Proof. Let J be an arbitrary index set, $i_0 \in J \backslash I$ and $(X_{i_0}, \mathcal{S}_{i_0}, \varXi_{i_0}, \nu_{i_0}) = (\Omega, \mathcal{T}, \Sigma, \mu)$. Put $I_0 := I \cup \{i_0\}$. Then the families $\langle (X_i, \mathcal{S}) \rangle_{i \in I_0}$ and $\langle \varXi_i \rangle_{i \in I_0}$ together with the probability measure $\mu \otimes \nu$ on $\varXi_I$ satisfy the assumptions of Theorem 5.3 and so the result follows.   □

The above corollary contains Theorem 3 of [**4**].

THEOREM 5.7. *Fix an index $i_0 \in I$ and assume that the measure ν_{i_0} admits a strong lifting ρ_{i_0}. Assume that for every $i \in I \backslash \{i_0\}$ there exists a family $\langle (X_j^i, \mathcal{S}_j^i) \rangle_{j \in I_i}$ of topological spaces and a family $\langle (\varXi_j^i) \rangle_{j \in I_i}$ of σ-algebras $\varXi_J^i$ on X_j^i, respectively, such that for every $j \in I_i$ the space $(X_j^i, \mathcal{S}_j^i)$ has a countable network consisting of sets in $\widehat{\varXi}_j^i$. Assume that $X_i = \otimes_{j \in I_i} X_j^i$, $\mathcal{S}_i = \otimes_{j \in I_i} \mathcal{S}_j^i$ and $\varXi_i = \otimes_{j \in I_i} \varXi_j^i$. Assume also that for every $J \in [I]^{<\kappa}$ and for every $i \in I \backslash J$ the measure $\widehat{\nu}_{J \cup \{i\}}$, as a measure on the completed product of the σ-algebras $\varXi_J$ and $\varXi_j^i$ for all $j \in I_i$, satisfies condition (IT). Then the measure $\widehat{\nu}$ admits a strong lifting.*

Proof. If I is finite then the result follows from Theorem 5.3 and finite induction. So we may assume that I is infinite. We may also assume that $I = \kappa := \kappa(I)$ and $i_0 = 0$.

Then $\langle X_{<\alpha}, \mathcal{S}_{<\alpha}, \varXi_{<\alpha}, \nu_{\alpha}, f_{(<\alpha)(<\beta)}, \kappa \rangle$ is a continuous projective system of topological probability spaces and $(X_I, \mathcal{S}_I, \varXi_I, \nu, \langle f_{<\alpha} \rangle_{\alpha<\kappa})$ is its projective limit.

We construct by induction on $\delta < \kappa$ a consistent family $\langle \rho_\delta \rangle_{\delta<\kappa}$ of strong liftings $\rho_\delta \in \varLambda(\widehat{\nu_\delta})$. The induction starts with $\rho_{i_0} \in \varLambda(\widehat{\nu_{i_0}})$. The rest of the proof follows in the same way as in Theorem 5.3, proof of the implication $(ii) \implies (iii)$. We have only to replace Theorem 2.7 by Theorem 5.3. $\qquad \square$

The first result on completion regularity is the following classical theorem of Kakutani [**15**]: If $\langle X_i \rangle_{i \in I}$ is a family of compact metric spaces and for every $i \in I$, ν_i is a Radon probability measure on X_i with full support, then the Radon product measure of $\langle \nu_i \rangle_{i \in I}$ on X_I is completion regular. An immediate consequence of Theorem 5.7 is the following generalization of the above classical result of [**15**].

COROLLARY 5.8. *Under the assumptions of Theorem 5.7 and the assumptions that* $\varXi_{i_0} = \mathcal{B}_0(X_{i_0})$, *for every* $i \in I \setminus \{i_0\}$ *the equality* $\varXi_j^i = \mathcal{B}_0(X_j^i)$ *holds true and* $(X_j^i, \mathcal{S}_j^i)$ *has a countable network consisting of sets from* $\mathcal{B}(X_j^i)$, *it follows that for every* $J \subseteq I$ *the equality* $\varXi_J = \mathcal{B}_0(X_J)$ *holds true, the measure* ν *is completion regular and there exists a strong completion Baire lifting for* $\widehat{\nu}$.

Proof. According to Theorem 5.3 we have that for each $i \in I$ the equality $\varXi_i = \mathcal{B}_0(X_i)$ holds true, the measure ν_i is completion regular and $\widehat{\nu_i}$ admits a strong lifting. Again by Theorem 5.3 and finite induction we get that $\mathcal{B}_0(X_F) = \otimes_{i \in F} \mathcal{B}_0(X_i)$ for every finite $F \subseteq I$. On the other hand, according to Theorem 5.7 the measure $\widehat{\nu}$ admits a strong p-lifting ρ, it is τ-additive, and the inclusion $\mathcal{S}_I \subseteq \widehat{\varXi}_I$ holds true. So we may apply Lemma 5.1 to get the equality $\mathcal{B}_0(X_J) = \otimes_{i \in J} \mathcal{B}_0(X_i)$ for each $J \subseteq I$, hence $\varXi_I = \otimes_{i \in I} \mathcal{B}_0(X_i) = \mathcal{B}_0(X_I)$. Consequently, $\widehat{\varXi}_I = \widehat{\mathcal{B}_0}(X_I)$, which combined with the inclusion $\mathcal{S}_I \subseteq \widehat{\varXi}_I$ implies $\widehat{\mathcal{B}}(X_I) \subseteq \widehat{\mathcal{B}_0}(X_I)$ hence $\widehat{\mathcal{B}}(X_I) = \widehat{\mathcal{B}_0}(X_I)$. Thus, ν is completion regular and ρ is in addition completion Baire. $\qquad \square$

The following result, which is a consequence of Theorem 5.7 and of [**12**, Theorem 3.1], is related with Question 5.5.

COROLLARY 5.9. *Let* $(\Omega, \mathcal{T}, \widehat{\mathcal{B}}(\Omega), \mu)$ *be a complete topological probability space such that* $(\Omega, \mathcal{T})$ *is completely regular and* μ *is completion regular and* τ-additive. Assume *that for every* $i \in I$ *there exists a family* $\langle (X_j^i, \mathcal{S}_j^i) \rangle_{j \in I_i}$ *of separable metric spaces and* ν_i *has full support, and that for every* $J \in [I]^{<\kappa}$ *and for every* $i \in I \setminus J$ *the measure* $\widehat{\nu}_{J \cup \{i\}}$, *as a measure on the completed product of* $\varXi_J$ *and* $\mathcal{B}(X_j^i)$ *for all* $j \in I_i$, *satisfies condition* (IT). *Then the* τ-additive product measure (see [**12**, page 331] for the definition) of μ and $\otimes_{i \in I} \nu_i$ *is completion regular.*

Proof. By Corollary 5.8 it follows that ν_I is completion regular. So the result follows from [**12**, Theorem 3.1]. $\qquad \square$

REMARKS 5.10. (a) It should be noted that according to [**14**, page 120], a measure μ satisfying the property (IT) as in Theorem 5.2 is not in general equivalent with a product measure.

(b) The measure μ of Example 4.3 does not satisfy condition (IT) of Theorems 5.2 and of 5.3.

In fact, assuming that μ satisfies condition (IT) and that the continuum hypothesis (CH) is satisfied, we get from the above corollary the completion regularity of μ, a contradiction.

It should be noted that subject to (CH) the measure μ admits a strong lifting (see [**7**] or [**28**]).

(c) Subject to (CH), the Wiener measure W defined on the completion of the Borel σ-algebra of $\Omega := \overline{\mathbf{R}}^{[0,1]} = \overline{\mathbf{R}}^{\omega_1}$ where $\overline{\mathbf{R}} = [+\infty, -\infty]$ does not satisfy condition (IT). In fact, let $\Omega_\alpha := \overline{\mathbf{R}}^\alpha$, f_α be the canonical projection of $\overline{\mathbf{R}}^{\omega_1}$ onto $\overline{\mathbf{R}}^\alpha$ for every $\alpha < \omega_1$ and, W_α be the completed image of W under f_α.

Since each W_α has the $USLP$, supports Ω_α, and is completion regular but W is not completion regular, it follows by Corollary 5.6 that μ does not satisfy condition (IT). The measure W has the $USLP$ because the support of W is the polish space $C(\overline{\mathbf{R}})$. Ω is not second countable, because if Ω were second countable then W should be completion regular, a contradiction. Moreover, Ω has no countable network (see [**13**] for the definition) since for compact spaces Ω the net weight, i.e. the least cardinality of a network for Ω, is equal to the weight of Ω (see Gruenhage [**13**, Chapter 10, Section 1)]. So $(X, \widehat{\mathcal{B}}(\Omega), W)$ is an example of a topological probability space with the $USLP$ but without countable base and without countable network.

(d) Fremlin's simplification [**8**] of Losert's counter-example to the strong lifting conjecture gives a Radon probability measure μ on $\Omega := \{0, 1\}^{\aleph_2}$ supporting Ω, being completion regular and having no a strong lifting. It follows by Theorem 5.3 and above that the measure μ does not satisfy condition (IT).

6. Consequences.

We recall the following definition taken from [**32**, Chapter I, Section 6, page 46]. Let ν be a Radon measure on a topological space $(\Theta, \mathcal{S})$. A ν-**concassage** of Θ is a partition of Θ into $N \cup \bigcup_{i \in I} K_i$, where N is a ν-null set and K_i are pairwise disjoint compact sets such that every point of Θ has a neighbourhood which meets at most a countable number of K_i.

Let κ be an infinite ordinal. We say that (Θ, ν) is **locally of order κ**, if there exists a concassage $\langle\langle K_i\rangle_{i\in I}, N\rangle$ of Θ such that the topology induced on every K_i has a basis of cardinality not exceeding κ (see [**2**]).

The following result has been proved in [**2**, page 22] for Radon measures of full support on a locally compact Hausdorff topological space, but we extend it to general (Hausdorff) topological spaces. Although the proof essentially repeats the arguments of [**2**] we give it for completeness.

LEMMA 6.1. *Let κ be an infinite ordinal. In order that every (completion regular and) Radon measure ν of full support on a (normal) topological space Θ, such that (Θ, ν) is locally of order κ, has a strong (completion Baire) lifting it is necessary and sufficient that every (completion regular and) Radon measure ν of full support on $[0, 1]^\kappa$ has a strong (completion Baire) lifting.*

Proof. The necessity is obvious. Let ν be a Radon measure of full support on Θ. Suppose that (Θ, ν) is locally of order κ. Applying [**32**, Chapter I, Section 6, Theorem 13], we may choose a concassage $\langle\langle K_i\rangle_{i\in I}, N\rangle$ of Θ such that the restriction $\nu \mid K_i$ of ν to each K_i is of full support. Indeed, let $\langle\langle L_i\rangle_{i\in I}, N\rangle$ be a concassage of Θ. For every $i \in I$ let $K_i \subseteq L_i$ be the support of $\nu|L_i$. Since each L_i has a basis of cardinality less or equal to κ, the same holds true for K_i. Exactly as in Schwartz [**32**] one proves that $\Theta\setminus\bigcup_i K_i$ is ν-measurable. We want to prove that $\nu(\Theta\setminus\bigcup_i K_i) = 0$. So take a compact $H \subset \Theta\setminus\bigcup_i K_i$ and take for every $\theta \in H$ a neighborhood U_θ such that U_θ has non-empty intersection only with countably many sets $L_{\theta,i}$. Due to the compactness of H there exist sets $U_1, \ldots, U_n \in \{L_{\theta,i} : \theta \in \Theta, i \in I\}$ covering H. Assume that

$U_m \cap L_{m,j} \neq \emptyset$, $j = 1, \ldots$ and $U_m \cap L_j = \emptyset$ for other L_j. Thus, $U_m \cap (\Theta \setminus \bigcup_{i,j} L_{i,j} \setminus N) = \emptyset$ for every $m \leq n$. Consequently, $H \cap (\Theta \setminus \bigcup_{i,j} L_{i,j} \setminus N) = \emptyset$, what yields $H \subset \bigcup_{i,j} L_{i,j} \cup N$. As $\nu(H \cap L_{i,j}) = 0$ for all i, j, we get $\nu(H) = 0$.

For each $i \in I$ there is a homeomorphism h_i from K_i onto a closed subset $h_i(K_i)$ of $[0, 1]^\kappa$. Denote by $\widetilde{\nu}_i$ the image measure of $\nu \mid K_i$ under h_i for $i \in I$, and by λ the Lebesgue measure on $[0, 1]$. Then $\nu_i := \lambda^\kappa + \widetilde{\nu}_i$ supports $[0, 1]^\kappa$ and by our assumption there exists a strong lifting $\overline{\rho}_i \in \Lambda(\nu_i)$.

Since $\nu \mid K_i$ supports K_i, it follows that $\widetilde{\nu}_i \mid h_i(K_i)$ supports $h_i(K_i)$ for each $i \in I$, hence we may apply [**17**, Lemma 2.3], to find a strong lifting in $\Lambda(\nu_i \mid \widehat{\mathcal{B}}([0, 1]^\kappa) \cap h_i(K_i))$, and again [**17**, Corollary 2.6], to find a strong lifting $\rho_i \in \Lambda(\widetilde{\nu}_i \mid h_i(K_i))$. Since h_i is a homeomorhism there exists an inverse image $\tau_i \in \Lambda(\nu \mid K_i)$ of ρ_i under h_i for each $i \in I$, which is a strong lifting. It then follows from [**17**, Theorem 2.4], that there exists a strong lifting in $\Lambda(\nu)$.

If in addition Θ is a normal topological space and ν is completion regular then it follows from [**6**, Lemma 2.3] that for each $i \in I$ we have $\mathcal{B}_0(K_i) = \mathcal{B}_0 \cap K_i$ according to [**6**, Lemma 2.3], hence the measure $\nu \mid K_i$ is completion regular, and so applying [**4**, Section 3, Lemma 1], we have that ν_i is completion regular, and again by [**6**, Lemma 2.3], $\nu_i \mid h_i(K_i)$ is completion regular. Consequently, the above liftings $\overline{\rho}_i$, $\widetilde{\rho}_i$, ρ_i, τ_i and τ are strong completion Baire and so the result follows. $\qquad\square$

QUESTION 6.2. Can we eliminate the completeness assumption for the topological measure spaces in Lemma 6.1?

DEFINITION 6.3. Let κ be an infinite ordinal and let ν be a Radon measure on a topological space $(\Theta, \mathcal{S})$ such that (Θ, ν) is locally of order κ. Let $\langle \nu_i \rangle_{i \in I}$ be the family of the measures ν_i on $[0, 1]^\kappa$ associated with (Θ, ν) as in the proof of Lemma 6.1. We say that the measure ν satisfies the property (GP^*), if for every $i \in I$ the measure ν_i satisfies the property (IT).

THEOREM 6.4. *Let ν be an arbitrary (completion regular and) Radon measure of full support on a (normal) toplogical space Θ satisfying the property (GP^*). Then ν admits a strong (completion Baire) lifting.*

Proof. Let (Θ, ν) be of locally order κ, where κ is an infinite ordinal. Then the result follows by Lemma 6.1 and by Theorem 5.3. $\qquad\square$

THEOREM 6.5. *Let μ and ν be Radon measures of full support on the topological spaces $(\Omega, \mathcal{T})$ and $(\Theta, \mathcal{S})$ respectively such that (Θ, ν) is locally of order κ, where κ is an arbitrary infinite ordinal. Let $\langle \nu_i \rangle_{i \in I}$ be the family of the measures ν_i on $[0, 1]^\kappa$ associated with (Θ, ν) as in the definition 6.2. Assume that μ admits a strong lifting ρ and $\mu \widehat{\otimes} \nu$ has the property (GP^*). Then $\mu \widehat{\otimes} \nu$ admits a strong lifting being an inverse image of ρ under the canonical projection.*

In particular, if μ and ν are completion regular and $(\Theta, \mathcal{S})$ is a normal topological space, then the measure $\mu \widehat{\otimes} \nu$ admits a strong completion Baire lifting being an inverse image of ρ under the canonical projection.

Proof. Let κ be an infinite ordinal such that $(\Theta, \mathcal{S})$ is locally of order κ. As in the proof of Lemma 6.1, there exists a concassage $\langle \langle K_i \rangle_{i \in I}, N \rangle$ of Θ such that the restriction $\nu \mid K_i$ of ν to each K_i is of full support and each topology $\mathcal{S} \cap K_i$ has a basis not exceeding κ. For each $i \in I$ there is a homeomorhism h_i from K_i onto a closed subset $h_i(K_i)$ of $[0, 1]^\kappa$. According to Corollary 5.6, for each $i \in I$ the measure

$\mu \,\widehat{\otimes}\, \nu_i$ has a strong p-lifting, and by [**17**, Lemma 2.3], the restriction of $\mu \,\widehat{\otimes}\, \nu_i$ to $\widehat{\mathcal{B}}(\Omega \times h_i(K_i))$, has a strong lifting. But again by [**17**, Corollary 2.6], the restriction of $\mu \,\widehat{\otimes}\, \widetilde{\nu}_i$ to $\widehat{\mathcal{B}}(\Omega \times h_i(K_i))$ has a strong lifting, and so since h_i is a homeomorphism we get that the measure $\mu \,\widehat{\otimes}\, (\nu \mid K_i)$ admits a strong lifting for each $i \in I$.

Consequently, applying [**17**, Theorem 2.4] we get a strong lifting for $\mu \,\widehat{\otimes}\, \nu$ being an inverse image of ρ .

In particular, if μ and ν are completion regular and $(\Theta, \mathcal{S})$ is a normal topological space, then according to Corollary 5.6, for each $i \in I$ the measure $\mu \,\widehat{\otimes}\, \nu_i$ has a strong completion Baire p-lifting, and by [**17**, Lemma 2.3], the restriction of $\mu \,\widehat{\otimes}\, \nu_i$ to $\widehat{\mathcal{B}}(\Omega \times h_i(K_i))$, which according to [**6**, Lemma 2.3], is completion regular, has a strong lifting. But again by [**17**, Corollary 2.6], the restriction of $\mu \,\widehat{\otimes}\, \widetilde{\nu}_i$ to $\widehat{\mathcal{B}}(\Omega \times h_i(K_i))$, which is completion regular, has a strong lifting, and so since h_i is a homeomorphism we get that the measure $\mu \,\widehat{\otimes}\, (\nu \mid K_i)$ admits a strong completion Baire lifting for each $i \in I$.

Consequently, applying [**17**, Theorem 2.4] we get a strong completion Baire lifting for $\mu \,\widehat{\otimes}\, \nu$ being an inverse image of ρ. This completes the proof. $\qquad\square$

The above theorem extends Corollary 5.6, while Theorem 6.4 improves Theorem 5.3.

Theorem 6.5 gives a positive answer to Kupka's problem for a wide class of topological probability spaces. The same theorem answers Question 2.3 to the positive for Radon measure spaces $(\Theta, \mathcal{S}, \widehat{\mathcal{B}}(\Theta), \nu)$ and

$$(\Omega, \mathcal{T}, \widehat{\mathcal{B}}(\Omega), \mu) = (\Theta, \mathcal{S}, \widehat{\mathcal{B}}(\Theta), \nu) \,\widehat{\otimes}\, (\widetilde{\Theta}, \widetilde{\mathcal{S}}, \widehat{\mathcal{B}}(\widetilde{\Theta}), \widetilde{\nu}),$$

where $(\widetilde{\Theta}, \widetilde{\mathcal{S}}, \widehat{\mathcal{B}}(\widetilde{\Theta}), \widetilde{\nu})$ is as in Theorem 6.4.

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