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ON ARITHMETIC SUMS OF CONNECTED SETS IN $\mathbb{R}^2$

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Abstract

We prove that for two connected sets $E, F \subset \mathbb{R}^2$ with cardinalities greater than 1, if one of E and F is compact and not a line segment, then the arithmetic sum $E + F$ has nonempty interior. This improves a recent result of Banakh *et al.* ['The continuity of additive and convex functions which are upper bounded on non-flat continua in $\mathbb{R}^n$', *Topol. Methods Nonlinear Anal.* **54**(1)(2019), 247–256] in dimension two by relaxing their assumption that E and F are both compact.

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1. Introduction

Given finitely many sets $E_1, \ldots, E_n \subset \mathbb{R}^d$, their arithmetic sum is defined by

$$E_1 + \cdots + E_n = \{x_1 + \cdots + x_n : x_i \in E_i \text{ for } 1 \le i \le n\}.$$

A fundamental question is to find suitable conditions on $E_1, \ldots, E_n$ under which their arithmetic sum has nonempty interior. There are two classical results on this question. First, if two sets $E, F \subset \mathbb{R}^d$ are large in the sense of having positive Lebesgue measure (and measurable), then the (generalised) Steinhaus theorem states that $E + F$ has nonempty interior. Second, Picard's theorem says that the same conclusion holds when E and F are large in the sense of being of second category in $\mathbb{R}^d$ and having the Baire property. The monograph of Oxtoby [10] gives a detailed account of these two results.

There are also many results on this question for sets that are small in the sense of both measure and topology, which are often fractal sets. Studies in this direction also date back to a work of Steinhaus, who in [15] first observed that the arithmetic sum of the middle-third Cantor set with itself is the interval $[0, 2]$. Subsequent generalisations were given by Hall [7], Newhouse [9] and Astels [2]. Very recently, Feng and the author [6] considered fractal sets in higher dimensional Euclidean spaces.

Recently, considerable attention has been given to the study of arithmetic sums involving connected sets. Chang [5] proved that if $E \subset \mathbb{R}^2$ is a curve connecting the points $(0, 0)$ and $(1, 0)$, and $F \subset \mathbb{R}^2$ is a curve connecting $(0, 0)$ and $(0, 1)$, then $E + F + \mathbb{Z}^2 = \mathbb{R}^2$. Simon and Taylor [13] studied the question when the arithmetic sum of a C^2 curve and a certain class of fractal sets in $\mathbb{R}^2$ has nonempty interior. In [14], they also studied the dimension and measure of the arithmetic sum of a C^2 curve and a set in the plane. Very recently, Banakh, Jabłońska and Jabłoński proved the following result, which motivated the present paper.

THEOREM 1.1 [3, Theorem 4]. *Let $K_1, \dots, K_d \subset \mathbb{R}^d$ be compact connected sets. Suppose that there exist $a_i, b_i \in K_i$, for $i = 1, \dots, d$, such that the d vectors $a_1 - b_1, \dots, a_d - b_d$ are linearly independent. Then, $K_1 + \cdots + K_d$ has nonempty interior.*

Banakh *et al.* proved Theorem 1.1 by making elegant use of a result in topology on products of continua (see [3, Proposition 1]). It is natural to ask whether the compactness assumption in Theorem 1.1 can be relaxed. In this note, we investigate this question in the case when $d = 2$. By using a completely different approach, we obtain the following result.

THEOREM 1.2. *Let $E, F \subset \mathbb{R}^2$ be connected sets with cardinalities greater than* 1*. If F is compact and not a line segment, then $E + F$ has nonempty interior.*

Theorem 1.2 improves Theorem 1.1 when $d = 2$, since we allow one of the two connected sets to be noncompact. We also show that the assumptions in Theorem 1.2 cannot be further relaxed. More precisely, we show that if $F \subset \mathbb{R}^2$ is a line segment, then there exists a noncompact connected set $E \subset \mathbb{R}^2$ not lying in a line such that $E + F$ has empty interior. Also, we will give examples of noncompact connected sets $E, F \subset \mathbb{R}^2$, neither of which is contained in a line, such that $E + F$ has empty interior. Theorem 1.2 combined with these examples gives a full answer to the above question on the compactness assumption in Theorem 1.1 for the case when $d = 2$.

Our strategy to prove Theorem 1.2 is as follows. First, we prove the conclusion when the complement of F has at least one bounded connected component. Then we refine this result by proving that if there is a compact set K lying in a line in $\mathbb{R}^2$ such that the complement of $F \cup K$ has at least one bounded connected component, then $E + F$ has nonempty interior as well. These two results are stated and proved in $\mathbb{R}^d$ (see Lemmas 2.1–2.2). Next, we prove that when F has empty interior, there exists a compact set K lying in a line in $\mathbb{R}^2$ such that the complement of $F \cup K$ has at least one bounded connected component (see Proposition 3.3). Finally, Theorem 1.2 follows by combining Lemma 2.2 and Proposition 3.3.

The paper is organised as follows. In Section 2, we give some preliminary lemmas. Then we prove Theorem 1.2 in Section 3. Finally, in Section 4, we present examples to show that the assumptions in Theorem 1.2 cannot be further relaxed.

2. Preliminary lemmas

For $A \subset \mathbb{R}^d$, let A^c, A^o, ∂A and $\overline{A}$ denote respectively the complement, interior, boundary and closure of A. We first prove a useful lemma.

LEMMA 2.1. *Let $E \subset \mathbb{R}^d$ be a connected set with cardinality greater than 1. Let $F \subset \mathbb{R}^d$ be a compact set so that F^c has at least one bounded connected component. Then the arithmetic sum $E + F$ has nonempty interior.*

PROOF. We assume that the interior of F is empty, otherwise there is nothing to prove. Let U be the unbounded connected component of F^c. Write $V = F^c \backslash U$. By the assumption that F^c has at least one bounded connected component, V is a nonempty bounded open subset of $\mathbb{R}^d$. Set

$$W = \{x \in \mathbb{R}^d : \text{there exists } a, b \in E \text{ such that } x \in (a + U) \cap (b + V)\}.$$

Clearly W is open. We claim that $W \neq \emptyset$. To prove this claim, it suffices to show that $(s + V) \cap U \neq \emptyset$ for each nonzero $s \in \mathbb{R}^d$. To this end, fix a nonzero $s \in \mathbb{R}^d$ and define $P_s : \mathbb{R}^d \to \mathbb{R}$ by $P_s(x) = \langle x, s\rangle$, where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. Since V is bounded,

$$\lambda := \sup_{x \in V} P_s(x) < \infty.$$

Pick $x_0 \in V$ so that $P_s(x_0) > \lambda - \|s\|^2/2$, and take a small $r \in (0, \|s\|/2)$ so that $B^o(x_0, r) \subset V$, where $B^o(x_0, r)$ stands for the open ball centred at x_0 of radius r. Then for each $y \in \mathbb{R}^d$ with $\|y - x_0\| < r$,

$$\begin{aligned}\langle s + y, s\rangle &= \langle s + x_0, s\rangle + \langle y - x_0, s\rangle \\ &\geq \langle x_0, s\rangle + \|s\|^2 - \|y - x_0\| \cdot \|s\| \\ &\geq \langle x_0, s\rangle + \|s\|^2/2 > \lambda,\end{aligned}$$

which implies that $s + y \notin V$ and so $s + B^o(x_0, r) \subset V^c$. Since F has empty interior, it follows that $(s + B^o(x_0, r)) \cap F^c \neq \emptyset$, equivalently, $(s + B^o(x_0, r)) \cap (U \cup V) \neq \emptyset$. Since $s + B^o(x_0, r) \subset V^c$, we get $(s + B^o(x_0, r)) \cap U \neq \emptyset$, so $(s + V) \cap U \neq \emptyset$, as desired. This proves $W \neq \emptyset$.

Finally, we prove that $W \subset E + F$, which immediately implies that $E + F$ has nonempty interior. Suppose this is not true, that is, there exists $x \in W$ so that $x \notin E + F$. By the definition of W, there exist $a, b \in E$ so that

$$x \in (a + U) \cap (b + V). \tag{2.1}$$

Since $x \notin E + F$, it follows that

$$(x - E) \subset F^c = U \cup V. \tag{2.2}$$

However, according to (2.1), $(x - E) \cap U \supset \{x - a\}$ and $(x - E) \cap V \supset \{x - b\}$. This together with (2.2) implies that $x - E$ is not connected, leading to a contradiction. □

The next lemma is a refined version of Lemma 2.1.

LEMMA 2.2. *Let $E \subset \mathbb{R}^d$ be a connected set with cardinality greater than 1. Let $F \subset \mathbb{R}^d$ be compact. Suppose that there exists a compact set $K \subset \mathbb{R}^d$ so that the following two properties hold:*

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(i) *K is contained in a hyperplane;*
(ii) *$(F \cup K)^c$ has at least one bounded connected component.*

Then the arithmetic sum $E + F$ has nonempty interior.

Lemma 2.2 is a direct consequence of the following result.

LEMMA 2.3. *Let $F, K \subset \mathbb{R}^d$ be as in Lemma 2.2. Let D be a dense subset of $\mathbb{R}^d$. Then the set*

$$A := \bigcap_{z \in D} (z - F)^c$$

is totally disconnected.

PROOF. By taking a suitable rotation and translation if necessary, we may assume that K is contained in the linear subspace $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d = 0\}$ and that at least one bounded connected component of $(F \cup K)^c$ has nonempty intersection with the half-space $H := \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$.

Let U be a bounded connected component of $(F \cup K)^c$ so that $V := U \cap H$ is nonempty. Since $\partial U \subset F \cup K$ and $K \subset \partial H$, we easily see that $\partial V \subset F \cup (\overline{U} \cap \partial H)$. Write $\tilde{K} = \overline{U} \cap \partial H$. Then $\tilde{K}$ is a compact subset of the linear subspace

$$\partial H = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d = 0\}$$

and

$$\partial V \subset F \cup \tilde{K}. \tag{2.3}$$

Set

$$h := \sup \Pi(V),$$

where $\Pi : \mathbb{R}^d \to \mathbb{R}$ is the mapping $(x_1, \ldots, x_d) \mapsto x_d$. Then h is positive and finite.

To prove that A is totally disconnected, we may assume that $\#A \geq 2$, since otherwise there is nothing to prove. Let $u, v \in A$ with $u \neq v$. In the following, we are going to construct an open set $W \subset \mathbb{R}^d$ such that

$$u \in W, \quad v \notin \overline{W} \quad \text{and} \quad \partial W \cap A = \emptyset. \tag{2.4}$$

Since $u, v \in A$ with $u \neq v$ are arbitrary, it will follow that A is totally disconnected.

To prove (2.4), first notice that the open set $(u + V) \setminus (v + \overline{V})$ is nonempty. Indeed, since $u \neq v$ and $(u + V) \setminus (v + \overline{V}) = ((u - v + V) \setminus \overline{V}) + v$, it suffices to show that $(a + V) \setminus \overline{V} \neq \emptyset$ for any nonzero $a \in \mathbb{R}^d$. To this end, fix $a \in \mathbb{R}^d$ with $a \neq 0$. Define

$$\lambda := \sup_{x \in V} \langle x, a \rangle,$$

where $\langle \cdot, \cdot \rangle$ is the standard inner product in $\mathbb{R}^d$. Then λ is finite as V is bounded. Also, it is clear that $\lambda = \sup_{x \in \overline{V}} \langle x, a \rangle$. Since $a \neq 0$, we have $\langle a, a \rangle > 0$. Hence, we can find $x_0 \in V$ such that $\langle x_0, a \rangle > \lambda - \langle a, a \rangle$. Thus, we have

$$\langle a + x_0, a \rangle = \langle x_0, a \rangle + \langle a, a \rangle > \lambda,$$

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which implies that $a + x_0 \in (a + V) \setminus \overline{V}$. Hence, $(u + V) \setminus (v + \overline{V})$ is a nonempty open set. By the density of D, we can pick a point $z \in D \cap [(u + V) \setminus (v + \overline{V})]$. Then we have

$$u \in z - V \quad \text{and} \quad v \notin z - \overline{V}. \tag{2.5}$$

Let

$$\tilde{D} = \{x \in D : \Pi(x) > \Pi(u) + h\}.$$

Since D is dense in $\mathbb{R}^d$, $\tilde{D}$ is clearly dense in the open half-space

$$\{x \in \mathbb{R}^d : \Pi(x) > \Pi(u) + h\}. \tag{2.6}$$

Let $x \in \mathbb{R}^d$ with $\Pi(x) > \Pi(u)$. Since $x + V$ is open and

$$\sup \Pi(x + V) = \Pi(x) + h > \Pi(u) + h,$$

we see from (2.6) that $(x + V) \cap \tilde{D} \neq \emptyset$. Equivalently, $x \in \tilde{D} - V$. Hence, we have

$$\tilde{D} - V \supset \{x \in \mathbb{R}^d : \Pi(x) > \Pi(u)\}. \tag{2.7}$$

Since $\tilde{K}$ is contained in the linear subspace $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d = 0\}$, V is contained in the open half-space $\{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$ and $z \in u + V$, we have

$$\Pi(z - \tilde{K}) = \{\Pi(z)\} \quad \text{and} \quad \Pi(z) > \Pi(u). \tag{2.8}$$

Then by (2.7), (2.8) and the compactness of $z - \tilde{K}$, we can find finitely many points $z_1, \ldots, z_k \in \tilde{D}$ so that

$$z - \tilde{K} \subset \bigcup_{i=1}^{k} (z_i - V). \tag{2.9}$$

Set

$$W := (z - V) \setminus \Big(\bigcup_{i=1}^{k} (z_i - \overline{V}) \Big). \tag{2.10}$$

See Figure 1 for an illustration of the definition of W, where for simplicity we assume that V is an open half disk. Below, we show that the open set W satisfies (2.4).

First notice that $v \notin \overline{W}$ since $v \notin z - \overline{V}$ (see (2.5)). Moreover, since $z_1, \ldots, z_k \in \tilde{D}$,

$$\inf \Pi\Big(\bigcup_{i=1}^{k} (z_i - \overline{V}) \Big) = \min_{1 \le i \le k} (\Pi(z_i) - h) > \Pi(u),$$

which implies that

$$u \notin \bigcup_{i=1}^{k} (z_i - \overline{V}).$$

Since $u \in z - V$, it follows that $u \in W$. In the following, we show that $\partial W \cap A = \emptyset$.

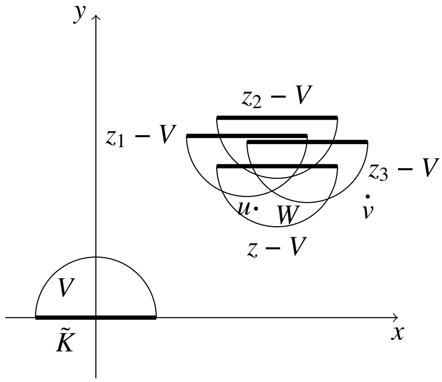

FIGURE 1. Definition of W in the proof of Lemma 2.3.

To see this, observe that by (2.3) and (2.10),

$$\partial W \subset (z - F) \cup (z - \tilde{K}) \cup \Big(\bigcup_{i=1}^{k} (z_i - F)\Big) \cup \Big(\bigcup_{i=1}^{k} (z_i - \tilde{K})\Big). \tag{2.11}$$

By (2.9), (2.10) and the compactness of $z - \tilde{K}$, we see that W is disjoint from a neighbourhood of $z - \tilde{K}$. In particular, this implies that

$$\partial W \cap (z - \tilde{K}) = \emptyset.$$

Next we show that

$$\partial W \cap \Big(\bigcup_{i=1}^{k} (z_i - \tilde{K})\Big) = \emptyset. \tag{2.12}$$

To see this, since $\tilde{K} \subset \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d = 0\}$, we have for $i \in \{1, \ldots, k\}$,

$$\Pi(z_i - \tilde{K}) = \{\Pi(z_i)\}. \tag{2.13}$$

Moreover, since $z_1, \ldots, z_k \in \tilde{D}$, we see that

$$\Pi(z_i) > h + \Pi(u), \quad i = 1, \ldots, k.$$

However, since $V \subset \{(x_1, \ldots, x_d) \in \mathbb{R}^d : x_d > 0\}$ and $z \in u + V$,

$$h + \Pi(u) \geq \Pi(z) \geq \sup \Pi(z - \overline{V}) \geq \sup \Pi(\overline{W}), \tag{2.14}$$

where the last inequality is by (2.10). Now (2.13)–(2.14) imply that for $i \in \{1, \ldots, k\}$,

$$\inf \Pi(z_i - \tilde{K}) = \Pi(z_i) > \sup \Pi(\partial W)$$

and from this, (2.12) follows.

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By (2.11)–(2.12), we see that

$$\partial W \subset (z - F) \cup \Big(\bigcup_{i=1}^{k} (z_i - F) \Big). \tag{2.15}$$

Since $z, z_1, \ldots, z_k \in D$, the definition of A implies that A has no intersection with the right-hand side of (2.15). As a consequence, $A \cap \partial W = \emptyset$. Hence, (2.4) is proved and we finish the proof of the lemma. For an illustration of the proof, see Figure 1. □

Now we deduce Lemma 2.2 from Lemma 2.3.

Proof of Lemma 2.2. We prove the lemma by contradiction. Suppose that $(E + F)^\circ = \emptyset$. Then we have, equivalently, that the set

$$D := (E + F)^c$$

is dense in $\mathbb{R}^d$. Hence, by Lemma 2.3, $\bigcap_{z \in D} (z - F)^c$ is totally disconnected. However, from the definition of D, we see that

$$E \subset \bigcap_{z \in D} (z - F)^c.$$

This contradicts the assumption that E is a connected set with cardinality greater than 1. Hence, we have $(E + F)^\circ \neq \emptyset$, completing the proof of the lemma. □

3. Proof of Theorem 1.2

We first state a classical result in convex analysis. The reader is referred to [11, Theorem 17.1] for a proof.

Theorem 3.1 (Carathéodory's theorem). *Let $S \subset \mathbb{R}^d$ and let* conv(S) *denote the convex hull of S. Then any $x \in$* conv(S) *can be represented as a convex combination of $d + 1$ elements of S.*

Now we apply the above theorem to prove the following fact.

Lemma 3.2. *Let S be a compact subset of $\mathbb{R}^2$. Suppose that $x \in \partial(\mathrm{conv}(S)) \backslash S$. Then there exist $u, v \in S$ with $u \neq v$ such that x is contained in the line segment connecting u, v, and moreover, S lies completely on one side of the line passing through u, v.*

Proof. The result might be well known. However, we are not able to find a reference, so we include a proof. By Carathéodory's theorem, x can be represented as a convex combination of three elements of S. Equivalently, x lies in a triangle with vertices in S. Since x is on the boundary of conv(S), it follows that x lies on one edge of the triangle. Let $u, v \in S$ be the endpoints of this edge. Since $x \notin S$, we have $u \neq v$.

Let $L_{u,v}$ denote the straight line passing through the points u, v. We show that S lies completely on one side of $L_{u,v}$. Suppose, in contrast, that S does not lie on one side of $L_{u,v}$. Then we can pick $w_1, w_2 \in S$ such that w_1, w_2 lie on different sides of $L_{u,v}$. Clearly, the point x lies in the interior of the quadrilateral with vertices u, v, w_1, w_2

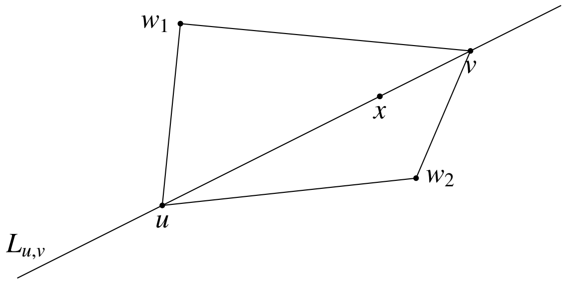

FIGURE 2. Illustration for the proof of Lemma 3.2.

(see Figure 2). However, this quadrilateral is a subset of conv(S), contradicting the assumption that $x \in \partial(\text{conv}(S))$. □

Next we prove the following proposition, which plays a key role in the proof of Theorem 1.2.

PROPOSITION 3.3. *Let F be a compact connected subset of $\mathbb{R}^2$ with empty interior. Suppose that F does not lie in a line. Then there exists a compact set $K \subset \mathbb{R}^2$ lying in a line such that $(F \cup K)^c$ has at least one bounded connected component.*

PROOF. We may assume that F^c has no bounded connected components, otherwise we simply take $K = \emptyset$.

Let conv(F) denote the convex hull of F. Since F is not contained in a straight line, conv(F) has nonempty interior and there exists a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ so that $h(\text{conv}(F))$ is the unit closed ball centred at the origin (see, for example, [4, Exercise 8.11]).

We claim $\partial(\text{conv}(F)) \not\subset F$. Suppose, in contrast, that $\partial(\text{conv}(F)) \subset F$. As $\partial(\text{conv}(F))$ is homeomorphic to the unit circle, $\mathbb{R}^2 \backslash \partial(\text{conv}(F))$ has exactly two connected components V_1, V_2, where V_1 is unbounded and V_2 bounded. Since $\partial(\text{conv}(F)) \subset F$, it follows that $F^c \subset V_1 \cup V_2$. Since F is compact, $F^c \cap V_1 \neq \emptyset$ and so $F^c \subset V_1$ by the connectedness of F^c. This implies that $V_2 \subset F$, contradicting the assumption that F has empty interior.

Pick $z \in \partial(\text{conv}(F)) \backslash F$. By Lemma 3.2, there exist $u, v \in F$ such that the straight line $L_{u,v}$ passes through z and F lies completely on one side of $L_{u,v}$. By taking a suitable rotation and translation to F, we may assume that $z = (0, 0)$, $L_{u,v}$ is the x-axis, u is on the negative part of the x-axis and v is on the positive part of the x-axis, and F lies entirely on the upper half plane.

Choose a large $R > 0$ such that F is contained in the closed half disc

$$S := \{(x, y) \in \mathbb{R}^2 : y \geq 0,\ x^2 + y^2 \leq R^2\}.$$

Let K be the line segment $[-R, R] \times \{0\}$, which is the bottom edge of S. Below, we show that $(F \cup K)^c$ has at least one bounded connected component.

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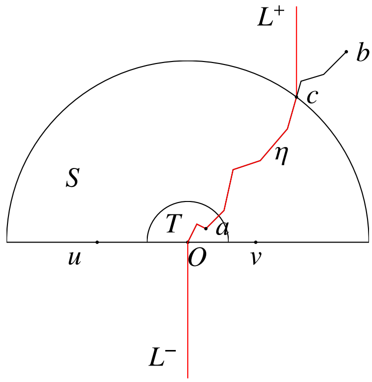

FIGURE 3. Illustration for the proof of Proposition 3.3.

Since the origin is not contained in F, there exists a small $r > 0$ such that the open half disc

$$T := \{(x, y) \in \mathbb{R}^2 : y > 0,\ x^2 + y^2 < r^2\}$$

is contained in $(F \cup K)^c$. Let V be the connected component of $(F \cup K)^c$ that contains T. Notice that S^c is contained in the unbounded connected component U of $(F \cup K)^c$. To show that V is bounded, it is enough to show that $V \neq U$ (keep in mind that $(F \cup K)^c$ has a unique unbounded connected component, due to the compactness of $F \cup K$).

Suppose, on the contrary, that $V = U$. Then $U \supset T \cup S^c$. Pick $a \in T$ and $b \in S^c$. Since U is open and connected, there exists a simple curve $\gamma \subset U$ such that γ consists of finitely many line segments and γ joins the points a, b (see, for example, [1, page 56] for a proof). Clearly, γ must intersect the open half circle

$$\Gamma := \{(x, y) : x^2 + y^2 = R^2,\ y > 0\}$$

at one or more than one points. As γ is a polygon, we may choose a sub-polygon γ_1 which joins a and a point $c \in \Gamma$ such that c is the unique intersection point of γ_1 and Γ. Connect the point a and the origin by a simple polygon $\gamma_2 \subset \overline{T}$ such that γ_2 intersects γ_1 only at the point a, and γ_2 intersects K only at the origin.

Let $\eta = \gamma_1 \cup \gamma_2$. Then η is a simple polygon, joining the origin and the point c. Except for the endpoints, points of η are contained in $U \cap S^o$. Hence, $\eta \cap F = \emptyset$.

Write $c = (c_1, c_2)$. Let L^+, L^- be the vertical half lines $L^+ := \{(c_1, y) : y \geq c_2\}$ and $L^- := \{(0, y) : y \leq 0\}$. Then the union $\eta \cup L^+ \cup L^-$ has no intersection with F. Moreover, its complement has two connected components, with u, v being contained in different components. This implies that F is disconnected, leading to a contradiction. See Figure 3 for an illustration of the proof. □

Now we combine Lemma 2.2 and Proposition 3.3 to prove Theorem 1.2.

PROOF OF THEOREM 1.2. We can assume that $F^\circ = \emptyset$, since otherwise there is nothing to prove. Since F is compact, connected, $F^\circ = \emptyset$ and F is not a line segment,

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by Proposition 3.3, there exists a compact set K contained in a line segment such that $(F \cup K)^c$ has a bounded connected component. Then it follows from Lemma 2.2 that $E + F$ has nonempty interior. □

4. Some examples

We have proved our main result Theorem 1.2 in the previous section: if $E, F \subset \mathbb{R}^2$ are connected sets with cardinalities greater than 1, and F is compact and not a line segment, then $E + F$ has nonempty interior. In this section, we present examples to show that the assumptions in this result cannot be further relaxed.

Our first example shows that there are noncompact connected sets $E, F \subset \mathbb{R}^2$, neither of which is contained in a line, such that $E + F$ has empty interior. Therefore, the compactness assumption for F in Theorem 1.2 cannot be dropped.

The example that we will give involves a result on additive functions on $\mathbb{R}$. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *additive* if $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$. It is well known that under some regularity assumptions, for instance continuity at a point or Lebesgue measurability, an additive function is necessarily linear. However, Jones [8, Theorem 5] proved the existence of discontinuous additive functions with connected graphs. Based on this result, we give the following example.

EXAMPLE 4.1. Let $f : \mathbb{R} \to \mathbb{R}$ be a discontinuous additive function whose graph $G_f := \{(x, f(x)) : x \in \mathbb{R}\}$ is connected. Let $E = F = G_f$. Then both E, F are connected and not contained in a line in $\mathbb{R}^2$. Moreover, since f is additive, it follows that $E + F = G_f$ and so $E + F$ has empty interior.

We next give examples to show that the conclusion of Theorem 1.2 may fail if $F \subset \mathbb{R}^2$ is a line segment and $E \subset \mathbb{R}^2$ is a connected set which is not contained in a line in $\mathbb{R}^2$. In our examples, we will take F to be a vertical line segment and E the graph of a certain function.

We first give a simple necessary and sufficient condition in terms of the oscillations of a function $f : \mathbb{R} \to \mathbb{R}$ for the existence of a vertical line segment L such that $G_f + L$ has empty interior.

Given a function $f : \mathbb{R} \to \mathbb{R}$, the oscillation of f at a point $x \in \mathbb{R}$ is defined by

$$\omega_f(x) = \lim_{\delta \to 0} [\sup_{y \in [x-\delta, x+\delta]} f(y) - \inf_{y \in [x-\delta, x+\delta]} f(y)].$$

We say that f is *uniformly oscillated* if $\inf_{x \in \mathbb{R}} \omega_f(x) > 0$. Clearly, f is not uniformly oscillated if f has a point of continuity.

LEMMA 4.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Then there exists a vertical line segment $L \subset \mathbb{R}^2$ such that $G_f + L$ has empty interior if and only if f is uniformly oscillated.*

PROOF. In one direction, assume that $\inf_{x \in \mathbb{R}} \omega_f(x) > 0$. Let L be a vertical line segment with length $0 < \ell < \inf_{x \in \mathbb{R}} \omega_f(x)$. Below, we show that $(G_f + L)^\circ = \emptyset$.

By applying a suitable translation, we can assume that $L = \{0\} \times [0, \ell]$. Suppose, in contrast, that $(G_f + L)^\circ \neq \emptyset$. Then, in particular, $G_f + L$ contains a horizontal line segment, say, $[a, b] \times \{c\}$ for some $a, b, c \in \mathbb{R}$ with $a < b$. Notice that

$$G_f + L = \bigcup_{x\in\mathbb{R}}((x, f(x)) + L) = \bigcup_{x\in\mathbb{R}}(\{x\} \times [f(x), f(x) + \ell])$$

is a disjoint union of vertical line segments. By this and our assumption that $G_f + L \supset [a, b] \times \{c\}$, we easily see that

$$(x, c) \in \{x\} \times [f(x), f(x) + \ell] \quad \text{for all } x \in [a, b],$$

and thus

$$c - \ell \leq f(x) \leq c \quad \text{for all } x \in [a, b]. \tag{4.1}$$

Let $x_0 = (a + b)/2$. Then (4.1) implies that $\omega_f(x_0) \leq \ell$, contradicting $\inf_{x\in\mathbb{R}} \omega_f(x) > \ell$. The contradiction yields $(G_f + L)^\circ = \emptyset$.

In the other direction, we will prove that if $\inf_{x\in\mathbb{R}} \omega_f(x) = 0$, then $(G_f + L)^\circ \neq \emptyset$ for any vertical line segment L. Assume that $\inf_{x\in\mathbb{R}} \omega_f(x) = 0$. Let L be a vertical line segment with length $\ell > 0$. Again, we can assume that $L = \{0\} \times [0, \ell]$.

Since $\inf_{x\in\mathbb{R}} \omega_f(x) = 0$, then by definition, we can find $x_0 \in \mathbb{R}$ and $\delta > 0$ such that

$$\sup_{x\in[x_0-\delta,x_0+\delta]} f(x) - \inf_{x\in[x_0-\delta,x_0+\delta]} f(x) < \ell/4.$$

This clearly implies that

$$f(x_0) - \ell/4 < f(x) < f(x_0) + \ell/4 \tag{4.2}$$

for all $x \in [x_0 - \delta, x_0 + \delta]$. We claim that $G_f + L$ contains the rectangle

$$R := [x_0 - \delta, x_0 + \delta] \times [f(x_0) + \ell/4, f(x_0) + (3\ell)/4].$$

To see this, let $(x, y) \in R$. Then, $f(x_0) + \ell/4 \leq y \leq f(x_0) + (3\ell)/4$. By this and (4.2), we see that $0 \leq y - f(x) \leq \ell$. Hence, $(0, y - f(x)) \in L$. Therefore, we have

$$(x, y) = (x, f(x)) + (0, y - f(x)) \in G_f + L.$$

Since $(x, y) \in R$ is arbitrary, it follows that $R \subset G_f + L$. This proves the above claim, and in particular, that $(G_f + L)^\circ \neq \emptyset$. □

According to Lemma 4.2, if $f : \mathbb{R} \to \mathbb{R}$ is a uniformly oscillated function with a connected graph, then taking $E = G_f$ and F an appropriate vertical line segment, $E + F$ has empty interior. Such functions do exist as shown in the following examples.

EXAMPLE 4.3. Jones [8, Theorems 1,2] proved that there are additive functions on $\mathbb{R}$ whose graphs are connected and dense in $\mathbb{R}^2$. Let f be such a function. Since G_f is dense in $\mathbb{R}^2$, it is easy to see that $\inf_{x\in\mathbb{R}} \omega_f(x) = \infty$. Let $E = G_f$ and let $F \subset \mathbb{R}^2$ be a vertical line segment. From the proof of Lemma 4.2, $E + F$ has empty interior.

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Recently, Rosen [12] proved that the set E in Example 4.3 has positive two-dimensional Lebesgue measure. Hence, by Fubini's theorem, E is not Lebesgue measurable. Below, we give another example in which E is Borel.

EXAMPLE 4.4. The Cesàro function $f : [0, 1] \to \mathbb{R}$ is defined by

$$f(x) := \limsup_{n\to\infty} \frac{a_1(x) + \cdots + a_n(x)}{n},$$

where

$$x = \sum_{n=1}^{\infty} \frac{a_n(x)}{2^n}$$

is the binary expansion of x. Here, we adopt the convention that $a_n(x) = 1$ for all large n if x has two different binary expansions.

Notice that f is a Borel function, and hence its graph G_f is a Borel subset of $\mathbb{R}^2$. Also, it is easy to check that $\inf_{x\in[0,1]} \omega_f(x) = 1$. Moreover, Vietoris [16] proved that G_f is connected.

Let $E = G_f$ and $F \subset \mathbb{R}^2$ be a vertical line segment of length less than 1. Then, we see from the proof of Lemma 4.2 that $E + F$ has empty interior.

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