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Electromagnetism of one-component plasmas of massless fermions

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We present a theoretical study of two- and three-dimensional massless Dirac one-component plasmas embedded in a constant uniform magnetic field. We determine the wavefunctions and Landau energy levels of a massless Dirac fermion in a constant magnetic field. On this basis we consider magnetism of Fermi fluids of massless charged particles. We show that such a three-dimensional Dirac plasma consisting of fermions with the same helicity has its own magnetic moment. We also consider the limit of strong magnetic fields and investigate the De Haas–van Alphen effect. We derive the Kubo formula for the electrical conductivity tensor of massless Dirac plasmas and consider the Shubnikov–de Haas effect. In addition, we propose a model of the static conductivity tensor and employ the matrix version of the classical method of moments to derive a Drude-like formula for the dynamic conductivity tensor for massless Dirac plasmas. We find that the electrical conductivity tensor for Dirac fermions with the right helicity is not isotropic in the plane perpendicular to the magnetic field.

Key words: quantum plasma, strongly coupled plasmas, plasma properties

1. Introduction

In Sarma & Hwang (2009), the investigation of dynamic properties of massless Dirac one-component plasmas was carried out in one-dimensional (1-D), two-dimensional (2-D) and three-dimensional (3-D) cases at zero temperature in the random-phase approximation. In the present work we carry out a theoretical study of the electromagnetic properties of 3-D finite-temperature plasmas of massless fermions embedded in a constant uniform magnetic field. We define such a Dirac plasma as a system of charged carriers with the linear energy dispersion $\epsilon_k = \hbar k v$ described by the Dirac–Weyl-like Hamiltonian

$$\hat{H} = -i\hbar v \hat{\boldsymbol{\sigma}} \cdot \nabla. \tag{1.1}$$

Here $\hbar$ is the Planck constant, $\hat{\boldsymbol{\sigma}}$ is the set of Pauli matrices and v is the constant speed of a Dirac fermion, which in (1.1) plays the role of the effective speed of light. Details regarding (1.1) are discussed in § 2. A very important new aspect of such a system is that

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the momentum $\hbar\boldsymbol{k}$ and the velocity magnitude v decouple from each other: the momentum satisfies the equation of motion, but the velocity determines the displacement. The velocity magnitude remains constant and interactions change its direction only. In addition, the spin of a Dirac fermion can take only two directions: along the momentum or against it. In the first case, the fermion has the positive helicity, and in the second, the negative one.

The most common and studied case of a plasma of massless particles is the 2-D system of Dirac fermions in graphene (Novoselov *et al.* 2005; Katsnelson, Novoselov & Geim 2006; Castro Neto *et al.* 2009), with the quasiparticles described by the Hamiltonian (1.1) as if they were massless relativistic particles with the momentum $\hbar\boldsymbol{k}$ (for example, photons) with the light speed c substituted by the Fermi speed $v \approx c/300$. In Andersen, Jacobsen & Thygesen (2014) and Enaldiev (2018), the existence of 1-D atomically confined plasmons at the edges of a zigzag nanoribbon was predicted, and collective excitations were considered in a two-component 1-D Dirac plasma. In Shaisultanov, Lyubarsky & Eichler (2012), the 3-D ultrarelativistic plasma with the linear energy dispersion of quasiparticles was also studied in the context of quark–gluon plasmas (Heinz 2009; Moldabekov *et al.* 2015). The spatial and magnetic confinement, the helical magnetic effect, and the chiral anomaly in Dirac plasmas were studied in Ren *et al.* (2021) and Yamamoto & Yang (2021).

In what follows we consider one-component 2-D and 3-D Dirac plasmas with a neutralizing background in a constant uniform magnetic field. The aim of this paper is to consider the electromagnetic properties of such a Dirac plasma. To this end, using the Dirac–Weyl equation, we determine the wavefunctions and Landau energy levels of a massless fermion in a constant magnetic field and demonstrate that these energy levels are formally similar to the energy spectrum of a 3-D relativistic electron in a constant magnetic field. However, unlike in the case of an electron with a non-zero mass, we find that the cyclotron frequency of massless fermions is explicitly non-classical being proportional to $1/\sqrt{\hbar}$. Further, we show that a 3-D Dirac plasma consisting of fermions with the same helicity in a weak magnetic field possesses its own magnetic moment, and we calculate this magnetic moment. In the case of strong magnetic fields we consider the De Haas–van Alphen (DHVA) effect, i.e. the giant oscillations of the Dirac plasma magnetization.

Next, within the framework of the linear response theory we derive a Kubo-like formula for the electrical conductivity tensor of a Dirac plasma. We apply it to consider the Shubnikov–de Haas (SdH) effect, the large amplitude oscillations of the 2-D and 3-D magnetized Dirac plasma conductivity. It stems from our study that both DHVA and SdH effects are direct consequences of the Landau diamagnetism. These results are complemented by numerical calculations of the magnetic moment and the static electrical conductivity of a Dirac plasma.

Finally, we apply the matrix version of the method of moments (see e.g. Kovalishina 1983; Adamyan & Tkachenko 2000) to construct the Drude–Lorentz-like model of the Dirac plasma dynamic conductivity tensor. Additional aspects, namely, the normalization condition of the Fermi–Dirac distribution density in the cases of 2D and 3-D massless fermions in a magnetic field and the mathematical details of the matrix method of moments are described in the Supplemental material available at https://doi.org/10.1017/S0022377823000752 and in appendix B.

The paper is organized as follows. Section 2 represents a brief overview of the theory of the Dirac and Weyl equations. In § 3 we solve the Dirac–Weyl equation for a massless Dirac fermion in a constant magnetic field. In § 4 we consider the magnetism of Dirac plasmas in weak magnetic fields and analyse the DHVA effect. Next, in § 5 we derive the Kubo formula for the electrical conductivity tensor of Dirac plasmas, consider the

SdH effect in strong magnetic fields, and employ the classical method of moments to construct the dynamic internal conductivity tensor in the long-wavelength case. Section 6 concludes the work. In appendix A some coefficients are defined. In appendix B the matrix Hamburger problem, Nevanlinna's formula and the Drude–Lorentz-type formula for the conductivity tensor of a magnetized Dirac plasma are considered. Appendix C presents the static electrical conductivity tensor of a 3-D Dirac plasma in a magnetic field and in appendix D the normalization in a magnetic field is considered. In the Supplemental material the following are considered: (1) the Kramers–Kronig relations; (2) some mathematical formulae.

2. Dirac and Weyl equations

Relativistically invariant equations for electrons were obtained by Dirac. They consist of the following system of homogeneous first-order differential equations for two spinor functions $\hat{\varphi}$ and $\hat{\chi}$ (Berestetskii, Pitaevskii & Lifshitz 1982):

$$i\hbar\frac{\partial\hat{\varphi}}{\partial t} = mc^2\hat{\varphi} + c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\hat{\chi}, \quad i\hbar\frac{\partial\hat{\chi}}{\partial t} = -mc^2\hat{\chi} + c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\hat{\varphi}. \tag{2.1a,b}$$

Here, m is the electron mass, $\hat{\boldsymbol{p}} = -\mathrm{i}\hbar\nabla$ is the momentum operator, c is the speed of light and $\hat{\boldsymbol{\sigma}}$ is the set of Pauli matrices

$$\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \hat{\sigma}_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.2a–c}$$

In accordance to the general relation between the infinitesimal rotation operator and the angular momentum the matrix $\frac{1}{2}\hat{\boldsymbol{\sigma}}$ is the spin operator. The commutational properties of the Pauli matrices are those of the angular momentum operator components,

$$\hat{\sigma}_x\hat{\sigma}_y - \hat{\sigma}_y\hat{\sigma}_x = 2\mathrm{i}\hat{\sigma}_z, \quad \hat{\sigma}_y\hat{\sigma}_z - \hat{\sigma}_z\hat{\sigma}_y = 2\mathrm{i}\hat{\sigma}_x, \quad \hat{\sigma}_z\hat{\sigma}_x - \hat{\sigma}_x\hat{\sigma}_z = 2\mathrm{i}\hat{\sigma}_y. \tag{2.3a–c}$$

They satisfy also the following relations:

$$\hat{\sigma}_i\hat{\sigma}_k + \hat{\sigma}_k\hat{\sigma}_i = 2\delta_{ik}, \tag{2.4}$$

where δ_{ik} is the Kronecker delta. Let us introduce the bispinor function

$$\hat{\Psi} = \begin{pmatrix} \hat{\varphi} \\ \hat{\chi} \end{pmatrix}. \tag{2.5}$$

The components of the bispinor (2.5) can be subjected to the unitary transformation

$$\hat{\Psi}' = \begin{pmatrix} \hat{\xi} \\ \hat{\eta} \end{pmatrix} = \hat{\mathrm{U}}\hat{\Psi}, \tag{2.6}$$

where $\hat{\mathrm{U}}$ is a certain unitary four-row matrix: $\hat{\mathrm{U}}\hat{\mathrm{U}}^\dagger = \hat{\mathrm{I}}$, for example

$$\hat{\mathrm{U}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \hat{I} & \hat{I} \\ \hat{I} & -\hat{I} \end{pmatrix}, \tag{2.7}$$

which makes the spinors $\hat{\xi}$ and $\hat{\eta}$ satisfy the following equations:

$$i\hbar\frac{\partial\hat{\xi}}{\partial t} = c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\hat{\xi} + mc^2\hat{\eta}, \quad i\hbar\frac{\partial\hat{\eta}}{\partial t} = -c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\hat{\eta} + mc^2\hat{\xi}. \tag{2.8a,b}$$

As it follows from (2.8a,b), due to the presence of the particle mass m, the first equation in (2.8a,b) for the spinor $\hat{\xi}$ contains the components of the spinor $\hat{\eta}$, while the second

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equation for $\hat{\eta}$ contains the components of the spinor $\hat{\xi}$. But if the particle mass is equal to zero ($m = 0$), there is no such mixing of the components of the spinors, and the equations in (2.8*a*,*b*) transform into two independent ones,

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\hat{\xi}\\ \hat{\eta}\end{pmatrix} = c\hat{\boldsymbol{\sigma}}\cdot\hat{\boldsymbol{p}}\begin{pmatrix}\hat{\xi}\\ -\hat{\eta}\end{pmatrix} \tag{2.9}$$

called the Weyl equations. These equations, like the original Dirac equations, are relativistically invariant. However, it can be easily seen that these equations will not be invariant under the transformation of space inversion. In other words, the two-component wavefunction satisfying (2.9), may be used to describe a particle of zero mass with the spin $1/2$ only if we drop the requirement of the invariance of the equations under space inversion. Such a situation exists in the case of the neutrinos which is described by (2.9); here the minus sign before $\hat{\eta}$ corresponds to the neutrino, while the plus sign corresponds to the antineutrino (Bjorken & Drell 1964). The behaviour of the massless Dirac fermion in an external electromagnetic field with the vector potential $\boldsymbol{A}$ can also be described by the Weyl equations

$$i\hbar\frac{\partial}{\partial t}\begin{pmatrix}\hat{\xi}\\ \hat{\eta}\end{pmatrix} = v\hat{\boldsymbol{\sigma}}\cdot\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}\right)\begin{pmatrix}\hat{\xi}\\ -\hat{\eta}\end{pmatrix}, \tag{2.10}$$

with the speed of light c replaced by the speed of a massless fermion v. The equation for the spinor $\hat{\xi}$ in (2.10) corresponds to the massless fermion with the right-handed helicity (the Dirac fermion spin is directed along the momentum $\boldsymbol{p}$) and the equation for the spinor $\hat{\eta}$ corresponds to the massless fermion with the left-handed helicity (with the spin directed against the momentum $\boldsymbol{p}$).

3. Massless fermions in a constant magnetic field

Let us determine the energy levels of a massless fermion in a constant uniform magnetic field. The vector potential of the uniform magnetic field $\boldsymbol{H}$ (with the z-axis directed along the magnetic field $\boldsymbol{H}$) is conveniently taken here as

$$A_x = -Hy, \quad A_y = A_z = 0. \tag{3.1a,b}$$

Then the equations in (2.10), in the 3-D case, can be written as

$$\left.\begin{aligned} \mathrm{i}\hbar\frac{\partial\hat{\xi}(\boldsymbol{r},t)}{\partial t} &= v\hat{\sigma}_x\left(\hat{p}_x - \frac{eH}{c}y\right)\hat{\xi}(\boldsymbol{r},t) + v\hat{\sigma}_y\hat{p}_y\hat{\xi}(\boldsymbol{r},t) + v\hat{\sigma}_z\hat{p}_z\hat{\xi}(\boldsymbol{r},t),\\ \mathrm{i}\hbar\frac{\partial\hat{\eta}(\boldsymbol{r},t)}{\partial t} &= -v\hat{\sigma}_x\left(\hat{p}_x - \frac{eH}{c}y\right)\hat{\eta}(\boldsymbol{r},t) - v\hat{\sigma}_y\hat{p}_y\hat{\eta}(\boldsymbol{r},t) - v\hat{\sigma}_z\hat{p}_z\hat{\eta}(\boldsymbol{r},t).\end{aligned}\right\} \tag{3.2}$$

First of all, we notice that the operators $\hat{p}_x$ and $\hat{p}_z$ commute with the right-hand sides of the equations in (3.2). This means that the x and z components of the generalized momentum are conserved. This is why we seek the solution of (3.2) in the form

$$\left.\begin{aligned} \hat{\xi}(\boldsymbol{r},t) &= \exp\left(\frac{\mathrm{i}}{\hbar}(p_x x + p_z z - \epsilon t)\right)\hat{\xi}(y), \quad \hat{\xi}(y) = \begin{pmatrix}u_+(y)\\ d_+(y)\end{pmatrix},\\ \hat{\eta}(\boldsymbol{r},t) &= \exp\left(-\frac{\mathrm{i}}{\hbar}(p_x x + p_z z + \epsilon t)\right)\hat{\eta}(y), \quad \hat{\eta}(y) = \begin{pmatrix}u_-(y)\\ d_-(y)\end{pmatrix},\end{aligned}\right\} \tag{3.3}$$

with the eigenvalues p_x and p_z being arbitrary real numbers. Since $A_z = 0$, the z-component of the generalized momentum can have any value, i.e. the motion of the massless Dirac

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fermion along the magnetic field is 'not quantized'. In view of (3.3), the system of (3.2), transforms into ($\boldsymbol{p} = \hbar \boldsymbol{k}$) the following:

$$\left.\begin{aligned} v\hbar\frac{\partial u_+(y)}{\partial y} &= (\epsilon_+ + v\hbar k_z)\, d_+(y) + v\left(\frac{e}{c}Hy - \hbar k_x\right) u_+(y), \\ v\hbar\frac{\partial d_+(y)}{\partial y} &= -\,(\epsilon_+ - v\hbar k_z)\, u_+(y) - v\left(\frac{e}{c}Hy - \hbar k_x\right) d_+(y), \\ v\hbar\frac{\partial u_-(y)}{\partial y} &= (\epsilon_- - v\hbar k_z)\, d_-(y) - v\left(\frac{e}{c}Hy - \hbar k_x\right) u_-(y), \\ v\hbar\frac{\partial d_-(y)}{\partial y} &= -\,(\epsilon_- + v\hbar k_z)\, u_-(y) + v\left(\frac{e}{c}Hy - \hbar k_x\right) d_-(y). \end{aligned}\right\} \tag{3.4}$$

Solving the system of differential equations in (3.4), after some algebra, we arrive at the following equations:

$$\frac{\partial^2 u_{n,\pm}(\zeta)}{\partial \zeta^2} + \frac{2}{\hbar^2\omega_{\rm H}^2}\left\{\epsilon_{n,\pm}^2(k_z) - \frac{\hbar^2\omega_{\rm H}^2}{2}\left[(\zeta - a_{\rm H}k_x)^2 \pm 1\right] - v^2\hbar^2k_z^2\right\} u_{n,\pm}(\zeta) = 0, \tag{3.5}$$

(with $\zeta = y/a_{\rm H}$), which are formally similar to the equations for a linear oscillator with the frequency $\omega_{\rm H} = v\sqrt{2eH/(c\hbar)} = \sqrt{2}v/a_{\rm H}$, $a_{\rm H} = \sqrt{c\hbar/(eH)}$ being the magnetic length and where

$$\epsilon_{n,+}(k_z) = \hbar\sqrt{\omega_{\rm H}^2(n+1) + v^2k_z^2}, \quad \epsilon_{n,-}(k_z) = \hbar\sqrt{\omega_{\rm H}^2 n + v^2k_z^2} \tag{3.6a,b}$$

are the energy levels of a massless Dirac fermion in a constant uniform magnetic field with the right- ($\epsilon_{n,+}(k_z)$) and left-handed ($\epsilon_{n,-}(k_z)$) helicities. The eigenfunctions $u_{n,\pm}$ and $d_{n,\pm}$ corresponding to the energy levels (3.6*a,b*) are given by the expressions

$$\left.\begin{aligned} &u_{n,\pm}(y) = C_{n,\pm}\exp\left\{\frac{(y-y_0)^2}{2a_{\rm H}^2}\right\} H_n\left(\frac{y-y_0}{a_{\rm H}}\right), \quad u_{-1,-}(y) = 0, \quad y_0 = a_{\rm H}^2k_x, \\ &d_{n,+}(y) = 2\frac{\hbar v}{a_{\rm H}^2}\frac{a_{\rm H}nu_{n-1,+}(y) - (y-y_0)u_{n,+}(y)}{\epsilon_{n,+}(k_z) + v\hbar k_z}, \quad d_{n,-}(y) = 2\frac{\hbar v}{a_{\rm H}}\frac{nu_{n-1,-}(y)}{\epsilon_{n,-}(k_z) - v\hbar k_z}, \end{aligned}\right\} \tag{3.7}$$

where $y_0 = a_{\rm H}^2k_x$, H_n are the Hermite polynomials and the coefficients $C_{n,\pm}$ are given by (A1). In the 2-D case the third dimension is suppressed, i.e. in (3.2)–(3.7) and in (A1) we must put $k_z = 0$. In particular, the energy levels (3.6*a,b*) of a massless Dirac fermion in a constant magnetic field in the 2D case, are

$$\epsilon_{n,+} = \hbar\omega_{\rm H}\sqrt{n+1}, \quad \epsilon_{n,-} = \hbar\omega_{\rm H}\sqrt{n}. \tag{3.8a,b}$$

Note that the expressions in (3.6*a,b*) for the energy levels are formally similar to the energy spectrum of a 3-D relativistic electron in a constant magnetic field (see e.g. Berestetskii *et al.* 1982). However, the cyclotron frequency $\omega_{\rm H}$ of a massless fermion, not like that of an electron, is proportional to the square root $\sqrt{H}$ and increases with the velocity v of the Dirac fermion. Besides, it must be emphasized, that the cyclotron frequency of a massless fermion is explicitly non-classical being proportional to $1/\sqrt{\hbar}$ (McCann & Fal'ko 2006). In addition, as follows from (A1), we do not have explicit expressions for the coefficients $C_{n,\pm}$, but only the recurrent relations.

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Since the energy levels in (3.6*a*,*b*) do not contain an arbitrary real k_x, this energy spectrum is continuously degenerate with the finite degeneracy degree for the motion in the xy-plane restricted to a large, but finite, area $S = L_x L_y$, and with the number of states (for given n) being $eHS/(2\pi\hbar c)$. Then the 2-D number of states is

$$\frac{eHS}{2\pi\hbar c}. \tag{3.9}$$

If, in addition, the motion is restricted in the z-direction also (dimension L_z), we observe that the number of possible values of k_z in an interval Δk_z is $(L_z/2\pi)\Delta k_z$ and the 3-Dnumber of states in this interval reads

$$\frac{eHS}{2\pi\hbar c}\frac{L_z}{2\pi}\Delta k_z = \frac{eHV}{4\pi^2\hbar c}\Delta k_z. \tag{3.10}$$

4. Magnetism of Dirac plasmas

4.1. *Weak magnetic fields*

Each Dirac fermion has a spin and in the presence of an external magnetic field the spins of these fermions are polarized and a net magnetic moment arises (the Pauli paramagnetism). Besides, when a gas consisting of massless fermions is placed in a magnetic field, the motion of the fermions changes: they begin to move along the helical trajectories. Hence, these magnetized Dirac fermions generate an additional magnetic field, whose direction is opposite to that of the external field. This is the Landau diamagnetism effect due to the variation of the orbital motion of fermions caused by the external field. This effect is of a quantum nature and is absent in the classical approximation. In fact, the diamagnetism occurs and results from the quantization of massless fermions levels in the magnetic field (see § 3). We conclude that the magnetization of a Dirac plasma of massless fermions results from the Pauli paramagnetism and the Landau diamagnetism.

Let us calculate the thermodynamic potential (Landau & Lifshitz 1980)

$$\Omega = -\beta^{-1}\sum_n \ln\left[1 + \exp\left(\eta - \beta\epsilon_n\right)\right], \tag{4.1}$$

where $\beta^{-1} = T$ is the system temperature in energy units (throughout the article we put the Boltzmann constant $k_B = 1$), and the sum is taken over all possible states, in the case of weak magnetic fields, i.e. when the inequality $\beta\hbar\omega_H \ll 1$ holds. The dimensionless chemical potential η in (4.1), in 3-D plasmas, is determined by the condition (D2). Introducing the density of states in accordance with (3.10), we obtain

$$\Omega_\pm = -\frac{V}{2\pi^2\hbar}\frac{eH}{\beta c}\int_0^\infty d\epsilon \sum_n \frac{dk_z}{d\epsilon}\ln\left\{1 + \exp\left[\eta - \beta\epsilon_{n,\pm}(k_z)\right]\right\}. \tag{4.2}$$

At small H we may replace the summation by integration using the Euler–Maclaurin sum formula (2.1) in the Supplemental material. In the case of a degenerate Dirac plasma ($\beta\epsilon_F \gg 1$), for the right helicity (+), we put in equation (2.1) in the Supplemental material $a = 0$, $b = \epsilon_F^2/(\hbar^2\omega_H^2) - 1$, and for the left helicity (−) we apply $a = 1$, $b = \epsilon_F^2/(\hbar^2\omega_H^2)$, where $\epsilon_F = \hbar v k_F$ is the Fermi energy with $k_F = \sqrt[3]{3\pi^2 n_3}$ (n_3 is the 3-D Dirac fermion

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density). As a result, we obtain

$$\Omega_\pm = -\frac{V}{8\pi^2}\frac{\epsilon_F^2\omega_H^2}{\hbar v^3}S_\pm, \quad S_+ = \sum_{n=0}^{[n_{0,+}]} f_+(n), \quad S_- = \sum_{n=1}^{[n_{0,-}]}\left[1+f_-(n)\right], \tag{4.3a–c}$$

where $[x]$ is the integer part of number x,

$$\left.\begin{aligned} f_+(n) &= \sqrt{\frac{n_{0,+}-n}{n_{0,+}+1}} - \frac{n+1}{n_{0,+}+1}\ln\left(\sqrt{\frac{n_{0,+}+1}{n+1}}+\sqrt{\frac{n_{0,+}-n}{n+1}}\right), \\ f_-(n) &= \sqrt{1-\frac{n}{n_{0,-}}} - \frac{n}{n_{0,-}}\ln\left(\sqrt{\frac{n_{0,-}}{n}}+\sqrt{\frac{n_{0,-}}{n}-1}\right) \end{aligned}\right\} \tag{4.4}$$

and

$$n_{0,+} = \epsilon_F^2/(\hbar^2\omega_H^2) - 1, \quad n_{0,-} = \epsilon_F^2/(\hbar^2\omega_H^2). \tag{4.5a,b}$$

The magnetic moment per unit volume is defined as

$$\mathfrak{M} = -\frac{1}{V}\left(\frac{\partial\Omega}{\partial \boldsymbol{H}}\right)_{\beta,V,\eta}. \tag{4.6}$$

From (4.3a–c) and (4.4), using the Euler–Maclaurin formula (2.1) in the Supplemental material, after some algebra, in the limit $H \to \mathbf{0}$ we arrive at the expression for the 3-D magnetic moments

$$\mathfrak{M}_{F,\pm} \simeq \mp\mathfrak{m}_F + \frac{5\alpha}{48\pi^2}\frac{v}{c}\left[1-\frac{4}{5}\ln\left(\frac{\hbar\omega_H}{2\epsilon_F}\right)\right]\boldsymbol{H}, \tag{4.7}$$

where $\alpha = e^2/(\hbar c) \simeq 1/137$ is the fine structure constant and

$$|\mathfrak{m}_F| = \frac{e}{8\pi^2}\frac{\epsilon_F^2}{\hbar^2 vc} \tag{4.8}$$

is the proper magnetic moment of the degenerate Dirac plasma. Then the total magnetic moment of the degenerate massless Dirac plasma is

$$\mathfrak{M}_{F,tot} = \mathfrak{M}_{F,+} + \mathfrak{M}_{F,-} \simeq \frac{5\alpha}{24\pi^2}\frac{v}{c}\left[1-\frac{4}{5}\ln\left(\frac{\hbar\omega_H}{2\epsilon_F}\right)\right]\boldsymbol{H}. \tag{4.9}$$

In addition, for the (linear) magnetic susceptibility of degenerate Dirac plasmas we obtain

$$\chi_{F,\pm} \simeq \frac{\alpha}{16\pi^2}\frac{v}{c}\left[1-\frac{4}{3}\ln\left(\frac{\hbar\omega_H}{2\epsilon_F}\right)\right] \tag{4.10}$$

and

$$\chi_{F,tot} \simeq \frac{\alpha}{8\pi^2}\frac{v}{c}\left[1-\frac{4}{3}\ln\left(\frac{\hbar\omega_H}{2\epsilon_F}\right)\right]. \tag{4.11}$$

In the opposite case of high temperatures ($\beta\epsilon_F \ll 1$) the Dirac fermions form a Boltzmann gas, and the magnetic moments are

$$\mathfrak{M}_{B,\pm} \simeq \mp\mathfrak{m}_B + \frac{5ne}{64}\frac{\omega_H}{c}l^2\left(1-\frac{4}{15}\beta\hbar\omega_H\right)\frac{\boldsymbol{H}}{H}, \tag{4.12}$$

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where

$$|\mathfrak{m}_{\mathrm{B}}| = \frac{lne}{8}\frac{v}{c} \tag{4.13}$$

is the proper magnetic moment of a Dirac plasma in the case of the Boltzmann statistics. The total magnetic moment of a Boltzmann massless Dirac plasma reads

$$\mathfrak{M}_{\mathrm{B,tot}} \simeq \frac{5ne}{32}\frac{\omega_{\mathrm{H}}}{c}l^2\left(1-\frac{4}{15}\beta\hbar\omega_{\mathrm{H}}\right)\frac{\boldsymbol{H}}{H}. \tag{4.14}$$

In turn, for the magnetic susceptibilities we have

$$\chi_{\mathrm{B},\pm} \simeq \frac{15}{64\pi\beta}\frac{l^2\omega_p^2}{\hbar\omega_{\mathrm{H}}c^2}\left(1-\frac{8}{15}\beta\hbar\omega_{\mathrm{H}}\right) \tag{4.15}$$

and

$$\chi_{\mathrm{B,tot}} \simeq \frac{15}{32\pi\beta}\frac{l^2\omega_p^2}{\hbar\omega_{\mathrm{H}}c^2}\left(1-\frac{8}{15}\beta\hbar\omega_{\mathrm{H}}\right), \tag{4.16}$$

where $l = \beta\hbar v$, $\omega_p = \sqrt{4\pi n_3 e^2/\mu}$ is the plasma frequency and $\mu = 3/(\beta v^2)$ is the effective Dirac 'mass'. As it stems from (4.7) and (4.12), the massless plasma, consisting of fermions with only positive or only negative helicity, possesses the proper magnetic moment in the absence of an external magnetic field. In other words, it possesses its own magnetization.

4.2. *Strong magnetic fields*

In the overdense plasma (e.g. at the critical electron number density $n_e \sim 10^{21}\ \mathrm{cm}^{-3}$), under the influence a laser pulse with the wavelength $\sim 1.054\ \mu\mathrm{m}$, of duration $\tau_0 \sim 1$ ps and with a high-intensity $\sim 10^{20}\ \mathrm{W\ cm}^{-2}$, were generated ultrastrong magnetic fields in the range 50–700 MG. In these experiments (Tatarakis *et al.* 2002*a*,2002*b*; Wagner *et al.* 2004; Sarri *et al.* 2012) atoms in the laser plasma are ionized in the tunnelling limit ($\gamma \ll 1$) for a time $\tau \sim 1$ fs ($\tau \ll \tau_0$), in the barrier suppression regime.

Let us now consider strong magnetic fields under the condition $1 \lesssim \beta\hbar\omega_{\mathrm{H}} \ll |\eta|$. In this case the magnetization of the Dirac plasma contains a contribution which oscillates with a large amplitude as a function of $\boldsymbol{H}$, DHVA effect (Landau & Lifshitz 1980). To separate the oscillatory parts of the thermodynamic potential (4.1) it is convenient to transform the sum in (4.2) by means of the Poisson formula (2.2) in the Supplemental material. In the 3-D case this leads to

$$\Omega_{\pm} = \Omega_{0,\pm}(\mu) + \frac{V}{\pi^2\hbar}\frac{eH}{\beta c}\mathrm{Re}\sum_{n=1}^{\infty} I_n, \tag{4.17}$$

$$I_n = -\int_{-\infty}^{\infty}\int_0^{\infty} \ln\left[1+\exp\left(\eta-\beta\epsilon_{x,\pm}(k_z)\right)\right]\exp\left(2\pi\mathrm{i}nx\right)\,\mathrm{d}x\,\mathrm{d}k_z, \tag{4.18}$$

where $\Omega_{0,\pm}(\mu)$ is the thermodynamic potential in the absence of the field. In the integrals I_n the variable x is taken to be $x = (\epsilon^2 - \hbar^2\omega_{\mathrm{H}}^2 - v^2\hbar^2k_z^2)/(\hbar^2\omega_{\mathrm{H}}^2)$ for the right-handed helicity and $x = (\epsilon^2 - v^2\hbar^2k_z^2)/(\hbar^2\omega_{\mathrm{H}}^2)$ for the left-handed helicity. Then, for the required

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oscillatory part of the integrals we have

$$I_{\mathrm{osc},n} = -\int_{-\infty}^{\infty}\int_{0}^{\infty} \ln\left[1+\exp\left(\eta-\beta\epsilon\right)\right]\exp\left(\frac{2\pi i n\epsilon^2}{\hbar^2\omega_{\mathrm{H}}^2}\right)\exp\left(-\frac{2\pi i n v^2 k_z^2}{\omega_{\mathrm{H}}^2}\right)\mathrm{d}\epsilon\,\mathrm{d}k_z. \tag{4.19}$$

The main contribution to the integral over k_z proceeds from the values of $k_z \sim \omega_{\mathrm{H}}/v$ (Landau & Lifshitz 1980). On the other hand, the oscillatory part of the integral is produced by the values of ϵ near η/β (see below); the lower limit of integration over ϵ is therefore taken to be equal to zero. The integration over k_z is separable and is carried out by means of the Poisson integral (2.4) in the Supplemental material. The remaining integral is taken by parts and as a result we obtain

$$\Omega_{\mathrm{osc},\pm} = \frac{\sqrt{2}V}{16\pi^3\beta}\frac{\omega_{\mathrm{H}}^3}{v^3}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\int_0^{\infty}\frac{\cos\left(\dfrac{2\pi y^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}+\dfrac{\pi}{4}\right)}{\exp(y-\eta)+1}\,\mathrm{d}y, \tag{4.20}$$

where the dimensionless chemical potential η is determined by the condition (D2). If the inequality holds $\eta \gg 1$, we make in this integral the change of variable $y-\eta=\xi$, and using the value of the complex integral J_1 in equation (2.5) in the Supplemental material, from (4.20), we finally get for the oscillatory part of $\Omega_{\pm}$,

$$\Omega_{\mathrm{osc},\pm} \simeq \frac{\sqrt{2}V}{16\pi^2\beta}\frac{\omega_{\mathrm{H}}^3}{v^3}\sum_{n=1}^{\infty}\frac{\sin\left(\dfrac{2\pi\eta^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}+\dfrac{\pi}{4}\right)}{n^{3/2}\sinh\left[4\pi^2 n\eta/(\beta^2\hbar^2\omega_{\mathrm{H}}^2)\right]}. \tag{4.21}$$

In the case if $\eta \lesssim 1$, the integral in (4.20) can be calculated only numerically.

The main contribution to the magnetic moment comes from the differentiation of the most rapidly varying factors in (4.21), i.e. the sines in the numerators. This leads to

$$\mathfrak{M}_{\mathrm{osc},\pm} \simeq \frac{e}{2\sqrt{2}\pi}\frac{\eta^2 V}{\beta^3\hbar^3\omega_{\mathrm{H}}vc}\sum_{n=1}^{\infty}\frac{\cos\left(\dfrac{2\pi\eta^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}+\dfrac{\pi}{4}\right)}{\sqrt{n}\sinh\left[4\pi^2 n\eta/(\beta^2\hbar^2\omega_{\mathrm{H}}^2)\right]}. \tag{4.22}$$

For 2-D Dirac plasmas, in view of equation (2.2) in the Supplemental material,

$$\Omega_{\pm} = \Omega_{0,\pm}(\mu) + \frac{2S}{\pi}\frac{eH}{\beta c}\mathrm{Re}\sum_{n=1}^{\infty} I_n \tag{4.23}$$

and the oscillatory parts of the thermodynamic potential are

$$I_{\mathrm{osc},n} = -\int_0^{\infty}\ln\left[1+\exp\left(\eta-\beta\epsilon\right)\right]\exp\left(\frac{2\pi i n\epsilon^2}{\hbar^2\omega_{\mathrm{H}}^2}\right)\mathrm{d}\epsilon. \tag{4.24}$$

The integral in (4.24) is taken by parts and then

$$\Omega_{\mathrm{osc},\pm} = \frac{\hbar S}{4\pi^2\beta}\frac{\omega_{\mathrm{H}}^2}{v^2}\sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\infty}\frac{\cos\left(\dfrac{2\pi y^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}\right)}{\exp(y-\eta)+1}\,\mathrm{d}y, \tag{4.25}$$

where the dimensionless chemical potential η is determined by the condition (D12). If the inequality holds $\eta \gg 1$, we make in this integral the change of variable $y-\eta=\xi$, and in

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view of equation (2.5) in the Supplemental material, we obtain

$$\Omega_{\mathrm{osc},\pm} \simeq \frac{\hbar S}{4\pi^2\beta}\frac{\omega_{\mathrm{H}}^2}{v^2}\sum_{n=1}^{\infty}\frac{\sin\left(\dfrac{2\pi\eta^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}\right)}{n\sinh\left[4\pi^2 n\eta/(\beta^2\hbar^2\omega_{\mathrm{H}}^2)\right]}. \tag{4.26}$$

Finally, this leads to the following result for the oscillatory part of the magnetic moment in the 2-D case:

$$\mathfrak{M}_{\mathrm{osc},\pm} \simeq \frac{e\eta^2 S}{\beta^3\hbar^2\omega_{\mathrm{H}}^2 c}\sum_{n=1}^{\infty}\frac{\cos\left(\dfrac{2\pi\eta^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}\right)}{\sinh\left[4\pi^2 n\eta/(\beta^2\hbar^2\omega_{\mathrm{H}}^2)\right]}. \tag{4.27}$$

The functions in (4.22) and (4.27) oscillate with high frequency, which is a manifestation of the DHVA effect in 3-D and 2-D massless Dirac plasmas. Its 'period' in the variable $1/H$ is

$$\Delta\left(1/H\right) = 2e\frac{\beta^2\hbar v^2}{c\eta^2} \tag{4.28}$$

and it increases with the Dirac fermion speed as v^2. The dimensionless chemical potential η in (4.28) is determined by the conditions (D2) and (D12). In the case of non-degenerate hot Dirac plasmas we have constructed analytical approximations (D10) and (D18) for the dimensionless chemical potential η. As it can be seen from the above calculations the DHVA effect is a direct consequence of the Landau diamagnetism.

Some results of numerical calculations for the value of the normalized magnetic moments $\mathfrak{M}^*$ (3-D) and $\mathfrak{M}_{2D}$ (2-D) of Dirac plasmas versus the magnetic field H for various densities and temperatures are presented in figures 1 and 2. The magnitude of the magnetic field H is measured in megagauss (MG), and n_3 and n_2 are the 3-D and 2-D Dirac fermion density, respectively. The dimensionless chemical potential η for these figures is determined by solving (D2) and (D12) numerically. These figures clearly demonstrate the DHVA effect in magnetized massless 3-D and 2-D Dirac plasmas.

5. Conductivity tensor of Dirac plasmas in an external constant magnetic field

5.1. *The Kubo linear reaction theory*

Let a system consisting of interacting massless fermions be in a state of statistical equilibrium, which is described by the Gibbs statistical operator (Akhiezer & Peletminskii)

$$\hat{w} = \exp\{\Omega - \beta\hat{\mathcal{H}}_0 - \eta\hat{N})\}, \tag{5.1}$$

where $\hat{\mathcal{H}}_0$ is the Hamiltonian of interacting Dirac fermions and $\hat{N}$ is the particle number operator. The Gibbs operator (5.1) acts in the Hilbert space of vectors describing the quantum states of massless fermions in a magnetic field. At some time t_0 the external field is switched on, so that the Hamiltonian operator of the system becomes

$$\hat{\mathcal{H}}(t) = \hat{\mathcal{H}}_0 + \hat{V}^{\mathrm{ext}}(t), \tag{5.2}$$

where

$$\hat{V}^{\mathrm{ext}}(t) = -\frac{1}{c}\int d\boldsymbol{r}\boldsymbol{A}^{\mathrm{ext}}(\boldsymbol{r},t)\cdot\hat{\boldsymbol{j}}(\boldsymbol{r}) \tag{5.3}$$

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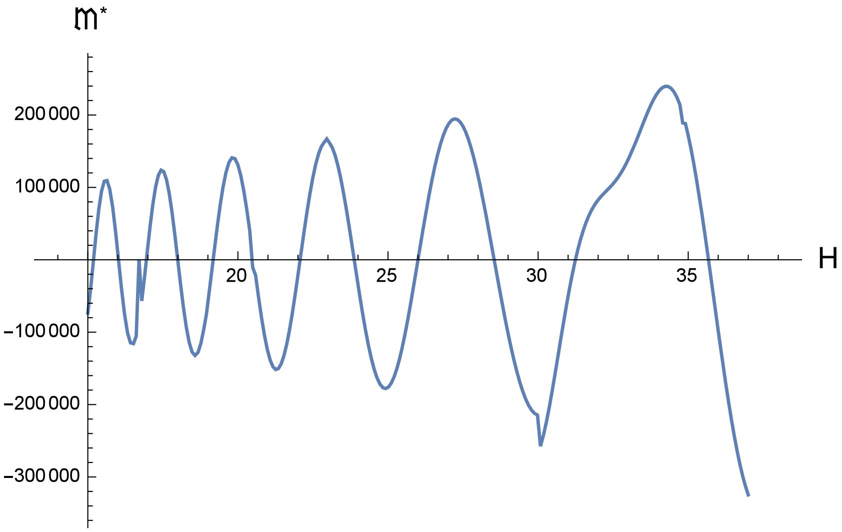

FIGURE 1. The function $\mathfrak{M}^*$ versus H for $n_3 = 10^{22}\,\mathrm{cm}^{-3}$, $T = 10\,\mathrm{K}$ and $v \approx c/300$.

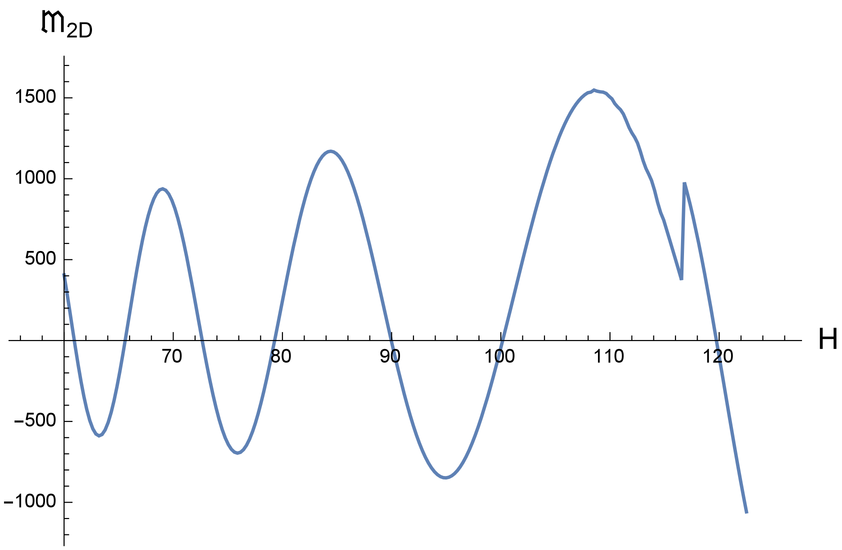

FIGURE 2. The function $\mathfrak{M}_{2D}$ versus H for $n_2 = 10^{18}\,\mathrm{cm}^{-2}$, $T = 100\,\mathrm{K}$ and $v \approx c/300$.

is the interaction Hamiltonian in the Weyl gauge with an external electromagnetic field characterized by the vector potential $\boldsymbol{A}^{\mathrm{ext}}(\boldsymbol{r}, t)$ and the current-density operator $\hat{\boldsymbol{j}}(\boldsymbol{r})$ is determined by (5.25*a*,*b*). The statistical operator $\hat{\rho}(t)$ of the system of Dirac fermions satisfies the Liouville equation

$$\mathrm{i}\hbar \frac{\partial \hat{\rho}(t)}{\partial t} = \left[\hat{\mathcal{H}}(t), \hat{\rho}(t)\right], \tag{5.4}$$

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where $[\hat{A}, \hat{B}]$ is the commutator of the operators $\hat{A}$ and $\hat{B}$. In order to find the statistical operator of the system $\hat{\rho}(t)$ for $t > t_0$, we introduce the operator

$$\tilde{\hat{\rho}}(t) = \exp(\mathrm{i}\hat{\mathcal{H}}_0 t/\hbar)\hat{\rho}(t)\exp(-\mathrm{i}\hat{\mathcal{H}}_0 t/\hbar) \tag{5.5}$$

in the interaction picture. Then according to (5.4) we obtain for $\tilde{\hat{\rho}}(t)$ the following equation:

$$\mathrm{i}\hbar\frac{\partial\tilde{\hat{\rho}}(t)}{\partial t} = \left[\tilde{\hat{V}}^{\mathrm{ext}}(t), \tilde{\hat{\rho}}(t)\right], \quad \tilde{\hat{V}}^{\mathrm{ext}}(t) = \exp(\mathrm{i}\hat{\mathcal{H}}_0 t/\hbar)\hat{V}^{\mathrm{ext}}(t)\exp(-\mathrm{i}\hat{\mathcal{H}}_0 t/\hbar). \tag{5.6a,b}$$

We assume that for $t = -\infty$ there was no external field and the system was in a statistical equilibrium state, i.e. $\hat{\rho}(-\infty) = \hat{w}$. Since $[\hat{\mathcal{H}}_0, \hat{w}] = 0$, then $\tilde{\rho}(-\infty) = \hat{w}$. The last equality is the initial condition for (5.6*a*,*b*). Then from (5.6*a*,*b*) we obtain the integral equation for the operator $\tilde{\rho}(t)$,

$$\tilde{\hat{\rho}}(t) = \hat{w} - \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}t^{'} \left[\tilde{\hat{V}}^{\mathrm{ext}}(t^{'}), \tilde{\hat{\rho}}(t^{'})\right]. \tag{5.7}$$

In the linear approximation for the interaction with the external field, we have

$$\tilde{\hat{\rho}}(t) \simeq \hat{w} - \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}t^{'} \left[\tilde{\hat{V}}^{\mathrm{ext}}(t^{'}), \hat{w}\right]. \tag{5.8}$$

Let us define the average value of the induced current-density in the system of massless fermions as

$$\bar{\boldsymbol{j}}(\boldsymbol{r}, t) = \mathrm{Tr}\left\{\hat{\rho}(t)\hat{\boldsymbol{j}}(\boldsymbol{r})\right\}. \tag{5.9}$$

Taking into account that in a statistical equilibrium the average current is equal to zero, from (5.8) we obtain

$$\bar{\boldsymbol{j}}(\boldsymbol{r}, t) = \frac{\mathrm{i}}{\hbar}\int_{-\infty}^{t} \mathrm{d}t^{'}\,\mathrm{Tr}\left\{\hat{w}\left[\tilde{\hat{V}}^{\mathrm{ext}}(t^{'}), \tilde{\hat{\boldsymbol{j}}}(\boldsymbol{r}, t)\right]\right\}, \tag{5.10}$$

where

$$\tilde{\hat{\boldsymbol{j}}}(\boldsymbol{r}, t) = \exp(\mathrm{i}\hat{\mathcal{H}}_0 t/\hbar)\hat{\boldsymbol{j}}(\boldsymbol{r})\exp(-\mathrm{i}\hat{\mathcal{H}}_0 t/\hbar). \tag{5.11}$$

In view of (5.3), the μ – component of the average current is

$$\overline{j_\mu}(\boldsymbol{r}, t) = -\frac{1}{c}\int_{-\infty}^{\infty} \mathrm{d}t^{'} \int \mathrm{d}\boldsymbol{r}^{'}\, G_{\mu\nu}(\boldsymbol{r} - \boldsymbol{r}^{'}, t - t^{'})A_\nu^{\mathrm{ext}}(\boldsymbol{r}^{'}, t^{'}), \tag{5.12}$$

where the Green function current–current $G_{\mu\nu}$ is defined as

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$$G_{\mu\nu}(\boldsymbol{r}, t) = -\frac{\mathrm{i}}{\hbar}\theta(t)\mathrm{Tr}\left\{w\left[j_\mu(\boldsymbol{r}, t), j_\nu(0)\right]\right\}, \tag{5.13}$$

where $\theta(t)$ is the Heaviside step function. Passing in (5.12) to the Fourier components according to equation (2.6) in the Supplemental material, we have

$$\overline{j_\mu}(\boldsymbol{k}, \omega) = -\frac{1}{c}G_{\mu\nu}(\boldsymbol{k}, \omega)A_\nu^{\mathrm{ext}}(\boldsymbol{k}, \omega). \tag{5.14}$$

Taking into account that at $\varphi(\boldsymbol{r}, t) = 0$

$$\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{r}, t) = -\frac{1}{c}\frac{\partial \boldsymbol{A}^{\mathrm{ext}}(\boldsymbol{r}, t)}{\partial t}, \tag{5.15}$$

or for the Fourier components,

$$\boldsymbol{E}^{\mathrm{ext}}(\boldsymbol{k}, \omega) = \mathrm{i}\frac{\omega}{c}\boldsymbol{A}^{\mathrm{ext}}(\boldsymbol{k}, \omega), \tag{5.16}$$

(5.14) can be rewritten as

$$\overline{j_\mu}(\boldsymbol{k}, \omega) = \frac{\mathrm{i}}{\omega}G_{\mu\nu}(\boldsymbol{k}, \omega)E_\nu^{\mathrm{ext}}(\boldsymbol{k}, \omega). \tag{5.17}$$

Then, according to the definition

$$\overline{j_\mu}(\boldsymbol{k}, \omega) = \kappa_{\mu\nu}(\boldsymbol{k}, \omega)E_\nu^{\mathrm{ext}}(\boldsymbol{k}, \omega), \tag{5.18}$$

the electrical susceptibility tensor of the system of massless Dirac fermions can be expressed in terms of the Green function as follows:

$$\kappa_{\mu\nu}(\boldsymbol{k}, \omega) = \frac{\mathrm{i}}{\omega}G_{\mu\nu}(\boldsymbol{k}, \omega), \tag{5.19}$$

or, in view of (5.13),

$$\kappa_{\mu\nu}(\boldsymbol{k}, \omega) = \frac{1}{\hbar\omega V}\int_0^\infty \exp(-\varepsilon t + \mathrm{i}\omega t)\left\langle\left[\hat{j}_{\boldsymbol{k},\mu}(t), \hat{j}_{-\boldsymbol{k},\nu}(0)\right]\right\rangle \mathrm{d}t \quad (\varepsilon > 0), \tag{5.20}$$

where $\langle\hat{f}(t)\rangle \equiv \mathrm{Tr}(\hat{w}\hat{f}(t))$. In equation (5.20) introduced is an infinitesimally small positive quantity ε for adiabatic switching on of a periodic external electric field. The limit $\varepsilon \downarrow 0$ is taken after the thermodynamic limit $V \to \infty$ ($V/N = \mathrm{const.}$), N is the number of plasma particles.

5.2. *Electrical conductivity tensor*

Oscillations similar to those of the DHVA effect are observed in kinetic phenomena, for example, in the electrical conductivity. Here we will consider conductivity oscillations of a Dirac plasma embedded in a constant strong magnetic field, i.e. the SdH effect (see e.g. Landau & Lifshitz 1980), using a static conductivity model within the framework of the Kubo linear response theory.

Along with the electrical susceptibility tensor in (5.18), the conductivity tensor $\hat{\sigma}(\boldsymbol{k}, \omega)$ can also be defined, as a reaction to the total electromagnetic field in the plasma. In the absence of the spatial dispersion, i.e. at $\boldsymbol{k} \to \boldsymbol{0}$, for the conductivity tensor $\hat{\sigma}(\omega) = \lim_{\boldsymbol{k}\to\boldsymbol{0}} \hat{\sigma}(\boldsymbol{k}, \omega)$ the Kubo formula (5.20) is valid (Zubarev 1974),

$$\sigma_{\mu\nu}(\omega) = \frac{1}{\hbar\omega V} \lim_{\varepsilon\downarrow 0} \lim_{\boldsymbol{k}\to\boldsymbol{0}} \int_0^\infty \exp(-\varepsilon t + \mathrm{i}\omega t) \left\langle \left[\hat{j}_{\boldsymbol{k},\mu}(t), \hat{j}_{-\boldsymbol{k},\nu}(0) \right] \right\rangle \, \mathrm{d}t. \tag{5.21}$$

Then, the Hermitian part of the electrical conductivity tensor (5.21) of a Dirac plasma in a magnetic field reads

$$\sigma_{\mu\nu,\mathrm{H}}(\omega) = \sigma_{\mu\nu,\mathrm{H},+}(\omega) + \sigma_{\mu\nu,\mathrm{H},-}(\omega), \tag{5.22}$$

where

$$\sigma_{\mu\nu,\mathrm{H},\pm}(\omega) = \frac{\pi}{\hbar\omega} \sum_{n,m} \left[f_{\mathrm{FD}}(\epsilon_{n,\pm}) - f_{\mathrm{FD}}(\epsilon_{n,\pm} + \hbar\omega) \right] \langle n | j_\mu | m \rangle_\pm \, \langle m | j_\nu | n \rangle_\pm \, \delta(\omega - \omega_{m,n,\pm}) \tag{5.23}$$

are the conductivity tensors corresponding to right-handed (+) and left-handed (−) helicities, the subscript H denotes the Hermitian matrix,

$$f_{\mathrm{FD}}(\epsilon_{n,\pm}) = \frac{1}{\exp(\beta\epsilon_{n,\pm} - \eta) + 1} \tag{5.24}$$

is the Fermi–Dirac distribution density, which obeys the normalization condition (D2) for the 3-D case and (D12) for the 2-D case, and $\omega_{n,m,\pm} = (\epsilon_{n,\pm} - \epsilon_{m,\pm})/\hbar$. In equation (5.23)

$$\langle n | j_\mu | m \rangle_\pm = \int \mathrm{d}\boldsymbol{r} \, \langle n | j_\mu(\boldsymbol{r}) | m \rangle_\pm, \quad \hat{\boldsymbol{j}}_+ = -ev\hat{\xi}^+ \hat{\boldsymbol{\sigma}} \hat{\xi}, \quad \hat{\boldsymbol{j}}_- = ev\hat{\eta}^+ \hat{\boldsymbol{\sigma}} \hat{\eta} \tag{5.25a,b}$$

are the matrix elements for the right-handed and the left-handed helicities, in the bra-ket (Dirac) notation, of the current-density operator $\boldsymbol{j}(\boldsymbol{r})$.

To account for the scattering of the Dirac fermions in the relaxation time approximation, the δ-function with the energy conservation law in (5.23) should be 'smeared out' by replacing the adiabatic parameter Δ with the collision frequency $\nu_\pm$ (Skobov & Kaner 1964; Blank & Kaner 1966). Then

$$\begin{aligned} \sigma_{\mu\nu,\mathrm{H},\pm}(\omega) = \sum_{n,m} & \frac{f_{\mathrm{FD}}(\epsilon_{n,\pm}) - f_{\mathrm{FD}}(\epsilon_{n,\pm} + \hbar\omega)}{\hbar\omega} \\ & \times \frac{\nu_\pm}{(\omega - \omega_{m,n,\pm})^2 + \nu_\pm^2} \langle n | j_\mu | m \rangle_\pm \langle m | j_\nu | n \rangle_\pm \end{aligned} \tag{5.26}$$

and in the static limit $\sigma_{0,\mu\nu,\pm} = \lim_{\omega\to 0} \sigma_{\mu\nu,\mathrm{H},\pm}(\omega)$, we obtain

$$\sigma_{0,\mu\nu,\pm} = -\frac{eH}{4\pi^2\hbar c} \int_{-\infty}^{\infty} \sum_{n,m=0}^{\infty} \left. \frac{\partial f_{\mathrm{FD}}(\epsilon, k_z)}{\partial \epsilon} \right|_{\epsilon=\epsilon_{n,\pm}} \frac{\nu_\pm}{\omega_{n,m,\pm}^2 + \nu_\pm^2} \langle n | j_\mu | m \rangle_\pm \, \langle m | j_\nu | n \rangle_\pm \, \mathrm{d}k_z. \tag{5.27}$$

The relaxation time approximation in (5.26) is equivalent to replacing the collision integral in the kinetic equation with the quantity $\sim \delta f / \tau_\pm$ (Abrikosov 1988), where δf is the small correction to the Fermi–Dirac distribution due to collisions, and the relaxation time $\tau_\pm = 1/\nu_\pm$. The collision frequency $\nu_\pm$ is calculated in (5.34).

Using the Dirichlet formula (2.7) in the Supplemental material the expression in (5.27) for the static electrical conductivity tensor can be transformed into

$$\sigma_{0,\mu\nu,\pm} = -\frac{eH}{4\pi^2\hbar c}\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\mathcal{F}_{n,n+m,\pm}(k_z)+\mathcal{F}_{n+m,n,\pm}(k_z)\right\}\,\mathrm{d}k_z, \tag{5.28}$$

where $\mathcal{F}_{n,n+m,\pm}(k_z)$ and $\mathcal{F}_{n+m,n,\pm}(k_z)$ are given by equation (2.8) in the Supplemental material.

Taking into account of (3.7), after some algebra, the Hermitian part of the static electrical conductivity tensor of a 3-D Dirac plasma in a magnetic field can be represented as

$$\hat{\sigma}_0 = \hat{\sigma}_{0,+} + \hat{\sigma}_{0,-}, \quad \hat{\sigma}_{0,\pm} = \begin{pmatrix} \sigma_{xx,\pm} & \mathrm{i}\sigma_{xy,\pm} & 0 \\ -\mathrm{i}\sigma_{xy,\pm} & \sigma_{yy,\pm} & 0 \\ 0 & 0 & \sigma_{zz,\pm} \end{pmatrix}, \tag{5.29}$$

where the Cartesian components $\sigma_{ij,\pm}$ $(i, j = x, y, z)$ are given by (C1).

In turn, the static electrical conductivity tensor of a 2-D magnetized massless Dirac plasma takes the form

$$\hat{\sigma}_0 = \hat{\sigma}_{0,+} + \hat{\sigma}_{0,-}, \quad \hat{\sigma}_{0,\pm} = \begin{pmatrix} \sigma_{xx,\pm} & \mathrm{i}\sigma_{xy,\pm} \\ -\mathrm{i}\sigma_{xy,\pm} & \sigma_{yy,\pm} \end{pmatrix}, \tag{5.30}$$

where

$$\left.\begin{aligned} \sigma_{xx,\pm} &= -2\pi\omega_{\mathrm{H}}^2\frac{\hbar^4 v^2}{R_k}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,\pm})+\mathfrak{F}(\epsilon_{n+1,\pm})\right]\mathcal{J}_{n,n+1,\pm}(x,x,0), \\ \sigma_{yy,\pm} &= -2\pi\omega_{\mathrm{H}}^2\frac{\hbar^4 v^2}{R_k}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,\pm})+\mathfrak{F}(\epsilon_{n+1,\pm})\right]\mathcal{J}_{n,n+1,\pm}(y,y,0), \\ \sigma_{xy,\pm} &= -2\pi\omega_{\mathrm{H}}^2\frac{\hbar^4 v^2}{R_k}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,\pm})-\mathfrak{F}(\epsilon_{n+1,\pm})\right]\mathcal{J}_{n,n+1,\pm}(x,y,0) \end{aligned}\right\} \tag{5.31}$$

and $\epsilon_{n,\pm}$ are given by (3.8*a,b*). In particular, for a degenerate Dirac plasma ($\beta\epsilon_{\mathrm{F}} \gg 1$), $\mathfrak{F}(\epsilon_{n,\pm}) = -\delta(\epsilon_{n,\pm}-\epsilon_{\mathrm{F}})$ and the 2-D electrical conductivity tensor (5.31) is

$$\left.\begin{aligned} \sigma_{ii,-} &= 4\pi\epsilon_{\mathrm{F}}\frac{\hbar^2 v^2}{R_k}\left[\mathcal{J}_{[n_0],[n_0]+1,-}(i,i,0)+\mathcal{J}_{[n_0]-1,[n_0],-}(i,i,0)\right], \\ \sigma_{ii,+} &= 4\pi\epsilon_{\mathrm{F}}\frac{\hbar^2 v^2}{R_k}\left[\mathcal{J}_{[n_0]-1,[n_0],+}(i,i,0)+\mathcal{J}_{[n_0]-2,[n_0]-1,+}(i,i,0)\right], \quad i = x, y \\ \sigma_{xy,-} &= 4\pi\epsilon_{\mathrm{F}}\frac{\hbar^2 v^2}{R_k}\left[\mathcal{J}_{[n_0],[n_0]+1,-}(x,y,0)-\mathcal{J}_{[n_0]-1,[n_0],-}(x,y,0)\right], \\ \sigma_{xy,+} &= 4\pi\epsilon_{\mathrm{F}}\frac{\hbar^2 v^2}{R_k}\left[\mathcal{J}_{[n_0]-1,[n_0],+}(x,y,0)-\mathcal{J}_{[n_0]-2,[n_0]-1,+}(x,y,0)\right], \end{aligned}\right\} \tag{5.32}$$

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where $[n_0]$ is the integer part of $n_0 = \epsilon_F^2/(\hbar^2\omega_H^2)$. For example, for the surface number density of Dirac fermions $n_s = 10^{21}\,\mathrm{cm}^{-2}$ and the magnetic field $H = 30\,\mathrm{MG}$, $[n_0] = 1$. Then, taking into account that $\omega_H \gg \nu_\pm$, we obtain

$$\sigma_{xx} \simeq \frac{\nu}{\omega_H}\frac{59}{R_k}, \quad \sigma_{yy} \simeq \frac{\nu}{\omega_H}\frac{61}{R_k}, \quad \sigma_{xy} \simeq \frac{\nu}{\omega_H}\frac{57}{R_k}, \tag{5.33a–c}$$

where $\nu \simeq \nu_- \simeq \nu_+$ and $\sigma_{ij} = \sigma_{ij,-} + \sigma_{ij,+}$ $(i, j = x, y)$.

Note, that the conductivity of a 'plasma consisting of charges with a non-zero mass' is not necessarily isotropic in the plane perpendicular to the magnetic field. This is a consequence of the fact that the currents $\hat{j}_x$ and $\hat{j}_y$ in (5.25*a*,*b*) are expressed in terms of the different Pauli matrices $\hat{\sigma}_x$ and $\hat{\sigma}_y$. However, as follows from (A2), $J_{x,n,-}(k_z) = J_{y,n,-}(k_z)$ and therefore $\sigma_{xx,-} = \sigma_{yy,-}$. In other words, for fermions with negative helicity, the static electrical conductivity is isotropic in the plane perpendicular to the magnetic field. But as it can be seen from (A2), $J_{x,n,+}(k_z) \neq J_{y,n,+}(k_z)$ and consequently for fermions with positive helicity $\sigma_{xx,+} \neq \sigma_{yy,+}$.

To calculate the collision frequency $\nu_\pm$ in (C1) and (C2*a*,*b*), we shall consider scattering from impurities and for the sake of simplicity we assume it to be isotropic. Within the first Born approximation the collision frequency in the presence of a magnetic field is (Abrikosov 1988)

$$\nu_\pm = \frac{\omega_H^2}{4\pi}\frac{n_i|v_i|^2}{\hbar^2 v^2}J_\pm, \quad J_\pm = -\hbar\sum_n \int \left.\frac{\partial f_{FD}(\epsilon)}{\partial\epsilon}\right|_{\epsilon=\epsilon_{n,\pm}} \mathrm{d}k_z = J_{0,\pm} + J_{osc,\pm}, \tag{5.34}$$

where n_i is the impurity atom number density and v_i is the interaction potential of an electron with an impurity atom. According to Poisson's formula (2.2) in the Supplemental material the sum $J_{0,\pm}$ in (5.34) is given by equation (2.3) in the Supplemental material. In turn, for the oscillatory part of the integral, in the 3-D case, we arrive at the following expression:

$$J_{osc,\pm} = \frac{4\beta}{\hbar\omega_H^2}\mathrm{Re}\sum_{n=1}^{\infty} J_n, \tag{5.35}$$

where

$$J_n = \int_{-\infty}^{\infty}\int_0^{\infty} \frac{\exp(\beta\epsilon - \eta)}{[\exp(\beta\epsilon - \eta) + 1]^2}\exp\left(\frac{2\pi i n\epsilon^2}{\hbar^2\omega_H^2}\right)\exp\left(-\frac{2\pi i n v^2 k_z^2}{\omega_H^2}\right)\epsilon\,\mathrm{d}\epsilon\,\mathrm{d}k_z. \tag{5.36}$$

The integration over k_z in (5.36) is carried out using equation (2.4) in the Supplemental material and we obtain

$$J_{osc,\pm} = \frac{\beta}{\sqrt{2}\hbar v\omega_H}\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}\int_0^{\infty}\frac{\exp(\beta\epsilon - \eta)}{[\exp(\beta\epsilon - \eta) + 1]^2}\sin\left(\frac{2\pi n\epsilon^2}{\hbar^2\omega_H^2} + \frac{\pi}{4}\right)\epsilon\,\mathrm{d}\epsilon, \tag{5.37}$$

where the dimensionless chemical potential η is determined by the condition (D2). If the inequality holds $\eta \gg 1$, we make in this integral the change of variable $y - \eta = \xi$ and using the value of the integral J_2 in equation (2.5) in the Supplemental material, from

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(5.37), we finally have for the oscillatory part

$$J_{\mathrm{osc},\pm} \simeq 8\sqrt{2}\pi^2 \frac{\eta^2}{\beta^3 \hbar^3 \omega_{\mathrm{H}}^3 v} \sum_{n=1}^{\infty} \frac{\sqrt{n} \sin\left(\dfrac{2\pi\eta^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2} + \dfrac{\pi}{4}\right)}{\sinh\left[4\pi^2 n\eta/(\beta^2\hbar^2\omega_{\mathrm{H}}^2)\right]}. \tag{5.38}$$

In the case if $\eta \lesssim 1$, the integral in (5.37) can be calculated only numerically.

For 2-D Dirac plasmas, the integral in (5.36) can be written as

$$J_n = \int_0^{\infty} \frac{\exp(\beta\epsilon - \eta)}{\left[\exp(\beta\epsilon - \eta) + 1\right]^2} \exp\left(\frac{2\pi i n \epsilon^2}{\hbar^2 \omega_{\mathrm{H}}^2}\right) \epsilon \, \mathrm{d}\epsilon \tag{5.39}$$

and $J_{\mathrm{osc},\pm}$ in (5.37) transforms into

$$J_{\mathrm{osc},\pm} = \frac{\beta}{\hbar\omega_{\mathrm{H}}^2} \sum_{n=1}^{\infty} \int_0^{\infty} \frac{\exp(\beta\epsilon - \eta)}{\left[\exp(\beta\epsilon - \eta) + 1\right]^2} \sin\left(\frac{2\pi n \epsilon^2}{\hbar^2\omega_{\mathrm{H}}^2}\right) \epsilon \, \mathrm{d}\epsilon, \tag{5.40}$$

where the dimensionless chemical potential η is determined by the condition (D12). If the inequality holds $\eta \gg 1$, we make in this integral the change of variable $y - \eta = \xi$, and in view of the complex integral J_2 in equation (2.5) in the Supplemental material, we finally obtain

$$J_{\mathrm{osc},\pm} \simeq 16\pi^2 \frac{\eta^2}{\beta^3\hbar^3\omega_{\mathrm{H}}^4} \sum_{n=1}^{\infty} \frac{n \sin\left(\dfrac{2\pi\eta^2 n}{\beta^2\hbar^2\omega_{\mathrm{H}}^2}\right)}{\sinh\left[4\pi^2 n\eta/(\beta^2\hbar^2\omega_{\mathrm{H}}^2)\right]}. \tag{5.41}$$

The functions in (5.38) and (5.41) oscillate with high frequency, which is a manifestation of the SdH effect in 3-D and 2-D massless Dirac plasmas. Its 'period' in the variable $1/H$ is the same as in DHVA effect,

$$\Delta(1/H) = 2\mathrm{e}\frac{\beta^2\hbar v^2}{c\eta^2} \tag{5.42}$$

and it increases with the Dirac fermion speed as v^2. The dimensionless chemical potential η in (5.42) is determined by the conditions (D2) and (D12). As it can be seen from the above calculations, the SdH effect is a direct consequence of Landau diamagnetism.

Some results of numerical calculations for the functions $\sigma = \sigma_{xx,\pm}, \sigma_{yy,\pm}, R_{zz} = (\sigma_{zz,\pm})^{-1}$ for the 3-D massless Dirac plasma, and for $\sigma_{2\mathrm{D}} = \sigma_{xx,\pm}, \sigma_{yy,\pm}$ for the 2-D Dirac plasma as a function of the magnetic field H for various densities and temperatures are shown in figures 3–5. Arbitrary units are used for the conductivity, the magnitude of the magnetic field H is measured in MG, and n_3 and n_2 are the 3-D and 2-D Dirac fermion density, respectively. The dimensionless chemical potential η for these figures is determined by solving (D2) and (D12) numerically. These figures clearly demonstrate the SdH effect in magnetized Dirac plasmas.

5.3. *Dynamic conductivity tensor. The Drude–Lorentz form*

As it is stressed in the Supplemental material (see § 1), in the long-wavelength case both the electrical susceptibility and the internal conductivity are response functions. With

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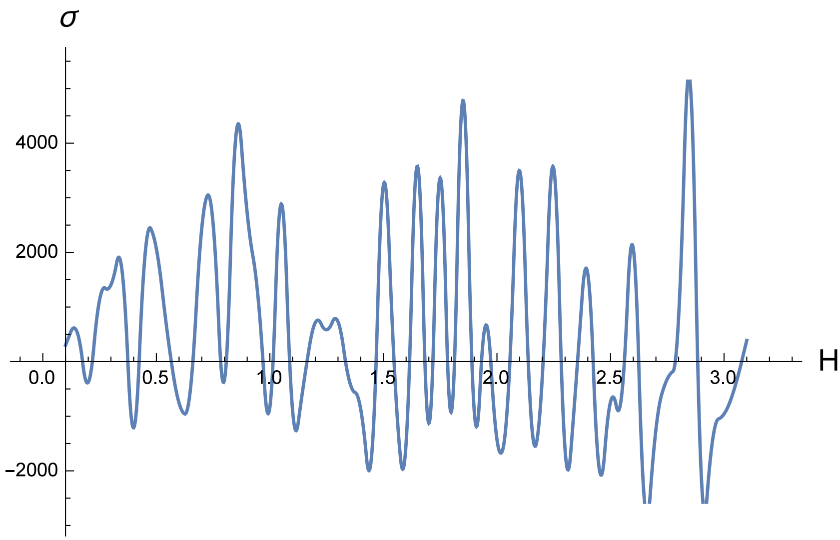

FIGURE 3. The function σ versus H for $n_3 = 10^{21}\ \mathrm{cm}^{-3}$, $T = 10\ \mathrm{K}$ and $v \approx c/300$.

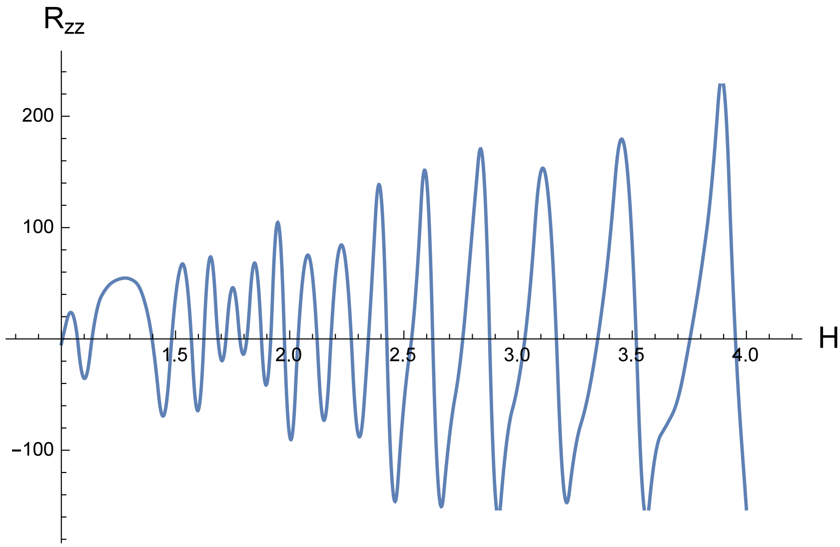

FIGURE 4. The function R_{zz} versus H for $n_3 = 10^{21}\ \mathrm{cm}^{-3}$, $T = 100\ \mathrm{K}$ and $v \approx c/300$.

$\sigma_{\mathrm{H}}(\omega)$ being the positive-definite Hermitian part (5.23)

$$\hat{\sigma}_{\mathrm{H}}(\omega) = \frac{1}{2}\left[\hat{\sigma}^{\dagger}(\omega) + \hat{\sigma}(\omega)\right] \tag{5.43}$$

of the conductivity tensor $\hat{\boldsymbol{\sigma}}(\omega)$ of a Dirac plasma, we introduce its power frequency moments,

$$\hat{\mathrm{T}}_{\ell} = \frac{1}{\pi}\int_{-\infty}^{\infty} \omega^{\ell}\hat{\sigma}_{\mathrm{H}}(\omega)\,\mathrm{d}\omega, \quad \ell = 0, 1. \tag{5.44}$$

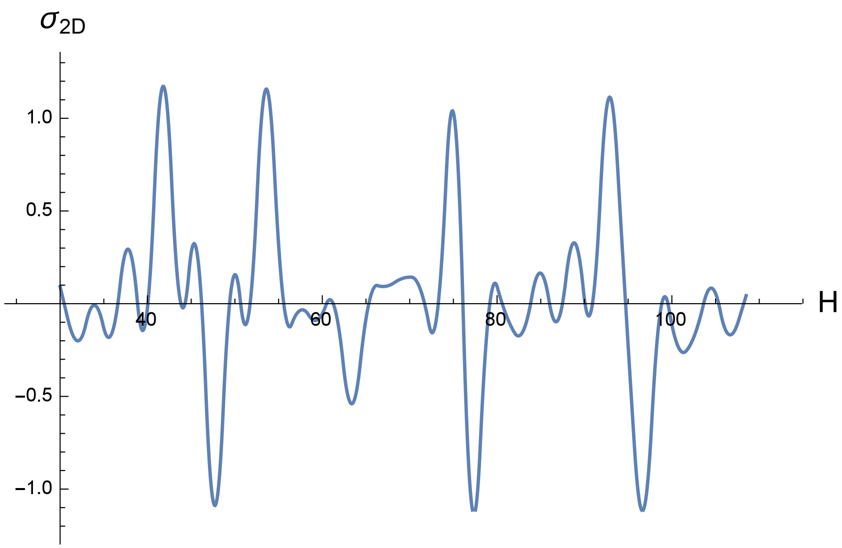

FIGURE 5. The function σ_{2D} versus H for $n_2 = 10^{19}\ \mathrm{cm}^{-2}$, $T = 10\ \mathrm{K}$ and $v \approx c/300$.

The moments (5.44) can be calculated using (5.23), as follows:

$$\left.\begin{aligned} \mathrm{T}_{0,\mu\nu} &= \frac{1}{\hbar}\sum_{n,m}\frac{f_{\mathrm{FD}}(\epsilon_{n,\pm}) - f_{\mathrm{FD}}(\epsilon_{n,\pm} + \hbar\omega_{m,n,\pm})}{\omega_{m,n,\pm}} \left\langle n|j_\mu|m\right\rangle_\pm \left\langle m|j_\nu|n\right\rangle_\pm , \\ \mathrm{T}_{1,\mu\nu} &= \frac{1}{\hbar}\sum_{n,m}\left[f_{\mathrm{FD}}(\epsilon_{n,\pm}) - f_{\mathrm{FD}}(\epsilon_{n,\pm} + \hbar\omega_{m,n,\pm})\right] \left\langle n|j_\mu|m\right\rangle_\pm \left\langle m|j_\nu|n\right\rangle_\pm . \end{aligned}\right\} \tag{5.45}$$

We wish to reconstruct the tensor $\hat{\sigma}(z)$ for Im $z \geq 0$ as a non-canonical solution of the truncated matrix Hamburger moment problem involving only two moments (5.45). Using the matrix version of the Nevanlinna theorem (Kovalishina (1983), Adamyan & Tkachenko (2000) and also see appendix B), we arrive at

$$\hat{\sigma}(z) = \hat{\sigma}_0\left(\hat{\mathrm{I}} + \mathrm{i}\hat{\varOmega}_1\hat{\tau}\right)\left[\hat{\mathrm{I}} - \mathrm{i}\left(z\hat{\mathrm{I}} - \hat{\varOmega}_1\right)\hat{\tau}\right]^{-1}, \tag{5.46}$$

where

$$\hat{\varOmega}_1 = \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1. \tag{5.47}$$

The dynamic conductivity tensor (5.46) is expressed in terms of the static conductivity tensor $\hat{\sigma}_0$ in (5.29) or (5.30) and the relaxation time tensor $\hat{\tau} = \hat{\nu}^{-1}$ ($\hat{\nu}$ is the collision frequency tensor). In the simplest approximation we can put $\hat{\nu} = \hat{\mathrm{I}}\nu_\pm$, where $\nu_\pm$ is given by (5.34). Thus, the dynamic conductivity tensor will also oscillate with high frequency, which is a manifestation of the SdH effect in magnetized massless Dirac plasmas.

The dielectric tensor can be constructed and the spectrum of collective excitations in the system under scrutiny can be studied, but this problem is beyond the scope of the present paper.

6. Conclusions

We have considered 2-D and 3-D one-component massless Dirac plasma embedded in a constant uniform magnetic field. First of all we have determined the wavefunctions and

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the energy Landau levels of the massless Dirac fermion in a constant magnetic field. Our solution is formally similar to the wavefunctions and energy spectrum of the relativistic electron in a constant magnetic field, but with the cyclotron frequency proportional to the velocity v of the massless Dirac fermion and the square root of the magnetic field $\sqrt{H}$. We also found that the cyclotron frequency of the Dirac fermion is explicitly non-classical, being proportional to $1/\sqrt{\hbar}$.

Next, we investigated the magnetism. We calculated the magnetic moment for weak magnetic fields and showed that a 3-D massless Dirac plasma consisting of fermions with the same helicity has its own magnetic moment even in the absence of a magnetic field. In the opposite limit of strong magnetic fields we have considered the DHVA effect – oscillations of the Dirac plasma magnetization with a large amplitude. In the case if the inequality $\eta = \mu/T \gg 1$ holds, the oscillatory part of the magnetic moment can be evaluated analytically and the period of these oscillations decreases with increasing speed of massless Dirac fermions.

Within the framework of the linear response theory we derived the Kubo formula for the electrical conductivity tensor. We proposed a model of the 3-D and 2-D static electrical conductivity tensor and used the matrix version of the classical method of moments to derive a Drude-like formula for the dynamic conductivity tensor of magnetized massless Dirac plasmas. Note that the case of 2-D plasma can have important applications in nanoelectronics. We find that the electrical conductivity tensor for 3-D and 2-D massless Dirac plasmas, in the absence of the spatial dispersion, is not isotropic in the plane perpendicular to the magnetic field. We have shown that both static and dynamic conductivities are characterized by high frequency oscillations which is a manifestation of the SdH effect in magnetized plasmas. The period of these oscillations decreases with increasing speed of massless Dirac fermions. Note, that oscillations of the conductivity are most convenient for experimental observations. It must be emphasized that both DHVA and SdH effects are a direct consequence of Landau diamagnetism.

To support our theory we performed numerical calculations of the magnetic moment and the static conductivity. Our figures clearly demonstrate peculiarities of DHVA and SdH effects in 3-D and 2-D magnetized massless Dirac plasmas. Besides, we have considered the question of a general connection between the external and internal formalisms in the electrodynamics of anisotropic homogeneous media, little reviewed in theliterature.

Supplementary material

Supplementary material is available at https://doi.org/10.1017/S0022377823000752 for the Kramers–Kronig relations and some mathematical formulae.

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Declaration of interests

The authors report no conflict of interest.

Appendix A. The coefficients $C_{n,\pm}$ and $J_{i,n,\pm}(k_z)$

The coefficients $C_{n,\pm}$ in (3.7) are defined as

$$\left.\begin{aligned}
C_{n,+} &= \frac{b_n C_{n-1,+} + \sqrt{a_n + b_n\,(b_n - a_n)\,C^2_{n-1,+}}}{a_n},\\
C_{0,+} &= \frac{1}{\sqrt{a_{\mathrm{H}}\sqrt{\pi}}\sqrt{1 + \dfrac{2\hbar^2 v^2}{a^2_{\mathrm{H}}\left[\epsilon_{0,+}(k_z) + v\hbar k_z\right]^2}}},\\
a_n &= a_{\mathrm{H}}\sqrt{\pi}2^n n!\left\{1 + \frac{2\hbar^2 v^2(2n+1)}{a^2_{\mathrm{H}}\left[\epsilon_{n,+}(k_z) + v\hbar k_z\right]^2}\right\},\\
b_n &= \frac{\sqrt{\pi}\hbar^2 v^2 2^{n+1} n n!}{a_{\mathrm{H}}\left[\epsilon_{n,+}(k_z) + v\hbar k_z\right]^2},\\
C_{n,-} &= \sqrt{\frac{1}{a_{\mathrm{H}}\sqrt{\pi}2^n n!} - \frac{2n\hbar^2 v^2}{a^2_{\mathrm{H}}\left[\epsilon_{n,-}(k_z) - \hbar v k_z\right]^2}\,C^2_{n-1,-}},\\
C_{0,-} &= \frac{1}{\pi^{1/4}\sqrt{a_{\mathrm{H}}}}.
\end{aligned}\right\} \tag{A1}$$

The functions $J_{i,n,\pm}(k_z)$ in (C1) and (C2*a*,*b*) are defined by the following relations:

$$\left.\begin{aligned}
J_{x,n,+}(k_z) &= 2^n(n+1)!\left[\left(\frac{1}{\epsilon_{n+1,+}(k_z) + v\hbar k_z} + \frac{1}{\epsilon_{n,+}(k_z) + v\hbar k_z}\right)C_{n+1,+}\right.\\
&\qquad\left. - \frac{C_{n,+}}{\epsilon_{n+1,+}(k_z) + v\hbar k_z}\right]C_{n,+},\\
J_{y,n,+}(k_z) &= -2^n(n+1)!\left[\left(\frac{1}{\epsilon_{n+1,+}(k_z) + v\hbar k_z} + \frac{1}{\epsilon_{n,+}(k_z) + v\hbar k_z}\right)C_{n+1,+}\right.\\
&\qquad\left. + \frac{C_{n,+}}{\epsilon_{n+1,+}(k_z) + v\hbar k_z}\right]C_{n,+},\\
J_{z,n,2,+}(k_z) &= 2^n(n+2)!\frac{\left(C_{n+2,+} - C_{n+1,+}\right)C_{n,+}}{\left(\epsilon_{n,+}(k_z) + v\hbar k_z\right)\left(\epsilon_{n+2,+}(k_z) + v\hbar k_z\right)},\\
J_{x,n,-}(k_z) &= J_{y,n,-}(k_z) = 2^n(n+1)!\frac{C_{n-1,-}C_{n+1,-}}{\epsilon_{n+1,-}(k_z) - v\hbar k_z},\\
J_{z,n,0,+}(k_z) &= 1 - \sqrt{\pi}a2^{n+1}n!C^2_{n,+},\\
J_{z,n,0,-}(k_z) &= 1 - \sqrt{\pi}a2^{n+1}n!C^2_{n,-}.
\end{aligned}\right\} \tag{A2}$$

Appendix B. The matrix Hamburger problem and Nevanlinna's formula

Here we consider the mathematical details of the matrix truncated Hamburger moment problem. The truncated Hamburger problem for matrix moments is formulated in the following way (Adamyan & Tkachenko (2000); see also Akhiezer 1965): Given a set of

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Hermitian $m \times m$ matrices

$$\left\{\hat{\mathrm{T}}_0, \hat{\mathrm{T}}_1, \hat{\mathrm{T}}_2, \ldots, \hat{\mathrm{T}}_{2\mathrm{n}}\right\}, \quad n = 0, 1, 2, \ldots, \tag{B1}$$

to find all matrix measures $\hat{\boldsymbol{\sigma}}(t)$ such that

$$\int_{-\infty}^{+\infty} t^k \, \mathrm{d}\hat{\boldsymbol{\sigma}}(t) = \hat{\mathrm{T}}_k, \quad k = 0, 1, 2, \ldots, 2n. \tag{B2}$$

In this paper we consider the case with $n = 1$, i.e. with only three known matrix moments $\{\hat{\mathrm{T}}_0, \hat{\mathrm{T}}_1, \hat{\mathrm{T}}_2\}$. In other words, the aim is to construct a tensorial generalization of the Drude–Lorentz formula. The Nevanlinna theorem establishes a bijection between the solutions of the truncated Hamburger matrix problem of moments $\{\hat{\mathrm{T}}_0, \hat{\mathrm{T}}_1, \hat{\mathrm{T}}_2\}$ and the Nevanlinna parameter tensor function $\hat{\mathrm{R}}(z)$ which is also a response or Nevanlinna class function (Krein & Nudel'man 1977) such that

$$\lim_{z\to\infty} \left(\hat{\mathrm{R}}(z)/z\right) = 0, \quad \operatorname{Im} z > 0. \tag{B3}$$

The coefficients of this linear-fractional transformation are the matrix orthogonal polynomials and their conjugates constructed as it is shortly described below (Adamyan & Tkachenko 2000).

Consider the invertible Hankel block matrices

$$\hat{\varGamma}_0 = \hat{\mathrm{T}}_0, \quad \hat{\varGamma}_1 = \begin{pmatrix} \hat{\mathrm{T}}_0 & \hat{\mathrm{T}}_1 \\ \hat{\mathrm{T}}_1 & \hat{\mathrm{T}}_2 \end{pmatrix}, \ldots \tag{B4}$$

and the matrices

$$\left.\begin{aligned} &\hat{\varXi}_0 = \hat{\mathrm{I}}, \quad \hat{\varTheta}_0(t) = \hat{\mathrm{I}}, \\ &\hat{\boldsymbol{\varXi}}_1 = \left(0, \hat{\mathrm{I}}\right)^T, \quad \hat{\boldsymbol{\varTheta}}_1(t) = \left(\hat{\mathrm{I}}, t\hat{\mathrm{I}}\right), \end{aligned}\right\} \tag{B5}$$

where $\hat{\mathrm{I}}$ is the 2×2 unit matrix. Let us introduce 2×2 matrix polynomials

$$\left.\begin{aligned} &\hat{\mathrm{D}}_s(t) = \hat{\boldsymbol{\varTheta}}_s(t)\hat{\varGamma}_s^{-1}\hat{\boldsymbol{\varXi}}_s, \quad \hat{\mathrm{D}}_0(t) = \hat{\varTheta}_0(t)\hat{\varGamma}_0^{-1}\hat{\varXi}_0 = \hat{\mathrm{T}}_0^{-1}, \\ &\hat{\mathrm{D}}_1(t) = \hat{\boldsymbol{\varTheta}}_1(t)\hat{\varGamma}_1^{-1}\hat{\boldsymbol{\varXi}}_1 = \left(\hat{\mathrm{I}}, t\hat{\mathrm{I}}\right)\begin{pmatrix} \hat{\mathrm{T}}_0 & \hat{\mathrm{T}}_1 \\ \hat{\mathrm{T}}_1 & \hat{\mathrm{T}}_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \hat{\mathrm{I}} \end{pmatrix}. \end{aligned}\right\} \tag{B6}$$

Consider the inversion problem

$$\begin{pmatrix} \hat{\mathrm{T}}_0 & \hat{\mathrm{T}}_1 \\ \hat{\mathrm{T}}_1 & \hat{\mathrm{T}}_2 \end{pmatrix}^{-1}\begin{pmatrix} 0 \\ \hat{\mathrm{I}} \end{pmatrix} = \begin{pmatrix} \hat{\mathrm{U}} \\ \hat{\mathrm{V}} \end{pmatrix}, \tag{B7}$$

which is equivalent to

$$\begin{pmatrix} \hat{\mathrm{T}}_0 & \hat{\mathrm{T}}_1 \\ \hat{\mathrm{T}}_1 & \hat{\mathrm{T}}_2 \end{pmatrix}\begin{pmatrix} \hat{\mathrm{U}} \\ \hat{\mathrm{V}} \end{pmatrix} = \begin{pmatrix} 0 \\ \hat{\mathrm{I}} \end{pmatrix}. \tag{B8}$$

The solution of this system of matrix equations is quite simple,

$$\hat{\mathrm{V}} = \left(\hat{\mathrm{T}}_2 - \hat{\mathrm{T}}_1\hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\right)^{-1} = \hat{\mathrm{C}}, \quad \hat{\mathrm{U}} = -\hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\hat{\mathrm{V}} = -\hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\hat{\mathrm{C}}, \tag{B9}$$

so that

$$\hat{\mathrm{D}}_1(t) = \left(t\hat{\mathrm{I}} - \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\right)\hat{\mathrm{C}}. \tag{B10}$$

The conjugate polynomials are defined as

$$\hat{\mathrm{E}}_s(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\omega - z}\hat{\sigma}_{\mathrm{H}}(\omega)\left(\hat{\mathrm{D}}_s(\omega) - \hat{\mathrm{D}}_s(z)\right) \tag{B11}$$

and we have that

$$\hat{\mathrm{E}}_0(z) = 0, \quad \hat{\mathrm{E}}_1(z) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{\omega - z}\hat{\sigma}_{\mathrm{H}}(\omega)\left(\hat{\mathrm{D}}_1(\omega) - \hat{\mathrm{D}}_1(z)\right) = \hat{\mathrm{T}}_0\hat{\mathrm{C}}. \tag{B12}$$

The Kramers–Kronig relations (see the Supplemental material) can be recast as

$$\hat{\sigma}(z) = \frac{1}{\pi\mathrm{i}}\int_{-\infty}^{\infty}\hat{\sigma}_{\mathrm{H}}(\omega)\frac{\mathrm{d}\omega}{\omega - z}, \tag{B13}$$

so that, always for Im $z > 0$, and due to the matrix version of the Nevanlinna theorem (Kovalishina 1983; Adamyan & Tkachenko 2000), we obtain the conductivity tensor in a convenient form analogous to that of the scalar case,

$$\hat{\sigma}(z) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\hat{\sigma}_{\mathrm{H}}(\omega)\frac{\mathrm{d}\omega}{z - \omega} = \mathrm{i}\left(\hat{\mathrm{E}}_1(z) + \hat{\mathrm{E}}_0(z)\hat{\mathrm{R}}(z)\right)\left(\hat{\mathrm{D}}_1(z) + \hat{\mathrm{D}}_0(z)\hat{\mathrm{E}}(z)\right)^{-1}, \tag{B14}$$

where $\hat{\mathrm{R}}(z)$ is the Nevanlinna parameter tensor function $\hat{\mathrm{R}}(z)$ mentioned above. Thus, we arrive at the following solution of the truncated matrix Hamburger problem $\{\hat{\mathrm{T}}_0, \hat{\mathrm{T}}_1, \hat{\mathrm{T}}_2\}$ with the third tensor moment $\hat{\mathrm{T}}_2$ to be treated as an immaterial element of the solution, i.e. it can be eliminated from the solution by a renormalization of the intrinsically unknown Nevanlinna parameter which possesses specific mathematical properties:

$$\begin{aligned}\hat{\sigma}(z) &= \mathrm{i}\hat{\mathrm{T}}_0\hat{\mathrm{C}}\left[\left(z\hat{\mathrm{I}} - \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\right)\hat{\mathrm{C}} + \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{R}}(z)\right]^{-1}\\ &= \mathrm{i}\hat{\mathrm{T}}_0\hat{\mathrm{C}}\left[\left(z\hat{\mathrm{I}} - \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\right)\hat{\mathrm{C}} + \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{R}}(z)\hat{\mathrm{C}}^{-1}\hat{\mathrm{C}}\right]^{-1}\\ &= \mathrm{i}\hat{\mathrm{T}}_0\hat{\mathrm{C}}\left\{\left[\left(z\hat{\mathrm{I}} - \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\right) + \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{R}}(z)\hat{\mathrm{C}}^{-1}\right]\hat{\mathrm{C}}\right\}^{-1}.\end{aligned} \tag{B15}$$

Then

$$\hat{\sigma}(z) = \mathrm{i}\hat{\mathrm{T}}_0\hat{\mathrm{C}}\hat{\mathrm{C}}^{-1}\left[\left(z\hat{\mathrm{I}} - \hat{\varOmega}_1 + \hat{\mathrm{R}}_1(z)\right)\right]^{-1} = \mathrm{i}\hat{\mathrm{T}}_0\left[\left(z\hat{\mathrm{I}} - \hat{\varOmega}_1 + \hat{\mathrm{R}}_1(z)\right)\right]^{-1}, \tag{B16}$$

where

$$\hat{\mathrm{C}} = \left(\hat{\mathrm{T}}_2 - \hat{\mathrm{T}}_1\hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1\right)^{-1}, \quad \hat{\mathrm{R}}_1(z) = \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{R}}(z)\hat{\mathrm{C}}^{-1} \tag{B17a,b}$$

and

$$\hat{\varOmega}_1 = \hat{\mathrm{T}}_0^{-1}\hat{\mathrm{T}}_1. \tag{B18}$$

Consider now the static approximation for the new Nevanlinna parameter

$$\hat{\mathrm{R}}_1(z) = \hat{\mathrm{R}}_1(0) = \mathrm{i}\hat{\tau}\hat{\varOmega}^2, \tag{B19}$$

where $\hat{\tau}$ is the relaxation time tensor. Then the general expression for the conductivity tensor takes the Drude–Lorentz form,

$$\hat{\sigma}(z) = \hat{\mathrm{T}}_0\hat{\tau}\left[\hat{\mathrm{I}} - \mathrm{i}\left(z\hat{\mathrm{I}} - \hat{\Omega}_1\right)\hat{\tau}\right]^{-1}. \tag{B20}$$

The relaxation time tensor can be further determined in terms of the static conductivity. Indeed, since

$$\hat{\sigma}_0 = \hat{\sigma}(0) = \hat{\mathrm{T}}_0\hat{\tau}\left(\hat{\mathrm{I}} + \mathrm{i}\hat{\Omega}_1\hat{\tau}\right)^{-1} \tag{B21}$$

we have that

$$\hat{\tau} = \left(\hat{\mathrm{T}}_0 - \mathrm{i}\hat{\sigma}_0\hat{\Omega}_1\right)^{-1}\hat{\sigma}_0 \tag{B22}$$

and arrive at

$$\hat{\sigma}(z) = \hat{\sigma}_0\left(\hat{\mathrm{I}} + \mathrm{i}\hat{\Omega}_1\hat{\tau}\right)\left[\hat{\mathrm{I}} - \mathrm{i}\left(z\hat{\mathrm{I}} - \hat{\Omega}_1\right)\hat{\tau}\right]^{-1}. \tag{B23}$$

The dynamic conductivity tensor (B23) is expressed in terms of the static conductivity tensor (5.31) or (5.32) (see Rylyuk & Tkachenko) and the relaxation time tensor $\hat{\tau} = \hat{\nu}^{-1}$ ($\hat{\nu}$ is the collision frequency tensor). These results lead to the Drude–Lorentz-type formula for the conductivity tensor of a magnetized plasma. Further generalizations are relatively cumbersome and they will be considered elsewhere. The same should be said with respect to the possibilities to avoid the static approximation of the Nevanlinna parameter function (B19).

Appendix C. Static electrical conductivity tensor of a 3-D Dirac plasma in a magnetic field

The Cartesian components of the static electrical conductivity tensor (5.29) of a 3-D Dirac plasma in a magnetic field are

$$\left.\begin{aligned}
\sigma_{zz,+} &= \sigma_{zz,1,+} + \sigma_{zz,2,+}, \quad \sigma_{zz,-} = \sigma_{zz,1,-},\\
\sigma_{xx,\pm} &= -\omega_{\mathrm{H}}^2\frac{\hbar^3 v^2}{R_k}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,\pm}(k_z)) + \mathfrak{F}(\epsilon_{n+1,\pm}(k_z))\right]\mathcal{J}_{n,n+1,\pm}(x,x,k_z)\,\mathrm{d}k_z,\\
\sigma_{yy,\pm} &= -\omega_{\mathrm{H}}^2\frac{\hbar^3 v^2}{R_k}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,\pm}(k_z)) + \mathfrak{F}(\epsilon_{n+1,\pm}(k_z))\right]\mathcal{J}_{n,n+1,\pm}(y,y,k_z)\,\mathrm{d}k_z,\\
\sigma_{xy,\pm} &= -\omega_{\mathrm{H}}^2\frac{\hbar^3 v^2}{R_k}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,\pm}(k_z)) - \mathfrak{F}(\epsilon_{n+1,\pm}(k_z))\right]\mathcal{J}_{n,n+1,\pm}(x,y,k_z)\,\mathrm{d}k_z,\\
\sigma_{zz,2,+} &= -2\omega_{\mathrm{H}}^4\frac{\hbar^5 v^2}{R_k}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\left[\mathfrak{F}(\epsilon_{n,+}(k_z)) + \mathfrak{F}(\epsilon_{n+2,+}(k_z))\right]\mathcal{J}_{n,n+2,\pm}(z,z,k_z)\,\mathrm{d}k_z,\\
\sigma_{zz,1,\pm} &= -\frac{\hbar\omega_{\mathrm{H}}^2}{2\pi R_k \nu_\pm}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\mathfrak{F}(\epsilon_{n,\pm}(k_z))J_{z,n,0,\pm}^2(k_z)\,\mathrm{d}k_z,
\end{aligned}\right\} \tag{C1}$$

$$\mathfrak{F}(\epsilon_{n,\pm}(k_z)) = \left.\frac{\partial f_{\mathrm{FD}}(\epsilon)}{\partial\epsilon}\right|_{\epsilon=\epsilon_{n,\pm}(k_z)}, \quad \mathcal{J}_{n,m,\pm}(i,j,k_z) = \frac{\nu_\pm J_{i,n,\pm}(k_z)J_{j,n,\pm}(k_z)}{\omega_{n,m,\pm}^2(k_z) + \nu_\pm^2}, \tag{C2a,b}$$

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where $\omega_{n,m,\pm}(k_z) = [\epsilon_{n,\pm}(k_z) - \epsilon_{m,\pm}(k_z)]/\hbar$, $R_k = 2\pi\hbar/e^2 = 25812.80745\ldots\ \Omega$ is the von Klitzing constant or the quantum of electrical resistance and $J_{i,n,\pm}(k_z)$ $(i = x, y, z)$ are given by (A2).

Appendix D. Normalization in a magnetic field

Here we calculate the chemical potential of a 3-D and 2-D Fermi fluid of a Dirac one-component plasma.

Since the 3-D number of states in a magnetic field is

$$\frac{eHV}{4\pi^2\hbar^2 c}, \tag{D1}$$

the normalization condition in a magnetic field reads

$$\frac{eHV}{4\pi^2\hbar c}\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\left\{\frac{1}{\exp(\beta\epsilon_{n,+}(k_z)-\eta)+1}+\frac{1}{\exp(\beta\epsilon_{n,-}(k_z)-\eta)+1}\right\}\mathrm{d}k_z = N_3, \tag{D2}$$

where N_3 is the total number of particles and $\epsilon_{n,+}$ and $\epsilon_{n,-}$ are given by (3.6*a*,*b*). This normalization condition can also be represented in the form

$$\frac{1}{2\pi^2 a_{\mathrm{H}}^2}\left\{-\frac{1}{l}\int_0^{\infty}\frac{\mathrm{d}t}{\exp(t-\eta)+1}+\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{\exp(\beta\epsilon_{n,-}(k_z)-\eta)+1}\right\} = n_3, \tag{D3}$$

where $a_{\mathrm{H}} = \sqrt{c\hbar/(eH)}$ being the magnetic length, $l = \beta\hbar v$ and $n_3 = N_3/V$. In the limit $H \to 0$ we may replace the summation in (D3) by integration using the Euler–Maclaurin sum formula

$$\sum_{a\le n<b} F(n) \simeq \int_a^b F(x)\,\mathrm{d}x. \tag{D4}$$

Then $\hbar^2\omega_{\mathrm{H}}^2 n \to \hbar^2\omega_{\mathrm{H}}^2 x \to v^2 h^2 k_\perp^2$ and $\mathrm{d}x = 2v^2/\omega_{\mathrm{H}}^2 k_\perp\,\mathrm{d}k_\perp$. Taking into account that in the limit $H \to 0$, $\epsilon_{n,-}(k_z) = \hbar\sqrt{\omega_{\mathrm{H}}^2 n + v^2 k_z^2} \to \hbar v\sqrt{k_\perp^2 + k_z^2} = \hbar v k$, we obtain

$$\frac{eH}{\pi^2\hbar c}\frac{v^2}{\omega_{\mathrm{H}}^2}\int_0^{\infty} k_\perp\,\mathrm{d}k_\perp\int_{-\infty}^{\infty}\frac{\mathrm{d}k_z}{\exp(\beta v h k-\eta)+1} = n_3. \tag{D5}$$

But $2\pi k_\perp\,\mathrm{d}k_\perp\,\mathrm{d}k_z = 4\pi k^2\,\mathrm{d}k$ and then

$$\frac{1}{\pi^2}\int_0^{\infty}\frac{k^2\,\mathrm{d}k}{\exp(\beta v h k-\eta)+1} = n_3. \tag{D6}$$

Making a change of variables $\beta v h k = t$ we arrive at the result

$$\int_0^{\infty}\frac{t^2\,\mathrm{d}t}{\exp(t-\eta)+1} = \pi^2 l^3 n_3. \tag{D7}$$

So, the dimensionless chemical potential η for 3-D massless Dirac fermions in the absence of a magnetic field is determined by the condition (D7). In non-degenerate, hot plasmas

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($\beta \to 0$), the Fermi–Dirac distribution in (D7) can be substituted by the Maxwell one and we have

$$\int_0^\infty \frac{t^2\,\mathrm{d}t}{\exp(t-\eta)+1} \simeq \mathrm{e}^\eta \int_0^\infty t^2 \mathrm{e}^{-t}\,\mathrm{d}t = 2\mathrm{e}^\eta \simeq \pi^2 l^3 n_3. \tag{D8}$$

Then for non-degenerate hot 3-D Dirac plasmas, in the absence of a magnetic field, $\eta \simeq \ln(\pi^2 l^3 n_3/2)$.

We can also get an analytical approximation for the dimensionless chemical potential η for non-degenerate hot 3-D Dirac plasmas, in the case of strong magnetic fields. Under the condition $1 \lesssim \beta\hbar\omega_H \ll \eta$, from (D3), we obtain

$$\frac{\mathrm{e}^\eta}{2\pi^2 a_H^2}\left\{\frac{1}{l} + 2\frac{\omega_H}{v}\sum_{n=1}^{\infty}\sqrt{n}\,\mathrm{K}_1(\beta\hbar\omega_H\sqrt{n})\right\} \simeq n_3 \tag{D9}$$

and then

$$\eta \simeq \ln\left\{\frac{2\pi^2 a_H^2 n_3}{1/l + 2(\omega_H/v)\sum_{n=1}^{\infty}\sqrt{n}\,\mathrm{K}_1(\beta\hbar\omega_H\sqrt{n})}\right\}, \tag{D10}$$

where

$$\mathrm{K}_\nu(z) = \int_0^\infty \mathrm{e}^{-z\coth(t)}\coth(\nu t)\,\mathrm{d}t \tag{D11}$$

is the Macdonald function. Let us give some numerical estimates. For $T = 10^4$ K, $n_3 = 10^{19}\,\mathrm{cm}^{-3}$, $H = 30$ MG, $v \simeq c/300$; $\beta\hbar\omega_H \simeq 2.3$ and $\eta \simeq -4$. For $T = 10^5$ K, $n_3 = 10^{21}\,\mathrm{cm}^{-3}$, $H = 700$ MG, $v \simeq c/300$; $\beta\hbar\omega_H \simeq 1.11$ and $\eta \simeq -6.14$.

Taking into account (3.9) the normalization condition for 2-D massless Dirac fermions in a magnetic field is

$$\frac{eHS}{2\pi\hbar c}\sum_{n=0}^{\infty}\left\{\frac{1}{\exp(\beta\epsilon_{n,+}-\eta)+1} + \frac{1}{\exp(\beta\epsilon_{n,-}-\eta)+1}\right\} = N_2, \tag{D12}$$

where $\epsilon_{n,+}$ and $\epsilon_{n,-}$ are given by (3.8*a*,*b*). This normalization condition can also be represented in the form

$$\frac{1}{2\pi a_H^2}\left\{-\frac{1}{\exp(-\eta)+1} + 2\sum_{n=0}^{\infty}\frac{1}{\exp(\beta\epsilon_{n,-}-\eta)+1}\right\} = n_2, \tag{D13}$$

where $n_2 = N_2/S$. In the limit $H \to 0$, in view of (D4), from (D13), we obtain

$$\frac{1}{\pi}\int_0^\infty \frac{k_\perp\,\mathrm{d}k_\perp}{\exp(\beta v h k_\perp - \eta)+1} = n_2. \tag{D14}$$

Making a change of variables $\beta v h k_\perp = t$ we have

$$\int_0^\infty \frac{t\,\mathrm{d}t}{\exp(t-\eta)+1} = \pi l^2 n_2. \tag{D15}$$

The dimensionless chemical potential η for 2-D massless Dirac fermions in the absence of a magnetic field is determined by (D15). In non-degenerate, hot plasmas ($\beta \to 0$), the

Fermi–Dirac distribution in (D15) can be substituted by the Maxwell one and we have

$$\int_0^\infty \frac{t\,\mathrm{d}t}{\exp(t-\eta)+1} \simeq \mathrm{e}^\eta \int_0^\infty t\mathrm{e}^{-t}\,\mathrm{d}t = \mathrm{e}^\eta \simeq \pi l^2 n_2. \tag{D16}$$

Then for non-degenerate hot 2-D Dirac plasmas, in the absence of a magnetic field, $\eta \simeq \ln(\pi l^2 n_2)$.

We can also get an analytical approximation for the dimensionless chemical potential η for non-degenerate hot 2-D Dirac plasmas, in the case of strong magnetic fields. Under the condition $1 \lesssim \beta\hbar\omega_{\mathrm{H}} \ll \eta$, from (D13), we obtain

$$\frac{\mathrm{e}^\eta}{2\pi a_{\mathrm{H}}^2}\left\{1+2\sum_{n=1}^{\infty}\exp(-\beta\epsilon_{n,-})\right\} \simeq n_2 \tag{D17}$$

and then

$$\eta \simeq \ln\left\{\frac{2\pi a_{\mathrm{H}}^2 n_2}{1+2\sum_{n=1}^{\infty}\exp(-\beta\hbar\omega_{\mathrm{H}}\sqrt{n})}\right\}. \tag{D18}$$

Let us give some numerical estimates. For $T = 10^5$ K, $n_2 = 10^{14}\ \mathrm{cm}^{-2}$, $H = 700$ MG, $v \simeq c/300$; $\beta\hbar\omega_{\mathrm{H}} \simeq 1.11$ and $\eta \simeq -4.1$. For $T = 10^5$ K, $n_2 = 10^{15}\ \mathrm{cm}^{-2}$, $H = 700$ MG, $v \simeq c/300$; $\beta\hbar\omega_{\mathrm{H}} \simeq 1.11$ and $\eta \simeq -1.8$.

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