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LOG CANONICAL THRESHOLDS ON GORENSTEIN CANONICAL DEL PEZZO SURFACES

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Abstract We classify all the effective anticanonical divisors on weak del Pezzo surfaces. Through this classification we obtain the smallest number among the log canonical thresholds of effective anticanonical divisors on a given Gorenstein canonical del Pezzo surface.

Keywords: del Pezzo surface; effective anticanonical divisor; log canonical threshold

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1. Introduction

Unless otherwise mentioned, all varieties are assumed to be projective, normal and defined over $\mathbb{C}$.

Let X be a variety with at worst log canonical singularities and let D be an effective divisor on X. The log canonical threshold $c_x(D)$ of D at a point x in X is defined by

$$c_x(D) = \sup\{c \mid \text{the log pair } (X, cD) \text{ is log canonical at the point } x\}.$$

The log canonical threshold $c(X, D)$ of the divisor D is defined by

$$c(X, D) = \sup\{c \mid \text{the log pair } (X, cD) \text{ is log canonical}\} = \inf_{x \in X}\{c_x(D)\}.$$

It is known that these numbers can be defined in other equivalent ways. For instance, if the divisor D is defined by a regular function f near a smooth point x, then the log canonical threshold $c_x(D)$ of D at the point x is the number defined by

$$c_x(D) = \sup\{c \mid |f|^{-2c} \text{ is locally integrable near } x\}.$$

It is also related to the Bernstein–Sato polynomial of the regular function f. The log canonical threshold $c(X, D)$ of the divisor D can be defined using a multiplier ideal sheaf as follows:

$$c(X, D) = \sup\{c \mid \mathcal{J}(cD) = \mathcal{O}_X\},$$

where $\mathcal{J}(cD)$ is the multiplier ideal sheaf of cD.

Log canonical thresholds, like multiplicity, measure how singular a divisor is. However, a log canonical threshold is a subtler invariant than multiplicity is. For instance, for the divisor D on $\mathbb{C}^2$ defined by $x^2 = y^n$ around the origin 0 that is of multiplicity 2 at the origin, the log canonical threshold $c_0(D)$ of the divisor D at the origin has different values as n varies. Also, a log canonical threshold is rather difficult to calculate in a general case. However, it has many amazing properties and provides important applications in areas such as birational geometry and Kähler geometry.

The following theorem is one of the motivations of this paper.

Theorem 1.1. *Suppose that X is an n-dimensional Fano orbifold. If there is a positive real number ϵ such that, for every effective $\mathbb{Q}$-divisor D that is numerically equivalent to $-K_X$, the log pair $(X, ((n+\epsilon)/(n+1))D)$ is Kawamata log terminal, then X has a Kähler–Einstein metric.*

Proof. See [**3**] and [**4**]. □

This result motivates the definition of the following numerical invariants.

Definition 1.2. Let X be a Fano variety with at worst log terminal singularities. The mth global log canonical threshold of X is defined by the number

$$\operatorname{lct}_m(X) = \sup\left\{\lambda \in \mathbb{Q} \;\middle|\; \begin{array}{c}\text{the log pair } (X, (\lambda/m)D) \text{ is log canonical}\\ \text{for any effective divisor } D \in |-mK_X|\end{array}\right\}.$$

The global log canonical threshold is defined by $\operatorname{lct}(X) = \inf\{\operatorname{lct}_m(X) \mid m \in \mathbb{N}\}$. Here, we do not define the mth global log canonical threshold of X if the linear system $|-mK_X|$ is empty.

We can see that $\operatorname{lct}(X)$ is the supremum of the values c such that the log pair (X, cD) is log canonical for every effective $\mathbb{Q}$-divisor that is numerically equivalent to $-K_X$. Using the global log canonical threshold, Theorem 1.1 can be reinterpreted as saying that the Fano manifold X admits a Kähler–Einstein metric if

$$\operatorname{lct}(X) > \frac{\dim(X)}{\dim(X)+1}.$$

The paper [**13**] also shows us that the global log canonical threshold plays an important role in rationality problems.

The first global log canonical threshold may be a cornerstone of the method to obtain $\operatorname{lct}(X)$. It is natural that we ask whether there is an integer m for which $\operatorname{lct}_m(X) = \operatorname{lct}(X)$. We can find some evidence in simple cases.

Theorem 1.3. *Let X be a smooth del Pezzo surface. Then*

$$\mathrm{lct}(X)=\mathrm{lct}_1(X)=\begin{cases}\frac{1}{3} & \text{if } X\cong\mathbb{F}_1 \text{ or } K_X^2\in\{7,9\},\\ \frac{1}{2} & \text{if } X\cong\mathbb{P}^1\times\mathbb{P}^1 \text{ or } K_X^2\in\{5,6\},\\ \frac{2}{3} & \text{if } K_X^2=4,\\ \frac{2}{3} & \text{if } X \text{ is a cubic in } \mathbb{P}^3 \text{ with an Eckardt point},\\ \frac{3}{4} & \text{if } X \text{ is a cubic in } \mathbb{P}^3 \text{ without Eckardt points},\\ \frac{3}{4} & \text{if } K_X^2=2 \text{ and } |-K_X| \text{ has a tacnodal curve},\\ \frac{5}{6} & \text{if } K_X^2=2 \text{ and } |-K_X| \text{ has no tacnodal curves},\\ \frac{5}{6} & \text{if } K_X^2=1 \text{ and } |-K_X| \text{ has a cuspidal curve},\\ 1 & \text{if } K_X^2=1 \text{ and } |-K_X| \text{ has no cuspidal curves}.\end{cases}$$

Proof. See [**1**] and [**10**]. □

Throughout this paper, an algebraic surface S with ample anticanonical divisor will be called a del Pezzo surface of degree d if it has at worst normal Gorenstein canonical singularities and if the self-intersection number of the anticanonical divisor is d. Let Σ be the set of singular points of S. For singular del Pezzo surfaces of degree 3, [**2**] shows us the following.

Theorem 1.4. *Suppose that S is a cubic del Pezzo surface in $\mathbb{P}^3$ and that $\Sigma\neq\varnothing$. Then*

$$\mathrm{lct}(S)=\mathrm{lct}_1(S)=\begin{cases}\frac{1}{6} & \text{if } \Sigma=\{\mathrm{E}_6\},\\ \frac{1}{4} & \text{if } \Sigma\supseteq\{\mathrm{A}_5\},\ \Sigma=\{\mathrm{D}_5\},\\ \frac{1}{3} & \text{if } \Sigma\supseteq\{\mathrm{A}_4\},\{2\mathrm{A}_2\},\ \Sigma=\{\mathrm{D}_4\},\\ \frac{2}{3} & \text{if } \Sigma=\{\mathrm{A}_1\},\\ \frac{1}{2} & \text{otherwise}.\end{cases}$$

For a del Pezzo surface S, the first global log canonical threshold $\mathrm{lct}_1(S)$ is meaningful by itself. It has a nice application to birational maps between del Pezzo fibrations (see [**10**] or [**11**]). The paper [**11**] has computed all the values of $\mathrm{lct}_1(S)$ for del Pezzo surfaces S of degree 1.

The aim of this paper is to find all the values of the first global log canonical thresholds of Gorenstein canonical del Pezzo surfaces that have singular points. This can be done by handling effective anticanonical divisors on the minimal resolutions of del Pezzo surfaces.

To this end, our first requirement is to find information on the singularities of del Pezzo surfaces. This information can be obtained from [**5**–**9**, **14**, **15**]. Instead of studying the singularities of del Pezzo surfaces, we are able to understand them by studying configurations of -2-curves on smooth surfaces with nef and big anticanonical divisors: so-called weak del Pezzo surfaces. We can also obtain information on effective anticanonical divisors on (weak) del Pezzo surfaces in such a way.

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We have the following geometric descriptions for del Pezzo surfaces. They show the relation between del Pezzo surfaces and weak del Pezzo surfaces. They also provide us with a way to construct effective anticanonical divisors on weak del Pezzo surfaces.

Theorem 1.5. *Let S be a del Pezzo surface of degree d. We then have the following.*

(1) $1 \leqslant d \leqslant 9$.

(2) *If $d = 9$, then $S \cong \mathbb{P}^2$.*

(3) *If $d = 8$, then either $S \cong \mathbb{P}^1 \times \mathbb{P}^1$ or $S \cong \mathbb{F}_1$ or S is a cone over a quadric in $\mathbb{P}^2$.*

(4) *If $1 \leqslant d \leqslant 7$, then there exists a set of points in almost general position on $\mathbb{P}^2$ such that the blow-up centred on the set is the minimal resolution of S.*

When we say that a finite set $\{p_1, p_2, \dots, p_n\}$ of points on the projective plane $\mathbb{P}^2$ (infinitely near points are allowed) is in almost general position, it means that

- *no four of them are on a line,*
- *no seven of them are on a conic, and*
- *for all j $(1 \leqslant j \leqslant n-1)$, the point p_{j+1} on the blow-up V_j of $\mathbb{P}^2$ centred at $\{p_1, p_2, \dots, p_j\}$ does not lie on any strict transform $\hat{E}_i$ of E_i $(1 \leqslant i \leqslant j)$ such that $\hat{E}_i^2 = -2$, where E_i is an exceptional divisor on V_i.*

Proof. See [**5**] or [**7**]. □

As we mentioned above, the singularities on a del Pezzo surface can be described by the configuration of -2-curves on the minimal resolution of the del Pezzo surface that is a weak del Pezzo surface. Meanwhile, the configurations of -2-curves can be seen effectively from their dual graphs (Dynkin diagrams).

Theorem 1.6. *The singularities of a del Pezzo surface of degree d are one of the following:*

$d = 7$, A_1,

$d = 6$, *any subgraph of* $\mathrm{A}_1 + \mathrm{A}_2$,

$d = 5$, *any proper subgraph of the extended Dynkin diagram* $\tilde{\mathrm{A}}_4$,

$d = 4$, *any proper subgraph of the extended Dynkin diagram* $\tilde{\mathrm{D}}_5$,

$d = 3$, *any proper subgraph of the extended Dynkin diagram* $\tilde{\mathrm{E}}_6$,

$d = 2$, $6\mathrm{A}_1$, $\mathrm{D}_4 + 3\mathrm{A}_1$ *or any proper subgraph of the extended Dynkin diagram* $\tilde{\mathrm{E}}_7$.

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Proof. See [**7**] and [**12**]. □

For the sake of the first global log canonical thresholds, we need to distinguish some singularity types of del Pezzo surfaces of degree 2 with the same dual graphs. To do so, we will separate A_5 singularities into two types. On the minimal resolution of the del Pezzo surface we have a chain of five -2-curves that comes from an A_5 singularity. One type has a -1-curve intersecting the -2-curve that is the middle -2-curve in the chain of five -2-curves. The other does not. The type of singularity will be denoted by A_5' in the former case and by A_5'' in the latter case. For singularity types A_5 and $\mathrm{A}_5+\mathrm{A}_1$ on del Pezzo surfaces of degree 2, there are two types for each (see [**14**]). One is for A_5' and the other is for A_5''. For singularity type $\mathrm{A}_5+\mathrm{A}_2$ on del Pezzo surfaces of degree 2, there is only one type (see [**14**]). The singularity A_5 in this type is A_5'.

Also, there are two types of singularity on del Pezzo surfaces of degree 2 with the dual graph $3\mathrm{A}_1$ (respectively $4\mathrm{A}_1$) (see [**14**]). One has a -1-curve on the del Pezzo surface that passes through three A_1 singular points (denoted by $(3\mathrm{A}_1)'$ (respectively $(4\mathrm{A}_1)'$)). The other (denoted by $(3\mathrm{A}_1)''$ (respectively $(4\mathrm{A}_1)''$)) does not. For singularity type $\mathrm{A}_2+3\mathrm{A}_1$ on del Pezzo surfaces of degree 2, there is only one type (see [**14**]). The singularities $3\mathrm{A}_1$ in this type are $(3\mathrm{A}_1)'$.

We are now at the stage where we can state the main theorem of this paper.

Theorem 1.7. *Let S_d be a del Pezzo surface of degree d and let Σ_d be the set of singular points in S_d. Suppose that $\Sigma_d \neq \emptyset$. We then have*

$$\mathrm{lct}_1(S_2)=\begin{cases}\frac{1}{6} & \text{if } \Sigma_2=\{\mathrm{E}_7\},\\ \frac{1}{4} & \text{if } \Sigma_2=\{\mathrm{E}_6\},\ \Sigma_2\supseteq\{\mathrm{D}_6\},\\ \frac{1}{3} & \text{if } \Sigma_2\supseteq\{\mathrm{D}_5\},\{\mathrm{A}_5'\},\\ \frac{1}{2} & \text{if } \Sigma_2\supseteq\{(3\mathrm{A}_1)'\},\{(4\mathrm{A}_1)'\},\{5\mathrm{A}_1\},\\ & \qquad\qquad \{\mathrm{A}_3\},\{\mathrm{A}_4\},\{\mathrm{A}_5''\},\{\mathrm{A}_6\},\{\mathrm{A}_7\},\{\mathrm{D}_4\},\\ \frac{2}{3} & \text{otherwise};\end{cases}$$

$$\mathrm{lct}_1(S_3)=\begin{cases}\frac{1}{6} & \text{if } \Sigma_3=\{\mathrm{E}_6\},\\ \frac{1}{4} & \text{if } \Sigma_3\supseteq\{\mathrm{A}_5\},\ \Sigma_3=\{\mathrm{D}_5\},\\ \frac{1}{3} & \text{if } \Sigma_3\supseteq\{\mathrm{A}_4\},\{2\mathrm{A}_2\},\ \Sigma_3=\{\mathrm{D}_4\},\\ \frac{2}{3} & \text{if } \Sigma_3=\{\mathrm{A}_1\},\\ \frac{1}{2} & \text{otherwise};\end{cases}$$

$$\mathrm{lct}_1(S_4)=\begin{cases}\frac{1}{6} & \text{if } \Sigma_4=\{\mathrm{D}_5\},\\ \frac{1}{4} & \text{if } \Sigma_4\supseteq\{\mathrm{A}_1+\mathrm{A}_3\},\ \Sigma_4=\{\mathrm{A}_4\},\ \Sigma_4=\{\mathrm{D}_4\},\\ \frac{1}{3} & \text{if } \Sigma_4=\{\mathrm{A}_3\},\ \Sigma_4\supseteq\{\mathrm{A}_1+\mathrm{A}_2\},\\ \frac{1}{2} & \text{otherwise};\end{cases}$$

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$$\mathrm{lct}_1(S_5) = \begin{cases} \frac{1}{6} & \textit{if } \Sigma_5 = \{\mathrm{A}_4\}, \\ \frac{1}{4} & \textit{if } \Sigma_5 = \{\mathrm{A}_3\},\ \Sigma_5 = \{\mathrm{A}_1 + \mathrm{A}_2\}, \\ \frac{1}{3} & \textit{if } \Sigma_5 = \{\mathrm{A}_2\}, \{2\mathrm{A}_1\}, \\ \frac{1}{2} & \textit{if } \Sigma_5 = \{\mathrm{A}_1\}; \end{cases}$$

$$\mathrm{lct}_1(S_6) = \begin{cases} \frac{1}{6} & \textit{if } \Sigma_6 = \{\mathrm{A}_1 + \mathrm{A}_2\}. \\ \frac{1}{4} & \textit{if } \Sigma_6 = \{\mathrm{A}_2\},\ \Sigma_6 = \{2\mathrm{A}_1\}, \\ \frac{1}{3} & \textit{if } \Sigma_6 = \{\mathrm{A}_1\}; \end{cases}$$

$$\mathrm{lct}_1(S_7) = \quad \frac{1}{4} \quad \textit{if } \Sigma_7 = \{\mathrm{A}_1\}.$$

The first log canonical thresholds of del Pezzo surfaces of degree 1 have been dealt with in [**11**].

Let $\pi\colon \tilde{S} \to S$ be the minimal resolution of S. Since we assume that the del Pezzo surface admits only Gorenstein canonical singularities, the resolution π is crepant, i.e. $K_{\tilde{S}} = \pi^*(K_S)$. Therefore, the pullback $\tilde{D} := \pi^*(D)$ of an effective anticanonical divisor D on S is an effective anticanonical divisor on $\tilde{S}$. We can write $\pi^*(D) = \bar{D} + E$, where $\bar{D}$ is the strict transform of D and $\mathrm{Supp}(E)$ consists of -2-curves.

Lemma 1.8. *If $D = \sum a_i D_i$ is an effective anticanonical divisor on a del Pezzo surface of degree d, then $\sum a_i \leqslant d$.*

Proof. This is easy to check. □

For a constant c,

$$\pi^*(K_S + cD) = K_{\tilde{S}} + c\tilde{D}.$$

Thus it is sufficient to consider $\tilde{D}$ on $\tilde{S}$ to compute $\mathrm{lct}_1(S)$. An effective anticanonical divisor D on S that does not pass through any singular point of S is not different from the pullback of D via π. When we consider effective anticanonical divisors that pass through singular points of S, it suffices to investigate effective anticanonical divisors on the weak del Pezzo surface $\tilde{S}$ that contains a -2-curve.

Since $\mathrm{lct}_1(S)$ is always at most 1, classifying all effective anticanonical divisors in $\tilde{S}$ that have either at least one component with multiplicity greater than or equal to 2 or components that are not normal crossing, we can prove the main theorem. The classification will be presented in the following section.

2. The configuration of the anticanonical divisors

Let us summarize the results of [**11**]. Let S be a del Pezzo surface of degree 1 and let $\pi\colon \tilde{S} \to S$ be the minimal resolution of S. Then, for an effective anticanonical divisor D on S, the configuration of its pullback divisor $\tilde{D}$ by the morphism π coincides with one of Kodaira's elliptic fibres. Lemma 1.8 implies that every effective anticanonical divisor on S is irreducible and reduced, since the degree of S is 1. If the divisor D passes through a singular point of S and the divisor $\tilde{D}$ has a multiple component, then we can see that the dual graph of the divisor $\tilde{D}$ must be one of those that appear in the tables in Propositions 2.1, 2.3, 2.5 and 2.12. For del Pezzo surfaces of degree 1 we can obtain all the configurations of $\tilde{D}$ from [**11**].

Every effective anticanonical divisor on a weak del Pezzo surface of degree d can be obtained from an effective anticanonical divisor on a weak del Pezzo surface of degree $(d+1)$ via a suitable blow-up.

Suppose that we have obtained the list of all the configurations of effective anticanonical divisors on weak del Pezzo surfaces of degree d. Let

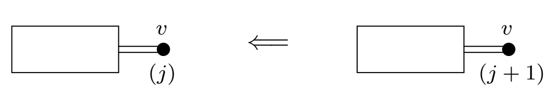
$$D_{d+1} = \sum a_i E_i + \sum b_j D_j \sim -K_{d+1}$$

be an effective anticanonical divisor of a weak del Pezzo surface of degree $(d+1)$, where each E_i is a -2-curve and D_j is a prime divisor that is not a -2-curve. In addition, let $\hat{D} = \sum D_j$. We then consider the blow-up ψ of the weak del Pezzo surface of degree $(d+1)$ at a point p in $D_k \setminus \bigcup E_i$ with $\mathrm{mult}_p(\hat{D}) = 1$. This produces a new effective anticanonical divisor on the weak del Pezzo surface of degree d that is the blow-up by ψ. To be precise, we obtain

$$\psi^*(-K_{d+1}) - F = \sum a_i \bar{E}_i + \sum b_j \bar{D}_j + (b_k - 1)F \sim -K_d,$$

where $\bar{E}_i$, $\bar{D}_j$ are the strict transforms of E_i, D_j, respectively, and F is the exceptional divisor of ψ. This effective anticanonical divisor should appear in the list of all the configurations of effective anticanonical divisors on weak del Pezzo surfaces of degree d that we have already obtained. Therefore, in order to obtain the list of all the configurations of effective anticanonical divisors on weak del Pezzo surfaces of degree $(d+1)$ from the list for degree d, we have to consider only two cases, those for $b_k = 1$ and $b_k > 1$, as follows.

(P-1) Suppose that we have an effective anticanonical divisor on a weak del Pezzo surface of degree d whose dual graph Γ_d has a vertex v with weight 1 and self-intersection number $j \geqslant -2$. There is then the possibility that we have an effective anticanonical divisor on a weak del Pezzo surface of degree $(d+1)$ whose dual graph is the same as Γ_d except that the vertex v has self-intersection number $j+1$. This is the case for $b_k = 1$ in the description above:



(P-2) Suppose that we have an effective anticanonical divisor on a weak del Pezzo surface of degree d whose dual graph Γ_d has a vertex w with weight $b \geqslant 1$ and self-intersection number -1 that has only one adjacent vertex v. We also suppose that the vertex v has weight $b+1$ and self-intersection number $j \geqslant -2$. There is then the possibility that we have an effective anticanonical divisor on a weak del Pezzo surface of degree $(d+1)$ whose dual graph is the same as Γ_d except that the vertex

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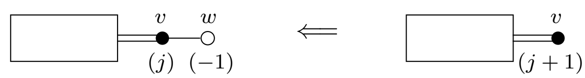
v has self-intersection number $j+1$ and it has no w. This is the case for $b_k > 1$ in the description above:



Here the double lines mean that the vertex v can be connected either to one vertex or to more than one vertex.

Let $\tilde{S}$ be a weak del Pezzo surface of degree $d \leqslant 7$. There are then $9-d$ points $p_{i,j}$, $i, j \geqslant 1$, on the projective plane $\mathbb{P}^2$ (infinitely near points allowed) in almost general position such that the blow-up centred at these points is the surface $\tilde{S}$. We have a birational morphism $\pi\colon \tilde{S} \to \mathbb{P}^2$ that is a composition of a sequence of blow-ups. Here each $p_{i,1}$ is a point on $\mathbb{P}^2$ and the point $p_{i,j+1}$ is a point on the exceptional divisor of the blow-up at the point $p_{i,j}$. The exceptional divisor of the blow-up at the point $p_{i,j}$ is denoted by $E_{i,j}$. We then see that

$$K_{\tilde{S}} = \pi(K_{\mathbb{P}^2}) + \sum_{ij} jE_{i,j}.$$

For a divisor D on $\mathbb{P}^2$, we have

$$\pi^*(D) = \bar{D} + \sum_{ij} (\mathrm{mult}_{p_{i,j}}(D))E_{i,j},$$

where $\bar{D}$ is the strict transform of D by π. In particular, a cubic curve $C = \sum a_h C_h$ (not necessarily irreducible nor reduced) on $\mathbb{P}^2$ defines an anticanonical divisor

$$\sum a_h \bar{C}_h + \sum_{i,j} \left(\sum_{k=1}^{j} (\mathrm{mult}_{p_{i,k}}(C) - 1) \right) E_{i,j} \tag{2.1}$$

on the surface $\tilde{S}$.

For all the dual graphs below we will use the following notation to differentiate smooth rational curves with various self-intersection numbers:

●	−2-curve
○	−1-curve
□	0-curve
◙	1-curve
◘	2-curve

The number near each vertex is the multiplicity of the curve corresponding to the vertex. The number 1 for multiplicity 1 will always be omitted.

In the tables of Propositions 2.1, 2.3, 2.5 and 2.12, some rows are marked with $\surd$. We shall see in § 3 that only those entries marked with $\surd$ need be considered in order to prove Theorem 1.7 (see Propositions 3.1, 3.3 and 3.6).

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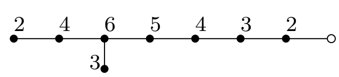
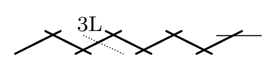
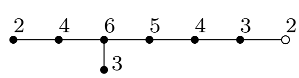
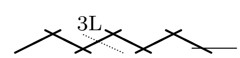
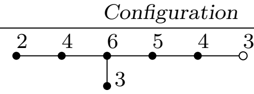
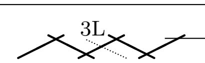
The column labelled 'Example' carries configurations of divisors on certain blow-ups of $\mathbb{P}^2$ in order to demonstrate the existence of anticanonical divisors of given types on weak del Pezzo surfaces. In each configuration, solid lines denote exceptional curves of blow-ups of $\mathbb{P}^2$. The configuration of the solid lines shows how to take blow-ups starting from $\mathbb{P}^2$ to obtain a weak del Pezzo surface for a given singularity type. Thin solid lines (always drawn horizontally) denote -1-curves and thick solid lines (always drawn on a slant) denote -2-curves. The dotted curves in each configuration denote the strict transform of a cubic curve (not necessarily reduced nor irreducible) via the blow-ups. The letters L and Q right beside the dotted curves denote that each corresponding dotted curve is the strict transform of a line and an irreducible conic, respectively. In addition, 2L and 3L denote the strict transforms of double lines and triple lines, respectively.

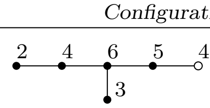
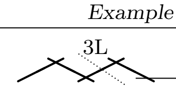
Since each curve in an effective anticanonical divisor on a weak del Pezzo surface of degree 1 has weight at most 6, so does a curve in an effective anticanonical divisor on a weak del Pezzo surface.

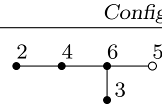
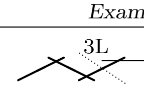
Proposition 2.1. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight* 6, *then it is exactly one of the following.*

	Degree 1	
	Configuration	*Example*
√ E_8		

	Degree 2	
	Configuration	*Example*
√ E_7		

	Degree 3	
	Configuration	*Example*
√ E_6		

	Degree 4	
	Configuration	*Example*
√ D_5		

	Degree 5	
	Configuration	*Example*
√ A_4		

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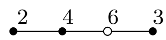
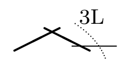
Degree 6		
	Configuration	*Example*
$\surd$ $\mathrm{A}_2 + \mathrm{A}_1$		

Proof. Since there is no E_8 on a del Pezzo surface of degree 2, we have only (P-2) possibilities. Furthermore, such a dual graph on a del Pezzo surface of degree greater than or equal to 2 has no vertex with weight 1. Therefore, we also have only (P-2) possibilities after degree 2. $\square$

The weight 5 never appears as a maximum weight in an effective anticanonical divisor on a weak del Pezzo surface of degree 1 and the maximum weight is preserved under the changes by (P-1) and (P-2). Therefore, if an effective anticanonical divisor on a weak del Pezzo surface has multiplicity 5 along a curve, then it must have another curve along which it has multiplicity 6.

Lemma 2.2. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight* 4 *as a maximum, then it contains at most one vertex with non-negative self-intersection number. In such a case, the vertex has self-intersection number* 0*.*

Proof. Let $\tilde{S}$ be a weak del Pezzo surface of degree $d \leqslant 7$. It is then obtained by suitable blow-ups $\pi\colon \tilde{S} \to \mathbb{P}^2$. Every effective anticanonical divisor on $\tilde{S}$ can be obtained from a cubic curve $C = \sum a_h C_h$ (not necessarily irreducible nor reduced) on $\mathbb{P}^2$ as (2.1) shows. Then, for some (i, j), we must have

$$\sum_{k=1}^{j} (\mathrm{mult}_{p_{i,k}}(C) - 1) = 4.$$

Since $\mathrm{mult}_{p_{i,k}}(C)$ is non-increasing as k grows and the curve C is cubic, the possible sequences for $\{\mathrm{mult}_{p_{i,k}}(C)\}_{k=1}^{j}$ have the following form:

$$(2, 2, 2, 2, *, *, \dots),\ (3, 2, 2, *, *, \dots),\ (3, 3, *, *, \dots).$$

Therefore, the curve C consists only of lines.

Since at most three $p_{i,k}$ can be on a line, the first form is impossible, the second must be $(3, 2, 2)$ and the last must be $(3, 3)$.

For the sequence $(3, 3)$, the curve C is a triple line. Exactly two points of $p_{i,k}$ are over C and hence the effective anticanonical divisor on $\tilde{S}$ given by C has no non-negative self-intersection curve.

For the sequence $(3, 2, 2)$, the curve C consists of one double line and one single line. We should take one blow-up at the intersection point of the double line and the single line. Two more blow-ups at some points over the double line must then follow. Therefore, the effective anticanonical divisor given by C has only one non-negative self-intersection curve and its self-intersection number is 0. $\square$

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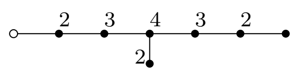
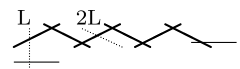
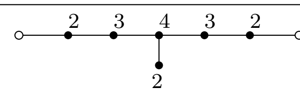
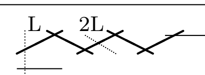
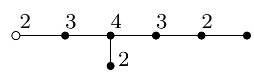
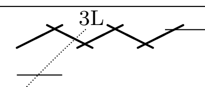
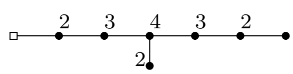
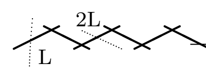
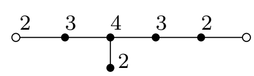
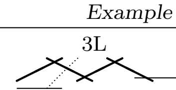
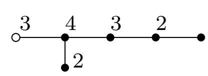
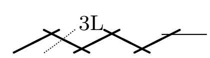
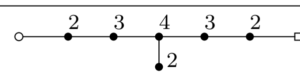
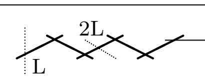
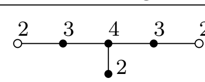
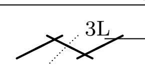
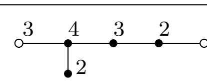
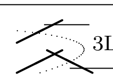
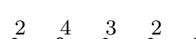
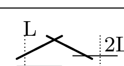
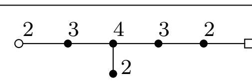
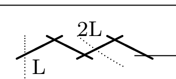
Proposition 2.3. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight 4 as a maximum, then it is exactly one of the following.*

Degree 1		
	Configuration	*Example*
√ E_7		

Degree 2		
	Configuration	*Example*
√ E_6		
√ D_6		
E_7		

Degree 3		
	Configuration	*Example*
√ D_5		
√ A_5		
E_6		

Degree 4		
	Configuration	*Example*
√ D_4		
√ A_4		
√ $A_3 + A_1$		
D_5		

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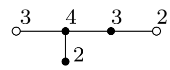
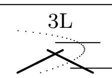
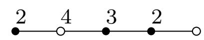
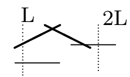
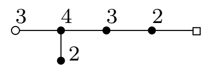
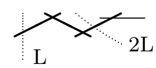
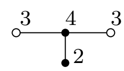
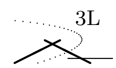
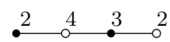
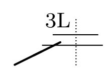
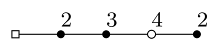
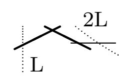
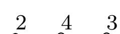
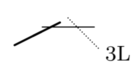
Degree 5		
	Configuration	*Example*
√ A_3		
√ $A_2 + A_1$		
A_4		

Degree 6		
	Configuration	*Example*
√ A_2		
√ $2A_1$		
$A_2 + A_1$		

Degree 7		
	Configuration	*Example*
√ A_1		

Proof. Starting from the E_7 dual graph for the degree 1 case, we apply (P-1) and (P-2) successively to get the possible dual graphs. From the obtained possible dual graphs, we then exclude the dual graphs that violate the properties in Lemma 2.2. □

Lemma 2.4. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight 3 as a maximum, the divisor $\tilde{D}$ does not have two 0-curves. In addition, if the divisor has a 1-curve, then its dual graph is obtained by suitable blow-ups from the following:*

Proof. The proof is similar to that of Lemma 2.2. In this case, the possible sequences for $\{\mathrm{mult}_{p_{i,k}}(C)\}_{k=1}^{j}$ have the following form:

$$(2, 2, 2, *, *, \dots), \ (3, 2, *, *, \dots).$$

Therefore, the curve C consists only of lines. We also see that the first form must be $(2, 2, 2)$ and that the second must be $(3, 2)$.

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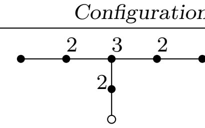
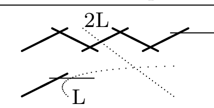
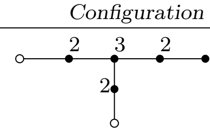
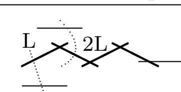
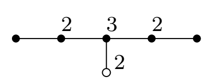
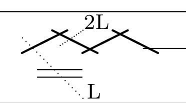
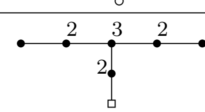
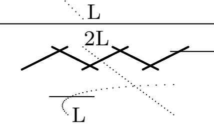
For the case $(2,2,2)$, the curve C must consist of one double line and one single line. We also see that one blow-up must be taken at a point that is on the double line but not on the single line. One blow-up at the intersection point of the exceptional divisor and the strict transform of the double line must then follow. And one more blow-up at the intersection point of the exceptional divisor of the second blow-up and the strict transform of the double line must then be taken.

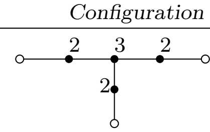
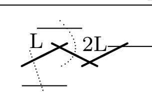
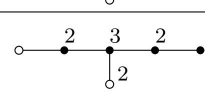
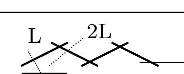
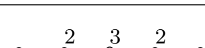
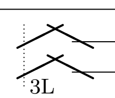
For the case $(3,2)$, the curve C must also consist of one double line and one single line. We must take the blow-up at the intersection point of the double line and the single line. The blow-up at a point on the strict transform of the double line but not on that of the single line must then follow.

In neither case does the dual graph contain two 0-curves. If it contains a 1-curve, such a dual graph must be obtained in the same way that it was for the case $(2,2,2)$. □

Proposition 2.5. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight 3 as a maximum, then it is exactly one of the following.*

Degree 1		
	Configuration	*Example*
√ E_6		

Degree 2		
	Configuration	*Example*
√ D_5		
√ $(A_5)'$		
E_6		

Degree 3		
	Configuration	*Example*
√ D_4		
√ A_4		
√ $2A_2$		

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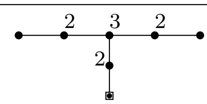
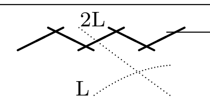
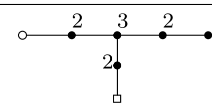
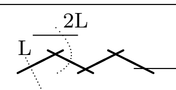
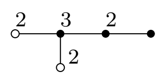
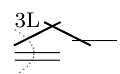
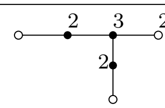
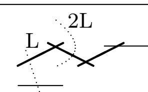
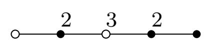
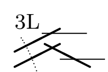
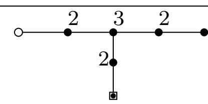
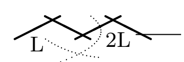
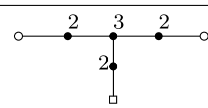
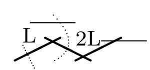
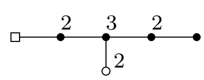
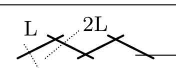
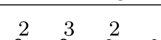
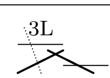
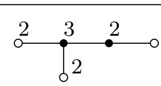
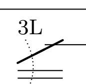
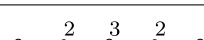
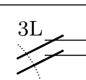
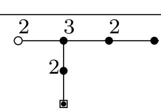
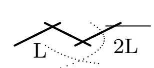
Degree 3 *(continued)*		
	Configuration	*Example*
E_6		
D_5		

Degree 4		
	Configuration	*Example*
√ A_3		
√ A_3		
√ $A_2 + A_1$		
D_5		
D_4		
A_4		

Degree 5		
	Configuration	*Example*
√ A_2		
√ A_2		
√ $2A_1$		
A_4		

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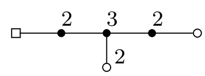
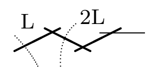
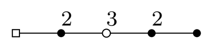
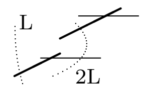
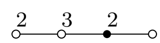
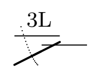
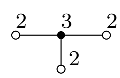
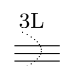
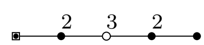
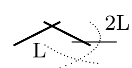
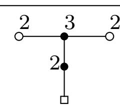
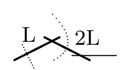
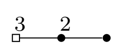
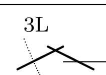
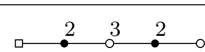
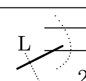
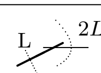
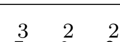
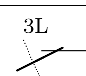
Degree 5 (continued)		
	Configuration	*Example*
A_3		
$A_2 + A_1$		

Degree 6		
	Configuration	*Example*
$\surd$ A_1		
$\surd$ A_1		
$A_2 + A_1$		
A_2		
A_2		
$2A_1$		

Degree 7		
	Configuration	*Example*
A_1		
A_1		

Proof. Starting from the E_6 dual graph for the degree 1 case, we apply (P-1) and (P-2) successively to get the possible dual graphs. From the obtained possible dual graphs, we then exclude the dual graphs that violate the properties in Lemma 2.4. □

For the dual graphs of effective anticanonical divisors on weak del Pezzo surfaces with a vertex of weight 2 as a maximum, let $\tilde{D}$ be such an anticanonical divisor. As in (2.1),

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the divisor $\tilde{D}$ is of the form

$$\sum a_h \tilde{C_h} + \sum_{i,j} \left(\sum_{k=1}^{j} (\mathrm{mult}_{p_{i,k}}(C) - 1) \right) E_{i,j}$$

with the same notation as (2.1).

For some (i, j), we must have

$$\sum_{k=1}^{j} (\mathrm{mult}_{p_{i,k}}(C) - 1) = 2.$$

The possible sequences for $\{\mathrm{mult}_{p_{i,k}}(C)\}_{k=1}^{j}$ have the following form:

$$(2, 2, *, *, \dots),\ (3, *, *, \dots).$$

Furthermore, we can see that only the sequences

$$\begin{gathered}(2, 2, 1, 1, 1, 1),\ (2, 2, 1, 1, 1),\ (2, 2, 1, 1),\ (2, 2, 1),\\ (2, 2),\\ (3, 1, 1),\ (3, 1),\ (3)\end{gathered}$$

can happen. For the four sequences in the first row, the curve C must consist of one irreducible conic and a line intersecting tangentially. For the sequence in the second row, the curve C consists either of one irreducible conic and a line intersecting tangentially or of one double line and one single line. For the three sequences in the last row, the curve C consists either of one double line and one single line or of three lines intersecting at a single point.

When the curve C is given with one of the sequences above, the way to take blow-ups starting from $\mathbb{P}^2$ is uniquely determined by the curve C and the given sequence except in the case where C consists of one irreducible conic and a line intersecting tangentially and the sequence is $(2, 2, 1)$. This exceptional case can take blow-ups in two ways. First, we take the blow-up at the intersection point of the conic and the line. The blow-up at the intersection point of the strict transforms of the conic and the line then follows. For the last blow-up we have two choices. One is to take the blow-up at the intersection point of the exceptional divisor of the second blow-up and the strict transform of the conic, and the other is to take the blow-up at the intersection point of the exceptional divisor of the second blow-up and the strict transform of the line.

Lemma 2.6. *If the divisor $\tilde{D}$ has either a 1-curve or a 2-curve, it cannot have any other curve with non-negative self-intersection number.*

Proof. For the divisor $\tilde{D}$ to have either a 1-curve or a 2-curve, the sequence must be either $(2, 2, 1)$ or $(2, 2)$. The curve C consists either of one irreducible conic and a line intersecting tangentially or of one double line and one single line. If the curve C consists of one irreducible conic and a line intersecting tangentially, then the conic becomes either

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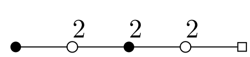
a 1-curve or a 2-curve and the single line becomes either a -1-curve or a -2-curve. If the curve C consists of one double line and one single line, then the single line becomes a 1-curve and the double line becomes a -1-curve. $\square$

Lemma 2.7. *If the dual graph of $\tilde{D}$ contains a chain consisting of five or four vertices with weight* 2 *and negative self-intersection, then it is obtained by suitable blow-ups from the following:*

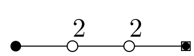
Proof. If the dual graph of the divisor $\tilde{D}$ contains a chain consisting of five or four vertices with weight 2 and negative self-intersection, then we have two possibilities. One possibility is that we have either $(2,2,1,1,1,1)$ or $(2,2,1,1,1)$ for the sequence $\{\mathrm{mult}_{p_{i,k}}(C)\}_{k=1}^{j}$ for some (i,j). In this case, the assertion is clear.

The other possibility is as follows. The curve C consists of one double line and one single line; for the sequence $\{\mathrm{mult}_{p_{i,k}}(C)\}_{k=1}^{j}$, $(3,1,1)$, $(3,1)$ or (3) is attained over the intersection point of the double line and the single line; the sequence $(2,2)$ is attained over a point on the double line but not on the single line. The blow-ups determined by each sequence then complete the proof. $\square$

Lemma 2.8. *For each of $k=0,1$, consider the set of dual graphs of all effective anticanonical divisors on weak del Pezzo surfaces such that they have exactly one vertex with weight* 1 *and self-intersection number k as a maximal self-intersection number. If the dual graph of the divisor $\tilde{D}$ has exactly one vertex with weight* 1 *and self-intersection number k as a maximum self-intersection number and has a longest chain consisting of vertices of weight* 2 *in the set, then it is obtained by suitable blow-ups from the following:*



for $k=0$,



for $k=1$.

Proof. For $k=0$, there are only two cases in which the dual graph of $\tilde{D}$ can satisfy the required conditions. The first is when we have the sequence $(2,2,1,1)$. The other is when the curve C consists of one double line and one single line; for the sequence $\{\mathrm{mult}_{p_{i,k}}(C)\}_{k=1}^{j}$, (3) is attained over the intersection point of the double line and the single line; the sequence $(2,2)$ is attained over a point on the double line but not on the single line. The blow-ups determined by each sequence complete the proof for the case $k=0$.

For $k=1$, there are only two cases in which the dual graph of $\tilde{D}$ can satisfy the required conditions. The first is when we have the sequence $(2,2,1)$. The other is when the curve C consists of one double line and one single line with the sequence $(2,2)$. $\square$

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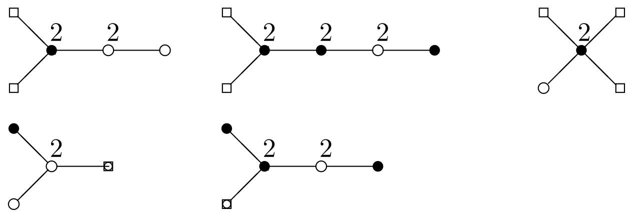
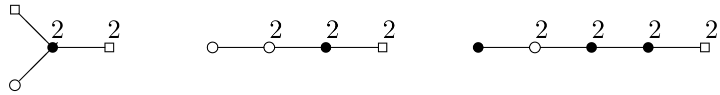
Lemma 2.9. *If the divisor $\tilde{D}$ has two reduced* 0*-curves, three reduced* 0*-curves or exactly one* 2*-curve as a maximal self-intersection number, then its dual graph is obtained by suitable blow-ups from the following:*



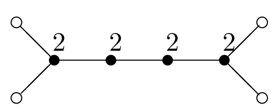
Proof. For the divisor $\tilde{D}$ to have either two reduced 0-curves or three reduced 0-curves, the curve C must consist of three lines intersecting at a single point with the sequence $(3,1,1)$, $(3,1)$ or (3). If the divisor $\tilde{D}$ has exactly one 2-curve, then the curve C consists of one irreducible conic and a line intersecting tangentially with the sequence $(2,2)$ or with $(2,2,1)$. For $(2,2,1)$, we have two ways to take blow-ups, as we mentioned right before Lemma 2.6. For the assertion, we must take blow-ups over three points over the line, not the conic. ☐

Lemma 2.10. *If the divisor $\tilde{D}$ has a* 0*-curve with multiplicity* 2 *as a maximal self-intersection number and the* 0*-curve intersects with only one component of $\tilde{D}$, then the dual graph is obtained by suitable blow-ups from the following:*



Proof. For the divisor $\tilde{D}$ to have a 0-curve with multiplicity 2, the curve C must consist of one double line and one single line. For such a double line to be a 0-curve, the sequence must be $(3,1,1)$, $(3,1)$ or (3). ☐

Lemma 2.11. *On a weak del Pezzo surface of degree* 4*, there is no effective anti-canonical divisor corresponding to the dual graph*



Proof. Contracting five -1-curves successively, we get a 0-curve on $\mathbb{P}^2$. This is a contradiction. ☐

Proposition 2.12. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight* 2 *as a maximum, then it is exactly one of the following.*

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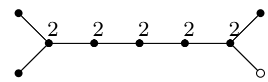
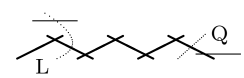
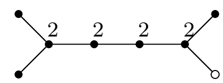
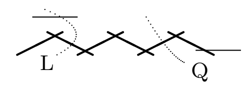
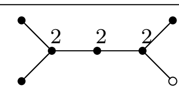
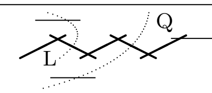
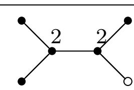
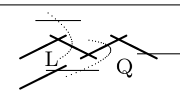
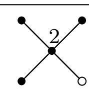
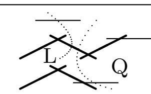
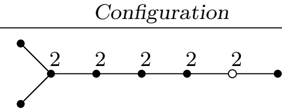
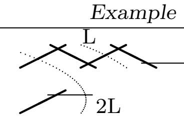
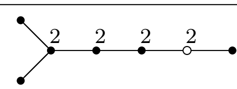
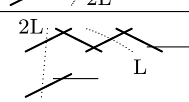
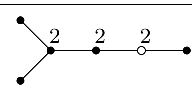
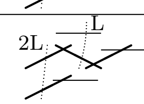
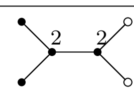
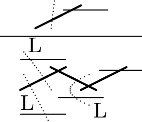
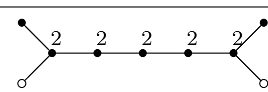
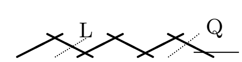
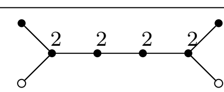
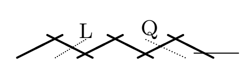
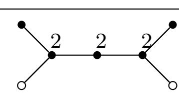
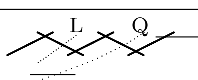
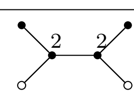
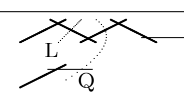
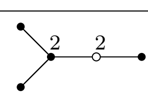
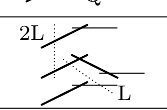
Degree 1		
	Configuration	*Example*
$\surd$ D_8		
$\surd$ D_7		
$\surd$ D_6		
$\surd$ D_5		
$\surd$ D_4		

Degree 2		
	Configuration	*Example*
$\surd$ $D_6 + A_1$		
$\surd$ $D_5 + A_1$		
$\surd$ $D_4 + A_1$		
$\surd$ D_4		
$\surd$ A_7		
$\surd$ A_6		
$\surd$ $(A_5)''$		
$\surd$ A_4		
$\surd$ $A_3 + A_1$		

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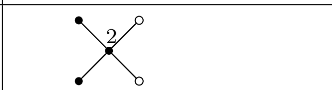
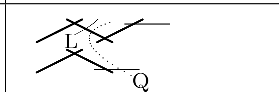
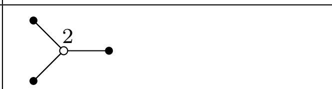
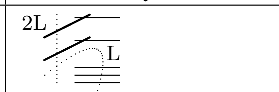
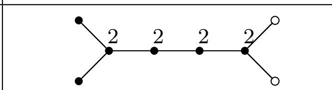
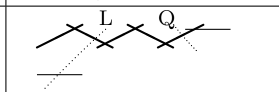
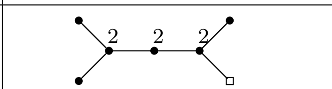
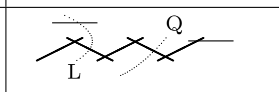
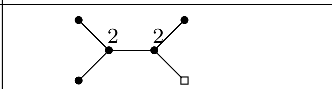
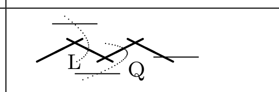
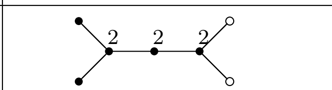
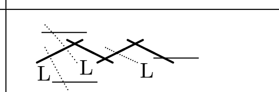
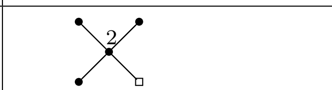
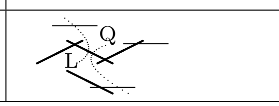
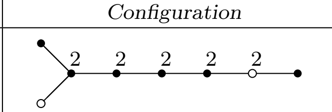
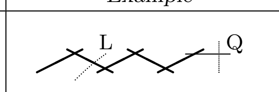
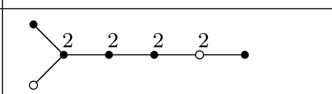
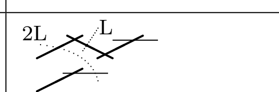
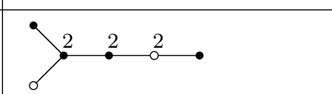
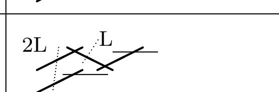
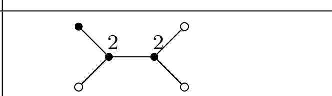
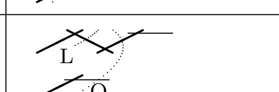
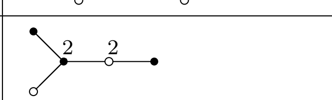
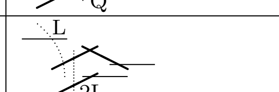
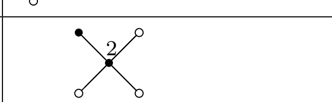
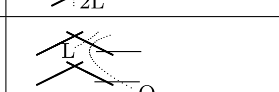
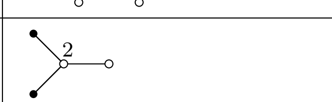
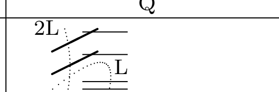
Degree 2 (continued)		
	Configuration	*Example*
$\surd$ A_3		
$\surd$ $(3A_1)'$		
D_6		
D_6		
D_5		
D_5		
D_4		

Degree 3		
	Configuration	*Example*
$\surd$ $A_5 + A_1$		
$\surd$ $A_4 + A_1$		
$\surd$ $A_3 + A_1$		
$\surd$ A_3		
$\surd$ $A_2 + A_1$		
$\surd$ A_2		
$\surd$ $2A_1$		

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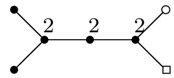
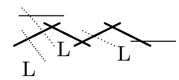
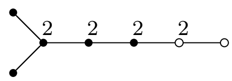
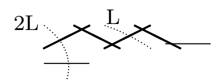
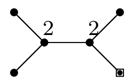
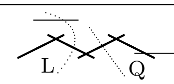
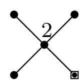
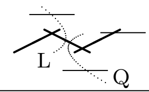
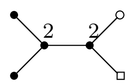
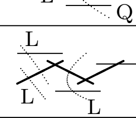
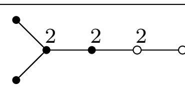
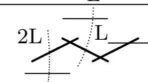
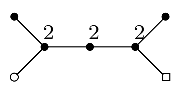
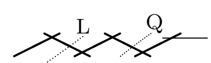
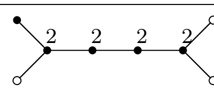
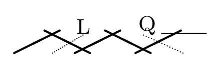
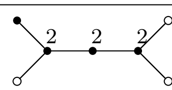
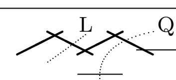
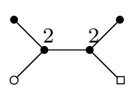
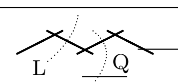
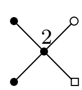
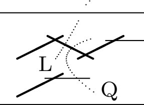
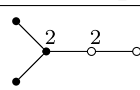
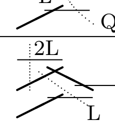
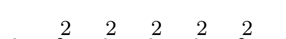
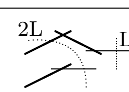
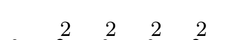
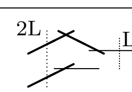
Degree 3 (continued)		
	Configuration	*Example*
D_5		
D_5		
D_5		
D_4		
D_4		
D_4		
A_5		
A_5		
A_4		
A_4		
A_3		
A_3		

Degree 4		
	Configuration	*Example*
$\surd$ $A_3 + 2A_1$		
$\surd$ $A_2 + 2A_1$		

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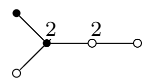
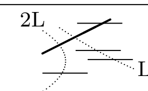
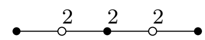
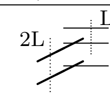
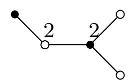
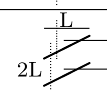
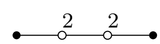
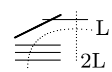
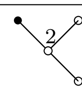
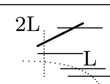
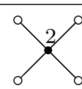
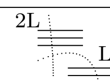
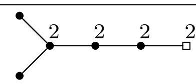
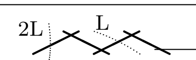
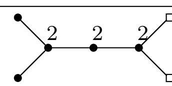
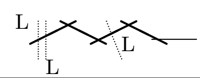
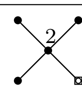
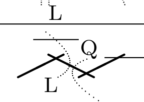
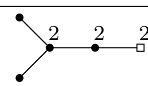
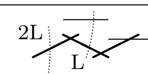
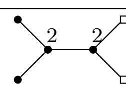
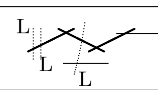
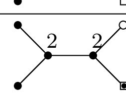
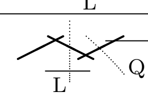
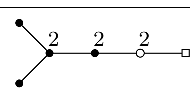
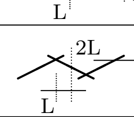
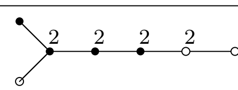
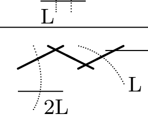
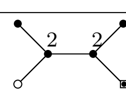
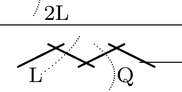
Degree 4 (continued)		
	Configuration	*Example*
√ A_2		
√ $3A_1$		
√ $2A_1$		
√ $2A_1$		
√ A_1		
√ A_1		
D_5		
D_5		
D_4		
D_4		
D_4		
D_4		
D_4		
A_4		
A_4		

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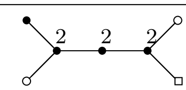
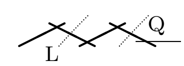
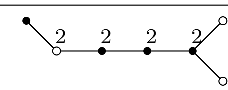
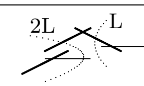
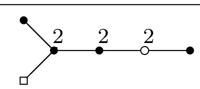
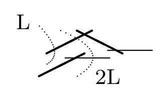
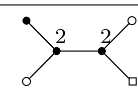
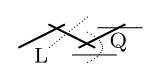
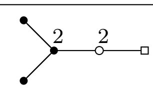
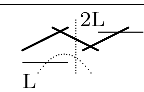
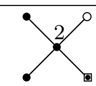
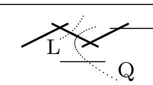
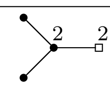
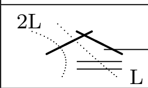
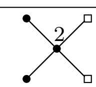
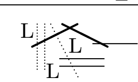
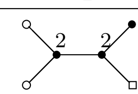
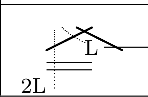
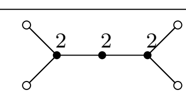
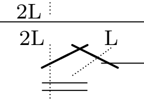
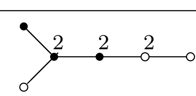
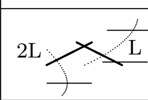
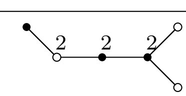
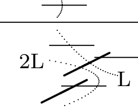
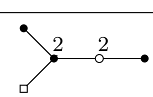
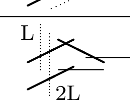
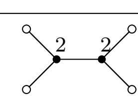
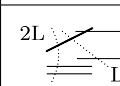
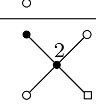
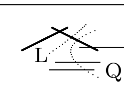
Degree 4 (continued)		
	Configuration	*Example*
A_4		
$A_3 + A_1$		
$A_3 + A_1$		
A_3		
A_3		
A_3		
A_3		
A_3		
A_3		
A_3		
A_3		
$A_2 + A_1$		
$A_2 + A_1$		
A_2		
A_2		

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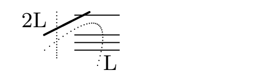
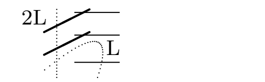
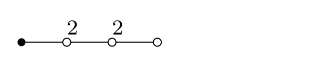
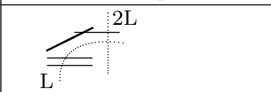
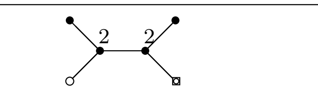
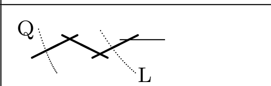
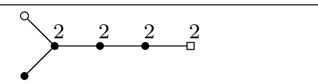
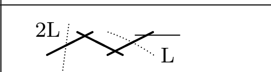
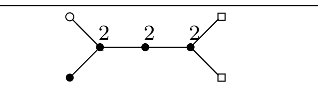
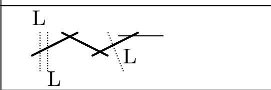
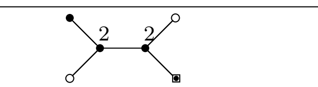
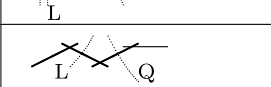
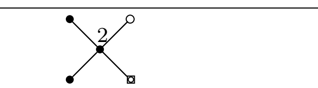
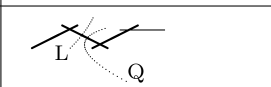
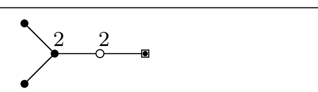
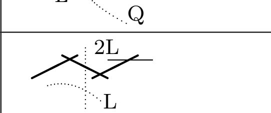
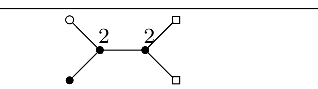
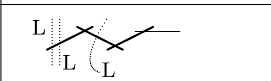
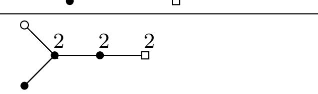
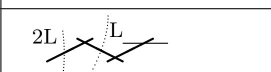
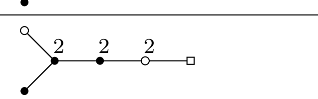
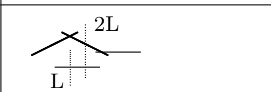
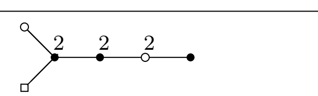
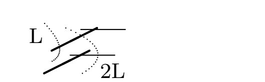
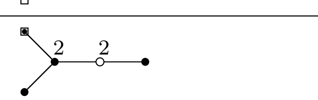
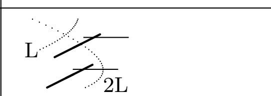
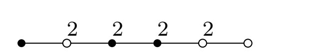
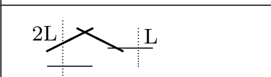
Degree 4 (continued)		
	Configuration	*Example*
$2A_1$		
$2A_1$		

Degree 5		
	Configuration	*Example*
$\surd$ A_1		
A_4		
A_4		
A_4		
A_3		
A_3		
A_3		
A_3		
A_3		
A_3		
$A_2 + A_1$		
$A_2 + A_1$		
$A_2 + A_1$		

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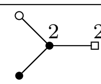
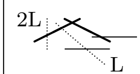
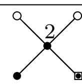
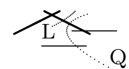
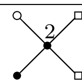
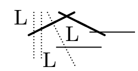
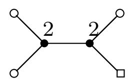
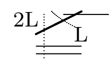
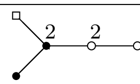
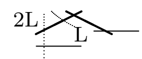
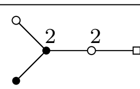
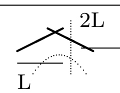
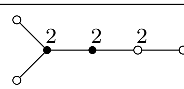
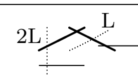
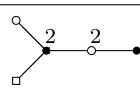
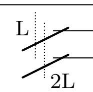
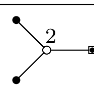
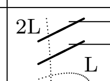
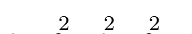
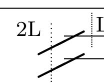
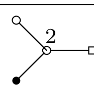
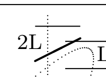
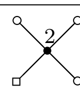
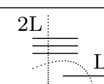
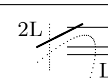
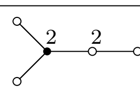
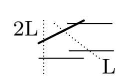
Degree 5 (continued)		
	Configuration	*Example*
A_2		
A_2		
A_2		
A_2		
A_2		
A_2		
A_2		
$2A_1$		
$2A_1$		
$2A_1$		
A_1		
A_1		
A_1		
A_1		

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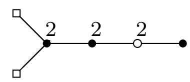
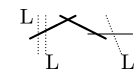
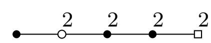
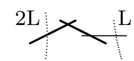
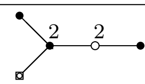
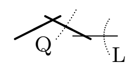
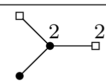
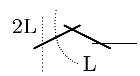
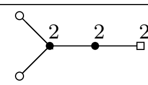
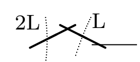
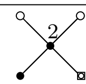
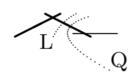
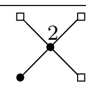
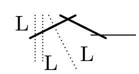
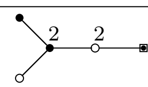
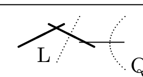
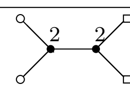
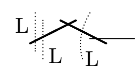
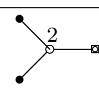
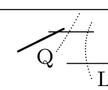
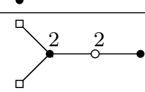
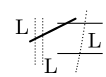
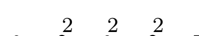
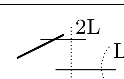
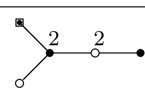
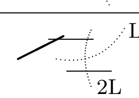
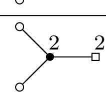
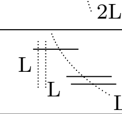
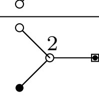
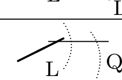
Degree 6		
	Configuration	*Example*
$A_2 + A_1$		
$A_2 + A_1$		
$A_2 + A_1$		
A_2		
A_2		
A_2		
A_2		
A_2		
A_2		
$2A_1$		
$2A_1$		
$2A_1$		
$2A_1$		
A_1		
A_1		

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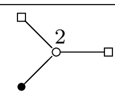
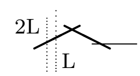
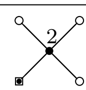
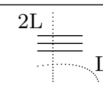
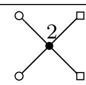
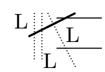
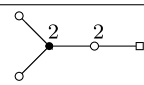
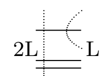
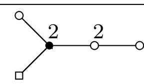
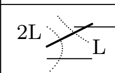
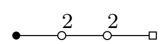
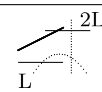
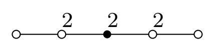
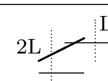
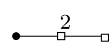
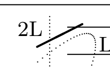
Degree 6 (continued)		
	Configuration	*Example*
A_1		
A_1		
A_1		
A_1		
A_1		
A_1		
A_1		
A_1		

Degree 7		
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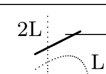
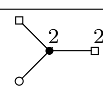
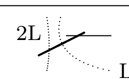
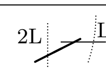
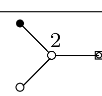
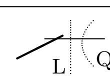
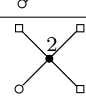
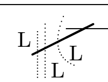
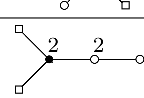
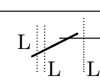
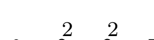
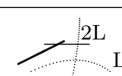
A_1		
A_1		
A_1		
A_1		
A_1		
A_1		
A_1		

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Proof. Starting from the dual graphs for the degree 1 case, we apply (P-1) and (P-2) successively to get the possible dual graphs. Then from the obtained possible dual graphs, we exclude the dual graphs that violate the properties in Lemmas 2.6–2.11. □

Proposition 2.13. *If the dual graph for an effective anticanonical divisor on a weak del Pezzo surface has a vertex of weight* 1 *as a maximum, the dual graph is circular except in the cases where the effective anticanonical divisor consists of either three curves intersecting transversally at a single point or two curves intersecting tangentially with intersection number* 2 *at a single point.*

Proof. The assertion holds for a weak del Pezzo surface of degree 1 (see [**11**]). Notice that we can only apply (P-1). The assertion is then clear. □

Proposition 2.14. *Let S be a del Pezzo surface of degree $d \geqslant 2$.*

(1) *If the surface S has only one singular point that is of type A_1, then there is an effective anticanonical divisor on its minimal resolution consisting of one $(-3+d)$-curve, one -1-curve and one -2-curve intersecting transversally at a single point.*

(2) *If the surface S has only one singular point that is of type A_2, then there is an effective anticanonical divisor on its minimal resolution consisting of one $(-2+d)$-curve and two -2-curves intersecting transversally at a single point.*

(3) *If the surface S is of degree* 2 *and it has only two singular points that are of type A_1 or A_2, then there is at least one of the following effective anticanonical divisors:*

- *one -1-curve, one -1-curve and one -2-curve intersecting transversally at a single point;*
- *two -2-curves and one* 0*-curve intersecting transversally at a single point.*

Proof. Considering successive suitable blow-ups of a cuspidal cubic, three lines intersecting at a single point and a conic and a line intersecting tangentially on $\mathbb{P}^2$, we can easily obtain the assertions. □

3. Log canonical thresholds

In this section we prove Theorem 1.7. For given singularity types, we consider all the possible effective anticanonical divisors. However, we do not have to consider all effective anticanonical divisors. It turns out that we have to consider only those that appear in the tables of Propositions 2.1, 2.3, 2.5 and 2.12 with the mark $\surd$ and those described in Proposition 2.14. In what follows we explain why this is.

Proposition 3.1. *Let D be an effective anticanonical divisor that contains a -2-curve on a weak del Pezzo surface $\tilde{S}$. Write $D = D_1 + D_2$, where D_1 and D_2 are effective divisors with $D_2^2 \geqslant 0$ and $D_1 \cdot D_2 = 2$. Suppose that there is a -1-curve L on $\tilde{S}$ such that $D_2^2 \geqslant D_2 \cdot L \geqslant 0$. There is then an effective divisor D_3 such that D_2 is linearly equivalent to $L + D_3$. In particular, $D_3^2 \leqslant D_2^2 - 1$. In addition, if $D_2^2 = 0$, then the divisor D_3 consists only of negative curves.*

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Proof. By the Riemann–Roch Theorem,

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(D_2)) \geqslant \frac{D_1 \cdot D_2}{2} + D_2^2 + 1 = 2 + D_2^2$$

since $h^0(\tilde{S}, \mathcal{O}(K_{\tilde{S}} - D_2)) = 0$. Consider the exact sequence

$$0 \to \mathcal{O}_{\tilde{S}}(D_2 - L) \to \mathcal{O}_{\tilde{S}}(D_2) \to \mathcal{O}_L(D_2|_L) \to 0.$$

We then see that $D_2 - L$ is linearly equivalent to an effective divisor D_3 since

$$h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(D_2 - L)) \geqslant h^0(\tilde{S}, \mathcal{O}_{\tilde{S}}(D_2)) - h^0(L, \mathcal{O}_L(D_2|_L)) \geqslant 1 + D_2^2 - D_2 \cdot L \geqslant 1.$$

Furthermore, $D_3^2 = (D_2 - L)^2 = D_2^2 + L^2 - 2D_2 \cdot L \leqslant D_2^2 - 1$.

If $D_2^2 = 0$, then $-K_{\tilde{S}} \cdot D_3 = (D_1 + D_2) \cdot (D_2 - L) = 1$. Since D_3 is a part of an effective anticanonical divisor with a -2-curve, it consists of rational curves. Therefore, we can conclude that D_3 consists of one -1-curve and some -2-curves □

Lemma 3.2. *Let $\tilde{S}$ be a weak del Pezzo surface of degree less than or equal to 7. Every non-negative non-singular rational curve C that appears in an effective anticanonical divisor on $\tilde{S}$ therefore has a -1-curve L with $L \cdot C = 0$.*

Proof. We have a sequence of blow-ups $\pi\colon \tilde{S} \to \mathbb{P}^2$. The curve C is then the strict transform of a line, a conic or a singular cubic on $\mathbb{P}^2$. If the curve C comes from a singular cubic, then the singular point must be the centre of an exceptional divisor of π. Suppose that there are at least two -1-curves on $\tilde{S}$ that are exceptional curves of π and that C intersects with all the -1-curves. Choose two points among the centres of all exceptional curves of π in such a way that if the curve C comes from a singular cubic, the singular point is one of the two points. The strict transform of the line passing through the chosen two points does not meet C and it is a -1-curve; otherwise the curve C would not be irreducible. On the other hand, if there is exactly one exceptional -1-curve of π such that C intersects with this line, then the centre of the exceptional curves on $\mathbb{P}^2$ is one point and the curve C must be the strict transform of either a conic or a singular cubic. If C is the strict transform of a conic, consider the line on $\mathbb{P}^2$ that is tangent to a conic at the centre. If C is the strict transform of a singular cubic, consider one of the lines on $\mathbb{P}^2$ that constitute the tangent cone to the cubic at the singular point. The strict transform of the tangent line on $\tilde{S}$ does not then intersect C and it is a -1-curve on $\tilde{S}$. □

From now on, for a divisor D on $\tilde{S}$ we define

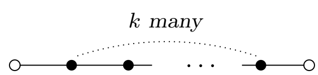
$$\operatorname{mult}(D) := \max\{\operatorname{mult}_C(D)\colon C \text{ is an irreducible curve on } \tilde{S}\}.$$

Proposition 3.3. *An effective anticanonical divisor D containing a -2-curve on $\tilde{S}$ is linearly equivalent to an effective anticanonical divisor D' containing a -2-curve such that*

- *it consists of only -1-curves and -2-curves;*
- $\operatorname{mult}(D') \geqslant \operatorname{mult}(D)$.

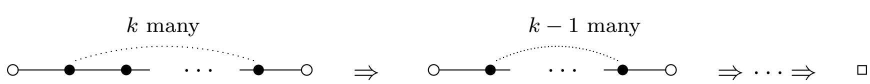
Proof. Suppose that the divisor D contains a non-negative curve C. Let $m = \mathrm{mult}_C(D)$. By Lemma 3.2, there is a -1-curve L with $C \cdot L = 0$. We then obtain an effective divisor F with $F^2 \leqslant C^2 - 1$ such that C is linearly equivalent to $L + F$ by Lemma 3.1. We replace D by the effective divisor $D - mC + m(L + F)$. Note that $\mathrm{mult}(D) \leqslant \mathrm{mult}(D - mC + m(L + F))$. Repeating this procedure a finite number of times we get the required effective anticanonical divisor. □

Lemma 3.4. *Let D be an effective anticanonical divisor on a weak del Pezzo surface $\tilde{S}$. Write $D = D_1 + D_2$, where D_1 and D_2 are effective divisors. Suppose that the divisor D_2 has the following dual graph:*



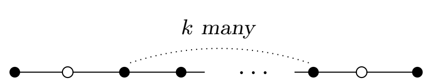
where $k \geqslant 0$. In addition, we suppose that the divisor D_1 has a -2-curve that is not a component of D_2 and does not intersect at least one -1-curve in D_2. There then exists a -1-curve that does not intersect the divisor D_2.

Proof. Let E be a -2-curve in D_1 that is not a component of D_2 and does not intersect at least one -1-curve in D_2. By contracting -1-curves $k + 1$ times from the -1-curve that does not intersect E, we get a 0-curve that is contained in an effective anticanonical divisor containing a -2-curve on a new weak del Pezzo surface:



We then apply Lemma 3.2 to obtain the required -1-curve. □

Lemma 3.5. *Let D be an effective anticanonical divisor on a weak del Pezzo surface $\tilde{S}$ of degree 2. Write $D = D_1 + D_2$, where D_1 and D_2 are effective divisors. Suppose that the divisor D_2 has the following dual graph:*



where $k \geqslant 0$. In addition, we suppose that the divisor D_1 has a -2-curve that is not a component of D_2 and is not connected to any -2-curve between two -1-curves in D_2. There then exists a -1-curve that does not intersect the divisor D_2.

Proof. Let E be a -2-curve in D_1 that is not a component of D_2 and is not connected to any -2-curve between two -1-curves in D_2. Applying Lemma 3.4 to the divisor obtained by subtracting two -2-curves at the ends of D_2 from D_2, we see that there is a -1-curve L that intersects either E or one of two -2-curves at the ends of D_2. If the -1-curve L intersects E, then we are done. Therefore, we suppose that the curve L intersects one of two -2-curves at the ends of D_2.

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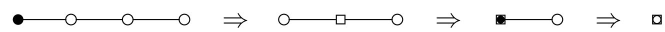
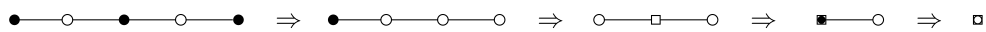
If $k = 0$, then we contract L and then blow down -1-curves as follows:

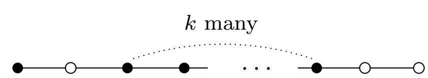


If $k = 1$, then we blow down -1-curves as follows:

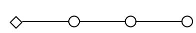


In both cases, we get a 2-curve that is contained in an effective anticanonical divisor on a new weak del Pezzo surface. The weak del Pezzo surface is of degree less than or equal to 7 and Lemma 3.2 implies the assertion.

Now we suppose that $k \geqslant 2$. By contracting L we get a divisor on a new weak del Pezzo surface whose dual graph is as follows:



We then blow down -1-curves k times from the -1-curve on the left to one on the right. We then obtain a divisor on a new weak del Pezzo surface whose dual graph is as follows:



where $\diamond$ is the curve from the -2-curve at the left end of D_2. Again we contract the -1-curve in the middle. We then get a divisor on a new weak del Pezzo surface whose dual graph is as follows:

Suppose that all -1-curves on this surface are connected to this divisor. We then contract all the -1-curves so that we obtain either $\mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. However, since the -2-curve E has never been touched by any -1-curve, this is a contradiction. Therefore, there is a -1-curve that does not intersect the divisor D_2. $\square$

Proposition 3.6. *Let $\tilde{S}$ be a weak del Pezzo surface of degree d. Suppose that there is an effective anticanonical divisor D with* $\mathrm{mult}(D) = 1$ *such that it contains at least one -2-curve, consists of only -1-curves and -2-curves and satisfies the following conditions:*

- *if $d \geqslant 3$, then it contains at least five curves;*
- *if $d = 2$, then it contains a chain of at least three -2-curves.*

There is then an effective anticanonical divisor D' on $\tilde{S}$ with $\mathrm{mult}(D') \geqslant 2$.

Proof. The dual graph of the divisor D must be circular.

First, we suppose that the divisor D contains at least three -1-curves.

If its dual graph contains either at least two chains of -1-curves or at least four -1-curves and only one chain of -1-curves, then we can obtain a divisor D_2 from D that satisfies the conditions of Lemma 3.4. Furthermore, we can pick a -1-curve L from the divisor D that does not intersect D_2. Proposition 3.1 then implies the assertion.

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If the dual graph contains exactly three -1-curves and only one chain of -1-curves, we let L_1, L_2 and L_3 be the three -1-curves with $L_1 \cdot L_3 = 0$. The divisor $D_2 = L_1 + L_2$ must then, by Lemma 3.4, have a -1-curve L that does not intersect D_2. If the -1-curve L intersects the -2-curve that intersects L_3, then the divisor $L_1 + L_2$ must be linearly equivalent to a divisor $L + R$, where R is an effective divisor, by Proposition 3.1. Then $D - L_1 - L_2 + L + R$ is an effective anticanonical divisor whose dual graph has a fork. It must therefore have a multiple component. If the curve L intersects the -2-curve that intersects L_1, then we replace the divisor $L_1 + L_2$ by $L_2 + L_3$ in the argument.

Second, we suppose that the divisor D contains only two -1-curves. Since the dual graph of D is circular we have two chains starting from one -1-curve and ending at the other -1-curve. Let $D_1 < D$ be the divisor corresponding to the chain that is not longer than the other chain. Note that D_1 contains no -2-curve if two -1-curves are connected. Lemma 3.5 shows that there is a -1-curve L that intersects a -2-curve not in D_1 and not connected to the -1-curves in D_1. Proposition 3.1 then implies that the divisor D is linearly equivalent to an effective anticanonical divisor whose dual graph has a fork. This completes the proof. □

Propositions 3.3 and 3.6 therefore show that we do not have to consider any effective anticanonical divisors without a multiple curve except those in Proposition 2.14. Furthermore, applying Proposition 3.1 with Lemmas 3.2 and 3.4 to effective anticanonical divisors with a multiple curve, we are able to obtain a short list of effective anticanonical divisors with a multiple curve to be considered for the first log canonical thresholds. Such divisors are marked by $\surd$ in the tables. This short list gives a proof of Theorem 1.7.

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