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On a Product Related to the Cubic Gauss Sum, III

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Abstract. We have seen, in the previous works [5], [6], that the argument of a certain product is closely connected to that of the cubic Gauss sum. Here the absolute value of the product will be investigated.

1 Introduction

Let $\rho = e^{2\pi i/3}$ and ω be the generator of a prime ideal of degree one in $\mathbf{Q}(\rho)$ which satisfies the congruence $\omega \equiv 1 \pmod 3$. Let p be the norm of ω. Define two real analytic functions $g(z)$ and $G(z)$ on the complex plane $\mathbf{C}$ by

$$g(z) = e(z) + \rho e(\rho z) + \rho^2 e(\rho^2 z),$$
$$G(z) = e(z) + e(\rho z) + e(\rho^2 z).$$

Here,

$$e(z) = \exp\bigl(2\pi i(z - \bar{z})/\lambda\bigr), \quad \lambda = \rho - \rho^2 = i\sqrt{3}$$

and $\bar{z}$ denotes the complex conjugate of z. These functions are periodic with respect to $\mathbf{Z}[\rho]$, the integer ring of $\mathbf{Q}(\rho)$. Take a $\frac{1}{3}$-representative system modulo ω and denote it by S; S consists of $(p-1)/3$ elements of $\mathbf{Z}[\rho]$ and the numbers

$$s, \rho s, \rho^2 s \quad (s \in S),$$

together with 0, form a complete representative system modulo ω. Define the cube root $\alpha(S)$ of -1 by the congruence

$$\alpha(S) \equiv \prod_{s \in S} s \pmod{\omega}.$$

The existence of $\alpha(S)$ is a consequence of Wilson's theorem. Let us consider the products

$$\delta(\omega) = \alpha(S) \prod_{s \in S} g\left(\frac{s}{\omega}\right), \quad \Delta(\omega) = \prod_{s \in S} G\left(\frac{s}{\omega}\right).$$

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These are independent of the choice of S and $\Delta(\omega)$ is a positive rational integer. In our previous papers, we studied the connection between these products and the cubic Gauss sum which is defined by

$$\tau(\omega) = \prod_{a=1}^{p-1} \left(\frac{a}{\omega}\right)_3 e^{2\pi i a/p}.$$

Here, $(\frac{a}{\omega})_3$ is the cubic residue symbol of $\mathbf{Q}(\rho)$.

Theorem 1 (**[5]**, **[6]**) *For any positive real number ε, we have*

$$\left(\frac{a}{\omega}\right)_3^{-1} \tau(\omega)\frac{\delta(\omega)}{\Delta(\omega)} = Ce^{O(p^{-\frac{1}{2}+\varepsilon})} \quad \text{as } p \to \infty$$

with a positive absolute constant C. In particular, we have

$$\arg\left\{\left(\frac{a}{\omega}\right)_3^{-1} \tau(\omega)\delta(\omega)\right\} = O(p^{-\frac{1}{2}+\varepsilon}) \quad \text{as } p \to \infty. \tag{1.1}$$

Now, as is well known, the absolute value of the Gauss sum $\tau(\omega)$ is $\sqrt{p}$ and it is natural to ask about the absolute value of the product $\delta(\omega)$ which is closely connected to the Gauss sum. The main purpose of this paper is to answer this question. Denote by D a fundamental domain of $\mathbf{C}/\mathbf{Z}[\rho]$ and μ the Lebesgue measure of $\mathbf{C}$.

Theorem 2 *Let*

$$I = \frac{2}{\sqrt{3}} \int_D \log|G(z)|\, d\mu.$$

(i) For any positive real number ε, we have

$$|\delta(\omega)| = e^{\frac{pI}{3}+O(p^{\varepsilon})}, \quad |\Delta(\omega)| = e^{\frac{pI}{3}+O(p^{\varepsilon})} \quad \text{as } p \to \infty.$$

(ii) The constant I is positive.

We see from Theorem 1 that two estimates in (i) above are equivalent to each other.

Let $\zeta = e^{2\pi i/p}$, $N = \mathbf{Q}(\zeta)$, K be the subfield of N of degree $(p-1)/3$ and O_N and O_K be the integer rings of N and K respectively. Take an integer f satisfying

$$f \equiv \rho \pmod{\omega}.$$

Then, as a special case of Theorem 2.1 (iii) of Brinkhuis [2], we see that

$$[O_N : O_K\zeta + O_K\zeta^f + O_K\zeta^{f^2}] = \Delta(\omega)\frac{\delta(\omega)}{\tau(\omega)}\overline{\left(\frac{\delta(\omega)}{\tau(\omega)}\right)} \tag{1.2}$$

and Theorem 2 gives the following assertion.

Corollary 3 *Notations being as above, we have, for any positive real number ε,*

$$[O_N : O_K\zeta + O_K\zeta^f + O_K\zeta^{f^2}] = e^{pI+O(p^\varepsilon)} \quad \textit{as } p \to \infty.$$

Note that the set $\{\zeta, \zeta^f, \zeta^{f^2}\}$ is the full set of conjugates of ζ with respect to the extension N/K. It is easy to see the facts corresponding to Theorem 2 and Corollary 3 in the quadratic case and we understand that the situation is quite different (*cf.* Fröhlich [3, pp. 221–222]). One of our fundamental interests lies in pursuing the analogue of Theorem 1 in higher degree cases. Related to this problem, it would be of some interest to investigate whether or not Theorem 2 and Corollary 3 can be extended to these cases. Computer calculation suggests the possibility of this extension. We hope to discuss this topic in the future.

In the following, we prove Theorem 2(i) in Sections 2 and 3 and show Theorem 2(ii) in Section 4. In Section 5, some comments related to the discussion in Brinkhuis [1], [2] and remarks concerning further problems will be added.

2 Estimation of $|\delta(\omega)|$ and $|\Delta(\omega)|$

We shall prove (i) of Theorem 2. As is mentioned in Section 1, it is enough to show the assertion about $|\delta(\omega)|$, and since

$$\delta(\omega)^3 = -\prod_{\substack{a \bmod \omega \\ a \not\equiv 0}} g\left(\frac{a}{\omega}\right)$$

it suffices to prove the following:

$$\frac{\sqrt{3}}{2p} \sum_{\substack{a \bmod \omega \\ a \not\equiv 0}} \log\left|g\left(\frac{a}{\omega}\right)\right| = \int_D \log|g(z)|\,d\mu + O(p^{-1+\varepsilon}) \quad (p \to \infty). \tag{2.1}$$

We note that

$$\int_D \log|G(z)|\,d\mu = \int_D \log|g(z)|\,d\mu, \tag{2.2}$$

which follows from the periodicity and $G(z) = g(z - \frac{1}{3})$. Put, as in [5] and [6],

$$D = \{z \in \mathbf{C}\,;\, |z| < |z-u| \ \ (0 \neq u \in \mathbf{Z}[\rho])\},$$

$$D(v) = v + \frac{1}{\omega}D \quad (v \in \mathbf{C})$$

and $R = \frac{1}{\omega}\mathbf{Z}[\rho] \cap D$. Then, since there is no point of $\frac{1}{\omega}\mathbf{Z}[\rho]$ on the boundary ∂D of D, R is a complete representative system of $\frac{1}{\omega}\mathbf{Z}[\rho]/\mathbf{Z}[\rho]$. It follows that

$$\sum_{\substack{a \bmod \omega \\ a \not\equiv 0}} \log\left|g\left(\frac{a}{\omega}\right)\right| = \sum_{0 \neq r \in R} \log|g(r)|,$$

$$\int_D \log|g(z)|\,d\mu = \sum_{r \in R} \int_{D(r)} \log|g(z)|\,d\mu.$$

The integral $\int_{D(r)} \log|g(z)|\,d\mu$ exists by Lemma 1 of [6] and hence the integrals in (2.2) also exist. We see that (2.1) is equivalent to

$$\frac{\sqrt{3}}{2p}\sum_{0\neq r\in R}\log|g(r)| = \sum_{r\in R}\int_{D(r)}\log|g(z)|\,d\mu + O(p^{-1+\varepsilon}) \quad (p\to\infty). \tag{2.3}$$

As in [6], let

$$M_1 = \left\{0, \pm\frac{1}{\lambda}, \frac{\rho^j}{3}(j=0,1,2)\right\},$$
$$M = \{z\in \mathbf{C}\,;\, z\equiv m \pmod{\mathbf{Z}[\rho]} \text{ for some } m\in M_1\}.$$

The function $g(z)$ vanishes if and only if z is contained in M. Set

$$d(z,M) = \inf_{m\in M}|z-m| \quad (z\in\mathbf{C}),$$
$$U = \left\{z\in\mathbf{C}\,;\, d(z,M) < \frac{1}{12}\right\},\quad V = \left\{z\in\mathbf{C}\,;\, d(z,M)\geq \frac{1}{12}\right\},$$
$$R_U = R\cap U,\quad R_V = R\cap V$$

and, for z with $d(z,M) < \frac{1}{6}$, define $m(z)$ to be the point of M which is nearest to z. Also, for every m in M_1, put

$$R_{U,m} = \{r\in R_U\,;\, m(r)\equiv m \pmod{\mathbf{Z}[\rho]}\}.$$

One has

$$R = R_U\cup R_V,\quad R_U = \bigcup_{m\in M_1} R_{U,m} \quad \text{(disjoint unions)}. \tag{2.4}$$

Furthermore, as in (2.13), (2.14) and (2.16) of [6], let

$$E(v) = \int_{D(v)}\big(\log|g(z)| - \log|g(v)|\big)\,d\mu,$$
$$E_0(u) = \int_{D(u)}\Big\{\big(\log|g(z)| - \log|z-m(u)|\big) - \big(\log|g(u)| - \log|u-m(u)|\big)\Big\}\,d\mu,$$
$$F(u) = \int_{D(u)}\big(\log|z-m(u)| - \log|u-m(u)|\big)\,d\mu.$$

Here, u and v are complex numbers with $0 < d(u,M) < \frac{1}{6}$ and $v\notin M$. As is seen in (2.12) of [6], we have, for each m in M,

$$\log|g(m+z)| - \log|z| = \log(2\sqrt{3}\pi) + O(|z|) \quad \text{as } |z|\to\infty.$$

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This is the motivation to introduce $E_0(u)$. Because the area of $D(v)$ is $\frac{\sqrt{3}}{2p}$, we see that

$$\frac{\sqrt{3}}{2p}\log|g(v)| = \int_{D(v)} \log|g(z)|\,d\mu - E(v),$$
$$\frac{\sqrt{3}}{2p}\log|g(u)| = \int_{D(u)} \log|g(z)|\,d\mu - F(u) - E_0(u).$$

It follows that the left hand side of (2.3) is equal to the following:

$$\begin{aligned}\frac{\sqrt{3}}{2p}\sum_{0\neq r\in R}\log|g(r)| &= \sum_{0\neq r\in R_U}\left\{\int_{D(r)}\log|g(z)|\,d\mu - F(r) - E_0(r)\right\}\\ &\quad+\sum_{r\in R_V}\left\{\int_{D(r)}\log|g(z)|\,d\mu - E(r)\right\}\\ &=\sum_{r\in R}\int_{D(r)}\log|g(z)|\,d\mu - \int_{D(0)}\log|g(z)|\,d\mu\\ &\quad-\sum_{0\neq r\in R_U}F(r) - \sum_{0\neq r\in R_U}E_0(r) - \sum_{r\in R_V}E(r).\end{aligned}\tag{2.5}$$

From Lemma 2 of [6],

$$\int_{D(0)}\log|g(z)|\,d\mu = -\frac{\sqrt{3}}{4p}\log p + \frac{1}{p}I_1 + \frac{\sqrt{3}}{2p}\log(2\sqrt{3}\pi) + O(p^{-\frac{3}{2}})\tag{2.6}$$

with $I_1 = \int_D \log|z|\,d\mu$. Here, the implied constant is absolute. Therefore, to prove (2.3) it suffices to show that

$$\sum_{0\neq r\in R_U}F(r) + \sum_{0\neq r\in R_U}E_0(r) + \sum_{r\in R_V}E(r) = O(p^{-1+\varepsilon})\quad (p\to\infty).\tag{2.7}$$

Now, from Lemma 6 of [6], we have, for every m in M_1,

$$p\sum_{0\neq r\in R_U,m}F(r) = \sum_{0\neq a\in \mathbf{Z}[\rho]-m}\int_D \log\left|1+\frac{z}{a}\right|\,d\mu + O(p^{-2}),\tag{2.8}$$

where the summation over a is absolutely convergent. Therefore, from (2.4),

$$\begin{aligned}\sum_{0\neq r\in R_U}F(r) &= \frac{1}{p}\sum_{m\in M_1}\sum_{0\neq a\in\mathbf{Z}[\rho]-m}\int_D\log\left|1+\frac{z}{a}\right|\,d\mu + O(p^{-3})\\ &= O(p^{-1}).\end{aligned}\tag{2.9}$$

Hence, we see from the following lemma that the estimate (2.7) holds and this implies that we have proved (2.3), (2.1) and Theorem 2(i).

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Lemma 1 *We have*

$$\sum_{r\in R_V} E(r) = O(p^{-1})$$

and

$$\sum_{0\neq r\in R_U} E_0(r) = O(p^{-1+\varepsilon})$$

for any positive real number ε*.*

We shall prove the above lemma in the next section.

Remark The estimate

$$\sum_{0\neq r\in R_U} F(r) + \sum_{0\neq r\in R_U} E_0(r) + \sum_{r\in R_V} E(r) = O(p^{-\frac{1}{2}}) \quad (p\to\infty), \tag{2.10}$$

which leads to the estimates

$$|\delta(\omega)| = e^{\frac{pI}{3}+O(p^{\frac{1}{2}})}, \quad |\Delta(\omega)| = e^{\frac{pI}{3}+O(p^{\frac{1}{2}})} \quad (p\to\infty),$$

follows directly from results in [6]. Namely, by Lemma 3 of [6], we have

$$\begin{gathered} E(v) = O(p^{-\frac{3}{2}}) \quad (v\in V),\\ E_0(u) = O(p^{-\frac{3}{2}}) \quad (0 < d(u,M) < \frac{1}{6}), \end{gathered} \tag{2.11}$$

and the trivial estimates

$$\#R_U \leq \#R = p, \quad \#R_V \leq \#R = p \tag{2.12}$$

give

$$\sum_{0\neq r\in R_U} E_0(r) = O(p^{-\frac{1}{2}}) \quad \sum_{r\in R_V} E(r) = O(p^{-\frac{1}{2}}),$$

establishing (2.10). The point of the next section is that we are trying to make the error terms as small as possible, partly because we hope to compare the results with an analogue of Theorem 2 in higher degree cases.

3 Proof of Lemma 1

Let us first introduce some notation. Taking an integer K greater than one, define U_k $(1\leq k\leq K)$, as in (2.17) of [6], by

$$\begin{gathered} U_1 = \{z\in \mathbf{C}\,;\, d(z,M) < p^{-\frac{1}{2}+\frac{1}{2K}}\},\\ U_k = \{z\in \mathbf{C}\,;\, p^{-\frac{1}{2}+\frac{k-1}{2K}} \leq d(z,M) < p^{-\frac{1}{2}+\frac{k}{2K}}\} \quad (2\leq k\leq K-1),\\ U_K = \left\{z\in \mathbf{C}\,;\, p^{-\frac{1}{2K}} \leq d(z,M) < \frac{1}{12}\right\} \end{gathered} \tag{3.1}$$

and put

$$R_k = R \cap U_k \quad (1 \le k \le K).$$

In the following, some of the arguments are valid only when p is sufficiently large with respect to K. We assume this. All the implied constants depend at most on K. We have

$$U = \bigcup_{k=1}^{K} U_k, \quad R_U = \bigcup_{k=1}^{K} R_k. \tag{3.2}$$

Also, it is easy to see that

$$\# R_k = O(p^{k/K}) \quad (1 \le k \le K), \tag{3.3}$$

cf. (2.18) of [6].

Now, from (2.11) and (3.3), we see that

$$\sum_{0 \ne r \in R_1} E_0(r) = O(p^{-\frac{3}{2}+\frac{1}{K}}). \tag{3.4}$$

Next, let $2 \le k \le K$, $r \in R_k$ and $m = m(r)$. Put

$$T(z) = T(z, m) = \log |g(z)| - \log |z - m|.$$

Then we have

$$E_0(r) = \int_{D(r)} \big(T(z) - T(r) \big)\, d\mu.$$

Consider now the Taylor expansion of $T(z)$ around r [6, (3.3)]. Note that $\log |g(z)|$ is differentiable infinitely many times outside M and we have, for every pair $(a, b) \ne (0, 0)$ of non-negative integers,

$$\frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b} \big(\log |g(m+z)| - \log |z| \big) = O(|z|^{1-a-b}) \quad \text{as } |z| \to 0, \tag{3.5}$$

cf. (2.10) of [6]. Then, due to the cancellation arising from the fact that $D(r)$ is a regular hexagon [6, (3.4)], we get, as in (3.5) of [6], that

$$E_0(r) = p^{-2} \int_D |z|^2\, d\mu \cdot \frac{\partial^2 T}{\partial z \partial \bar{z}}(r) + O(p^{-\frac{3}{2}-\frac{k-1}{K}}). \tag{3.6}$$

Therefore, from (3.3) it follows that

$$\sum_{r \in R_k} E_0(r) = p^{-2} \int_D |z|^2\, d\mu \cdot \sum_{r \in R_k} \frac{\partial^2 T}{\partial z \partial \bar{z}}(r) + O(p^{-\frac{3}{2}+\frac{1}{K}}) \quad (2 \le k \le K).$$

From (3.2) and (3.4),

$$(3.7) \qquad \sum_{0\neq r\in R_U} E_0(r) = p^{-2}\int_D |z|^2\, d\mu \cdot \sum_{k=2}^{K}\sum_{r\in R_k} \frac{\partial^2 T}{\partial z\partial \bar z}(r) + O(p^{-\frac{3}{2}+\frac{1}{K}}).$$

For $E(r)$ ($r \in R_V$) also, we get, in the same way as we get (3.6),

$$(3.8) \qquad E(r) = p^{-2}\int_D |z|^2\, d\mu \cdot \frac{\partial^2}{\partial z\partial \bar z}\log|g(r)| + O(p^{-\frac{5}{2}}),$$

cf. [6, p. 14]. Corresponding to the aid of (3.5) in deriving (3.6), we use in the above the fact that $\frac{\partial^{a+b}}{\partial z^a\partial \bar z^b}\log|g(z)|$ is continuous outside M and in particular is bounded on V [6, p. 14]. We see from (2.12) that

$$(3.9) \qquad \sum_{r\in R_V} E(r) = p^{-2}\int_D |z|^2\, d\mu \cdot \sum_{r\in R_V}\frac{\partial^2}{\partial z\partial \bar z}\log|g(r)| + O(p^{-\frac{3}{2}}).$$

Now, (3.1) says that

$$|r-m| \geq p^{-\frac{1}{2}+\frac{k-1}{2K}}$$

if $k \geq 2$, $r \in R_k$ and $m = m(r)$ and from (3.5) we see that

$$\frac{\partial^2 T}{\partial z\partial \bar z}(z) = O(|z-m|^{-1}) \quad \text{as } z \to m.$$

Hence,

$$\frac{\partial^2 T}{\partial z\partial \bar z}(r) = O(p^{\frac{1}{2}-\frac{k-1}{2K}}).$$

By (3.3),

$$(3.10) \qquad \sum_{r\in R_k}\frac{\partial^2 T}{\partial z\partial \bar z}(r) = O(p^{\frac{1}{2}+\frac{k+1}{2K}}) = \begin{cases} O(p) & \text{if } 2 \leq k \leq K-1 \\ O(p^{1+\frac{1}{2K}}) & \text{if } k = K \end{cases}$$

and therefore,

$$\sum_{k=2}^{K}\sum_{r\in R_k}\frac{\partial^2 T}{\partial z\partial \bar z}(r) = O(p^{1+\frac{1}{2K}}).$$

We have from (3.7) that

$$(3.11) \qquad \sum_{0\neq r\in R_U} E_0(r) = O(p^{-1+\frac{1}{2K}}).$$

This proves the second assertion of the lemma since K can be arbitrarily large. Let us consider the right hand side of (3.9). Because $\frac{\partial^2}{\partial z\partial \bar z}\log|g(z)|$ is bounded on V, we see from (2.12) that

$$\sum_{r\in R_V}\frac{\partial^2}{\partial z\partial \bar z}\log|g(r)| = O(p).$$

Hence we get from (3.9) that

$$(3.12) \qquad \sum_{r\in R_V} E(r) = O(p^{-1}),$$

which is the first assertion of the lemma. We have proved Lemma 1.

4 The Integral $\int_D \log|G(z)|\,d\mu$

Let us prove (ii) of Theorem 2. The point of the assertion is that the constant I is not zero since it can not be negative by (i) of Theorem 2. We observe that $e(-z) = \overline{e(z)}$, $e(\bar{z}) = \overline{e(z)}$ and that

$$G(\rho z) = G(z), \quad G(-z) = \overline{G(z)}, \quad G(\bar{z}) = \overline{G(z)}.$$

Hence,

$$|G(-\rho z)| = |G(\bar{z})| = |G(z)|$$

and it follows that

$$\int_D \log|G(z)|\,d\mu = 12\int_A \log|G(z)|\,d\mu$$

with

$$A = \left\{ z = x + iy\,;\, 0 < x < \frac{1}{2},\ y > 0,\ x - \sqrt{3}y > 0 \right\}.$$

Writing $z = x + iy$, we see

$$\begin{aligned} e(z)^{-1}G(z) &= 1 + e\big((\rho - 1)z\big) + e\big((\rho^2 - 1)z\big) \\ &= 1 + e^{2\pi i(x-\sqrt{3}y)} + e^{2\pi i(-x-\sqrt{3}y)} \end{aligned}$$

and

$$|G(z)|^2 = 3 + 2\big(\cos 2\pi(x - \sqrt{3}y) + \cos 2\pi(x + \sqrt{3}y) + \cos 4\pi x\big)$$

$$= 1 + 4\cos 2\pi x(\cos 2\pi\sqrt{3}y + \cos 2\pi x) \tag{4.1}$$

$$= 1 + 8\cos 2\pi x \cos\pi(x + \sqrt{3}y)\cos\pi(x - \sqrt{3}y). \tag{4.2}$$

Let

$$\begin{aligned} A_1 &= \left\{ z\,;\, 0 < x < \frac{1}{4},\ y > 0,\ x - \sqrt{3}y > 0 \right\} \\ A_2 &= \left\{ z\,;\, \frac{1}{4} < x < \frac{1}{2},\ y > 0,\ x + \sqrt{3}y < \frac{1}{2} \right\} \\ A_3 &= \left\{ z\,;\, \frac{1}{4} < x < \frac{1}{2},\ \frac{1}{2} - x < \sqrt{3}y < x \right\}. \end{aligned}$$

Then, we see from (4.2) that

$$\begin{aligned} |G(z)|^2 &> 1 \quad \text{if } z \in A_1 \cup A_3, \\ |G(z)|^2 &< 1 \quad \text{if } z \in A_2. \end{aligned}$$

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First, we shall estimate the integral of $\log|G(z)|$ over $A_1 \cup A_3$. Here, $0 < |G(z)|^2 - 1 \leq 8$ and we have

$$\begin{aligned} 2\log|G(z)| &= \log|G(z)|^2 \\ &\geq \frac{\log 8}{8}\big(|G(z)|^2 - 1\big) = \frac{\log 8}{2}\cos 2\pi x(\cos 2\pi\sqrt{3}y + \cos 2\pi x) \end{aligned}$$

by (4.1). It follows that

$$\begin{aligned} \int_{A_1} \log|G(z)|\,d\mu &\geq \frac{\log 8}{4}\int_0^{\frac{1}{4}} dx \int_0^{\frac{x}{\sqrt{3}}} \cos 2\pi x(\cos 2\pi\sqrt{3}y + \cos 2\pi x)\,dy \\ &= \frac{(\pi^2+4)\log 8}{256\sqrt{3}\pi^2} \end{aligned}$$

and

$$\begin{aligned} \int_{A_3} \log|G(z)|\,d\mu &\geq \frac{\log 8}{4}\int_{\frac{1}{4}}^{\frac{1}{2}} dx \int_{-\frac{x}{\sqrt{3}}+\frac{1}{2\sqrt{3}}}^{\frac{x}{\sqrt{3}}} \cos 2\pi x(\cos 2\pi\sqrt{3}y + \cos 2\pi x)\,dy \\ &= \frac{(\pi^2+4)\log 8}{128\sqrt{3}\pi^2}. \end{aligned}$$

Hence, we have

$$\int_{A_1\cup A_3} \log|G(z)|\,d\mu \geq \frac{\sqrt{3}(\pi^2+4)\log 8}{256\pi^2}. \tag{4.3}$$

In order to estimate the integral $\int_{A_2}\log|G(z)|\,d\mu$, we expand $G(z+\frac{1}{3}) = g(z)$ into a power series in z and $\bar{z}$. Put, for $j \in \mathbf{Z}/3\mathbf{Z}$,

$$g_j(z) = e(z) + \rho^j e(\rho z) + \rho^{2j} e(\rho^2 z).$$

Then,

$$\frac{\partial}{\partial z} g_j(z) = \frac{2\pi}{\sqrt{3}} g_{j+1}(z), \qquad \frac{\partial}{\partial \bar{z}} g_j(z) = -\frac{2\pi}{\sqrt{3}} g_{j-1}(z)$$

and so

$$\frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b} g_j(z) = (-1)^b \left(\frac{2\pi}{\sqrt{3}}\right)^{a+b} g_{j+a-b}(z).$$

In particular, for $g(z) = g_1(z)$, we see that

$$\frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b} g(z) = (-1)^b \left(\frac{2\pi}{\sqrt{3}}\right)^{a+b} g_{1+a-b}(z)$$

and

$$\frac{\partial^{a+b} g}{\partial z^a \partial \bar{z}^b}(0) = \begin{cases} 3(-1)^b(\frac{2\pi}{\sqrt{3}})^{a+b}, & 1+a-b \equiv 0 \pmod 3 \\ 0, & \text{otherwise.} \end{cases}$$

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Hence, we have an expansion

$$
\begin{aligned}
g(z) &= 3\sum_{a,b=0}^{\infty}\frac{1}{a!\,b!}(-1)^b\left(\frac{2\pi}{\sqrt{3}}\right)^{a+b} z^a\bar{z}^b \quad (1+a-b\equiv 0 \pmod 3) \\
&= -2\pi\sqrt{3}\bar{z}\bigl(1-B(z)\bigr)
\end{aligned}
\tag{4.4}
$$

with

$$
B(z)=\sum_{a+b\geq 2}\frac{1}{a!\,b!}(-1)^b\left(\frac{2\pi}{\sqrt{3}}\right)^{a+b-1} z^a\bar{z}^{b-1} \quad (1+a-b\equiv 0 \pmod 3).
$$

The power series converges absolutely for every z.

Assume that $|z|\leq R$. Then, we see that

$$
\begin{aligned}
|B(z)| &\leq \sum_{a+b\geq 2}\frac{1}{a!\,b!}\left(\frac{2\pi}{\sqrt{3}}\right)^{a+b-1} R^{a+b-1} \quad (1+a-b\equiv 0 \pmod 3) \\
&= \frac{\sqrt{3}}{2\pi R}\sum_{a,b\geq 0}\frac{1}{a!\,b!}\left(\frac{2\pi R}{\sqrt{3}}\right)^{a+b} - 1 \quad (1+a-b\equiv 0 \pmod 3) \\
&= -1+\frac{\sqrt{3}}{2\pi R}\sum_{j=0}^{2}\sum_{\substack{a\geq 0\\ a\equiv j(3)}}\frac{1}{a!}\left(\frac{2\pi R}{\sqrt{3}}\right)^{a}\sum_{\substack{b\geq 0\\ b\equiv 1+j(3)}}\frac{1}{b!}\left(\frac{2\pi R}{\sqrt{3}}\right)^{b} \\
&= -1+\frac{\sqrt{3}}{2\pi R}\frac{1}{9}\sum_{j=0}^{2}\bigl(e^{\frac{2\pi R}{\sqrt{3}}}+\rho^{-j}e^{\frac{2\pi R}{\sqrt{3}}\rho}+\rho^{-2j}e^{\frac{2\pi R}{\sqrt{3}}\rho^2}\bigr) \\
&\qquad\times\bigl(e^{\frac{2\pi R}{\sqrt{3}}}+\rho^{-j-1}e^{\frac{2\pi R}{\sqrt{3}}\rho}+\rho^{-2j-2}e^{\frac{2\pi R}{\sqrt{3}}\rho^2}\bigr) \\
&= -1+\frac{\sqrt{3}}{2\pi R}\frac{1}{9}3\bigl(e^{\frac{4\pi R}{\sqrt{3}}}+\rho^{-1}e^{\frac{2\pi R}{\sqrt{3}}(\rho+\rho^2)}+\rho^{-2}e^{\frac{2\pi R}{\sqrt{3}}(\rho+\rho^2)}\bigr) \\
&= -1+\frac{1}{2\sqrt{3}\pi R}\bigl(e^{\frac{4\pi R}{\sqrt{3}}}-e^{-\frac{2\pi R}{\sqrt{3}}}\bigr).
\end{aligned}
$$

In particular, we have

$$
|B(z)|\leq c \quad \text{if } |z|\leq\frac{1}{12}
$$

with

$$
c=-1+\frac{2\sqrt{3}}{\pi}\bigl(e^{\frac{\pi}{3\sqrt{3}}}-e^{-\frac{\pi}{6\sqrt{3}}}\bigr)=0.20344\cdots. \tag{4.5}
$$

It follows from (4.4) that

$$
\log|g(z)|\geq\log\bigl(2\pi\sqrt{3}(1-c)\bigr)+\log|z| \quad \text{if } 0<|z|\leq\frac{1}{12}. \tag{4.6}
$$

Now, set

$$A_{20} = \left\{ z \in A_2 \,;\, \left| z - \frac{1}{3} \right| < \frac{1}{12} \right\},$$

$$A_{21} = \left\{ z \in A_2 \,;\, \left| z - \frac{1}{3} \right| < \frac{1}{12}, \frac{1}{4} < x < \frac{5}{12} \right\},$$

$$\begin{aligned} A_{22} &= \left\{ z \in A_2 \,;\, \frac{5}{12} < x \right\} \\ &= \left\{ z \,;\, \frac{5}{12} < x < \frac{1}{2}, y > 0, x + \sqrt{3}y < \frac{1}{2} \right\}. \end{aligned}$$

It holds that

$$\int_{A_2} \log|G(z)|\, d\mu = \sum_{k=0}^{2} \int_{A_{2k}} \log|G(z)|\, d\mu.$$

By (4.6), we see that

$$\begin{aligned} \int_{A_{20}} \log|G(z)|\, d\mu &= \int_{\substack{|z| \le \frac{1}{12} \\ y>0}} \log|g(z)|\, d\mu \\ &\ge \int_{\substack{|z| \le \frac{1}{12} \\ y>0}} \Big(\log\big(2\pi\sqrt{3}(1-c)\big) + \log|z| \Big)\, d\mu \\ &= \log\big(2\pi\sqrt{3}(1-c)\big) \cdot \pi \frac{1}{12^2 2} + \int_0^{\frac{1}{12}} dr \int_0^{\pi} r \log r\, d\theta \\ &= \frac{\pi}{288} \log\big(2\pi\sqrt{3}(1-c)\big) - \pi \left(\frac{\log 12}{288} + \frac{1}{576} \right) \\ &= \frac{\pi}{288} \log\left(\frac{\pi(1-c)}{2\sqrt{3}} \right) - \frac{\pi}{576}. \end{aligned} \tag{4.7}$$

By (4.1), $|G(z)|$ is monotonically increasing with respect to y in A_2. Therefore, from (4.6) we have in A_{21} that

$$\log|G(z)| \ge \log\big(2\pi\sqrt{3}(1-c)\big) + \log\frac{1}{12} = \log\frac{\pi(1-c)}{2\sqrt{3}}.$$

In the same way, since $-1 < \cos 2\pi x < -\frac{\sqrt{3}}{2}$ in A_{22}, we have by (4.1) that

$$\begin{gathered} |G(z)|^2 \ge 1 + 4\cos 2\pi x(1 + \cos 2\pi x) > 4 - 2\sqrt{3}, \\ \log|G(z)| \ge \frac{1}{2}\log(4 - 2\sqrt{3}). \end{gathered}$$

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Since the area of A_{21} is $\frac{1}{36\sqrt{3}} - \frac{\pi}{288}$, we get

$$\int_{A_{21}} \log|G(z)|\,d\mu \geq \left(\frac{1}{36\sqrt{3}} - \frac{\pi}{288}\right) \log \frac{\pi(1-c)}{2\sqrt{3}}. \tag{4.8}$$

Also, the area of A_{22} is $\frac{1}{288\sqrt{3}}$ and

$$\int_{A_{22}} \log|G(z)|\,d\mu \geq \frac{\log(4-2\sqrt{3})}{576\sqrt{3}}. \tag{4.9}$$

From (4.7), (4.8) and (4.9) follows that

$$\int_{A_2} \log|G(z)|\,d\mu \geq -\frac{\pi}{576} + \frac{1}{36\sqrt{3}} \log \frac{\pi(1-c)}{2\sqrt{3}} + \frac{\log(4-2\sqrt{3})}{576\sqrt{3}}$$

and together with (4.3) we conclude that

$$\int_{A} \log|G(z)|\,d\mu \geq \frac{\sqrt{3}(\pi^2+4)\log 8}{256\pi^2} - \frac{\pi}{576} + \frac{1}{36\sqrt{3}} \log \frac{\pi(1-c)}{2\sqrt{3}} + \frac{\log(4-2\sqrt{3})}{576\sqrt{3}}.$$

By (4.5), we can see that

$$\int_{A} \log|G(z)|\,d\mu \geq 0.008476\cdots > 0,$$

which proves (ii) of Theorem 2.

Remark Computer calculation by J. Sato shows that

$$I = 0.32306593\cdots,$$

although we have no information on the accuracy of the computation.

5 Remarks

1. We mention a consequence of Theorem 2. Because $p \equiv 1 \pmod 3$, the multiplicative group $\mathbf{F}_p^\times$ of the finite field $\mathbf{F}_p$ contains the subgroup of order 3, which we denote by μ_3. Put, for $j = 0, 1$,

$$b_j = \#\{T\,;\, T \text{ is a complete representative system for } \mathbf{F}_p^\times/\mu_3 \text{ and } \sum_{t\in T} t = j\}.$$

It is known that

$$b_0 - b_1 = \Delta(\omega) > 1,$$

cf. [2, Theorem 3.2 (iii), Corollary 4.7 (ii), Proposition 6.1 (iii) and Remark 6.2]. On the other hand, the number of complete representative systems for $\mathbf{F}_p^\times/\mu_3$ is $3^{(p-1)/3}$ and from this follows that

$$b_0 + (p-1)b_1 = 3^{(p-1)/3},$$

cf. also Remark 6.3 of [2]. Hence, Theorem 2 implies the following.

$$b_0 = \frac{1}{p}3^{\frac{p-1}{3}} + \frac{p-1}{p}e^{\frac{pl}{3}+O(p^\varepsilon)} \quad \text{as } p \to \infty,$$

$$b_1 = \frac{1}{p}3^{\frac{p-1}{3}} - \frac{1}{p}e^{\frac{pl}{3}+O(p^\varepsilon)} \quad \text{as } p \to \infty.$$

2. A comment on (1.2). We can see easily that the right hand side of (1.2) can be written as $p^{-1}\operatorname{Norm}_{K/\mathbf{Q}}(a)$ with some a in O_K. Also, it is known that the right hand side is prime to p [2, Proposition 7.1]. Now, because $\operatorname{Norm}(\mathbf{q})^3 \equiv 1 \pmod p$ for any prime ideal $\mathbf{q}$ of K prime to p, we see the following: decompose the index (1.2) as the product of prime numbers and write

$$[O_N : O_K\zeta + O_K\zeta^f + O_K\zeta^{f^2}] = \prod_q q^{e_q}\ ;$$

then it holds that

$$(q^{e_q})^3 \equiv 1 \pmod p$$

for each q. This can be extended to a more general situation. Namely, let n be an odd prime number and p be a prime number with $p \equiv 1 \pmod n$. Let $\zeta = e^{2\pi i/p}$, $N = \mathbf{Q}(\zeta)$ and K be the subfield of N with $[N:K] = n$. Let ν be an arbitrary integer of N. Then, if a prime power q^{e_q} exactly divides the index $[O_N : \sum_{\tau\in\operatorname{Gal}(N/K)} O_K\nu^\tau]$ and $q \neq p$, the congruence $(q^{e_q})^n \equiv 1 \pmod p$ holds. Now, it is known [1, Theorem 2] that the integer ring O_N does not have normal integral basis over K. It follows that

$$\Big[O_N : \sum_{\tau\in\operatorname{Gal}(N/K)} O_K\nu^\tau\Big] > p^{\frac{1}{n}}.$$

3. Finally, we shall point out some problems related to the topic discussed in this paper. First problem is to make clear the relation between the uniform distribution of the argument of the cubic Gauss sum (Heath-Brown and Patterson [4]) and the product approximation (Theorem 1) or a product expression (Matthews [8]) of the Gauss sum. A result of McGettrick [9] is helpful. However, finding some nice way for taking a $\frac{1}{3}$-representative system modulo ω becomes a problem. Second, our proof of Theorem 1 (or of (1.1)) essentially depends on the result of [8]. It is desirable, if possible, to give a direct proof for Theorem 1 (or for (1.1)). The result of [8] and (1.1) lie approximately at the same depth, and the former follows rather easily from the latter. Thirdly, because $\tau(\omega)^3 = -p\omega$, it follows from (1.1) that

$$\arg\Big\{-\omega\prod_{a=1}^{p-1}(\zeta^a + \rho\zeta^{af} + \rho\zeta^{af^2})\Big\} = O(p^{-\frac{1}{2}+\varepsilon}) \quad \text{as } p \to \infty \tag{5.1}$$

with ζ and f being the same as in (1.2). This is a result of Loxton [7]. The proof for (5.1) is at present essentially unique and it is done by approximating the logarithm of the ω-division value $g(\frac{a}{\omega})$ by a suitable integral. To find a different way for proving (5.1) will be of interest. The treatment of Reshetukha [10], for example, may deserve some attention. All these problems are related to the problem of extending Theorem 1 to higher degree cases.

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