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MAXIMUM PRINCIPLES FOR PARABOLIC EQUATIONS

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Abstract

Let $u(x,t)$ be a smooth function in the domain $Q = \Omega \times (0, L]$, Ω in $\mathbb{R}^n$, let Du be the spatial gradient of $u(x,t)$ and let $\nabla u = (Du, u_t)$. If $u(x,t)$ satisfies the parabolic equation $F(u, Du, D^2u) = u_t$, we define $w(x,t)$ by $g(w) = |\nabla u|^{-1} G(\nabla u)$ (g is positive and decreasing, G is concave and homogeneous of degree one) and we prove that $w(x,t)$ attains its maximum value on the parabolic boundary of Q. If $u(x,t)$ satisfies the equation $\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t (q^2 = |Du|^2, 1 + 2q^2 h(q^2) > 0)$ we prove that $qf(u)$ takes its maximum value on the parabolic boundary of Q provided f satisfies a suitable condition. If $u(x,t)$ satisfies the parabolic equation $a^{ij}(Du)u_{ij} - b(x,t,u,Du) = u_t$ (b is concave with respect to (x,t,u)) we define $C(x,y,t,\tau) = u(z,\theta) - \alpha u(x,t) - \beta u(y,\tau)$ ($0 < \alpha$, $0 < \beta$, $\alpha + \beta = 1$, $z = \alpha x + \beta y$, $\theta = \alpha t + \beta \tau$) and we prove that if $C(x,y,t,\tau) \le 0$ when $x, y, z \in \Omega$ and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z, \in \partial\Omega$, then it is $C(x,y,t,\tau) \le 0$ everywhere.

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1. Introduction

For elliptic and parabolic equations various maximum principles have been known for a long time [11]. In recent years some maximum principles have been obtained for expressions involving the gradient of solutions. In [12] it is proved that if u is a smooth solution of the elliptic equation $F(u, Du, D^2u) = 0$ in a bounded domain Ω of $\mathbb{R}^n$ and ν is a fixed direction, then the angle between ν and the gradient Du (assumed to be non-vanishing) takes its maximum value on the boundary of Ω. The previous result has been extended to parabolic equations

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(defined in $\Omega \times [0, L] \subset \mathbb{R}^n \times \mathbb{R}$) in [13] in case $Du = (u_1, \dots, u_n)$ (the spatial gradient), and in [1] in case $\nabla u = (u_1, \dots, u_n, u_t)$. Here $u_i (i = 1, \dots, n)$ is the derivative of u with respect to x_i and u_t is the derivative with respect to t, $t \in (0, L]$.

In [10] the elliptic equation $\Delta u + 2h(q^2)u_i u_j u_{ij} = 0$ is investigated. Here Δ is the Laplace operator, the summation convention (from 1 to n) is in effect, $q^2 = u_i u_i$ and $h(q^2)$ is a smooth function satisfying $1 + 2q^2 h(q^2) > 0$. In [10] it is proved that if u is a smooth solution of this equation in a bounded domain Ω then the function $qf(u)$ takes its maximum value on the boundary of Ω provided f satisfies a suitable condition. Similar results were previously obtained in [7, 9, 8] especially for harmonic and p-harmonic functions. (If $h(q^2) = (p-2)/q^2$ then the previous equation reads as $(q^{p-2}u_i)_i = 0$ and its solutions are the usual p-harmonic functions.)

In Section 3 of this paper we consider a solution u of the equation $u_t = F(u, Du, D^2u)$ and we define a function $w(x, t)$ by $\rho g(w) = G(u_1, \dots, u_n, u_t)$, where $\rho = |\nabla u|$, g is a smooth positive decreasing function and G is a smooth concave function, positively homogeneous of degree one. We prove that $w(x, t)$ assumes its maximum value on $\partial\Omega$. If ν is a fixed direction in $\mathbb{R}^{n+1}$, $G = \nu \cdot \nabla u$ and $g(w) = \cos(w)$ then we obtain the result of [1]. This maximum principle may be used to investigate the shape of the level sets of $u(x, t)$.

Furthermore we extend the result of [10] to a parabolic equation. Namely, let u satisfy $\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t$, where the left hand side acts on the spatial variables $x_1, \dots, x_n$ only and $q = |Du|$. We prove that if $u(x, t)$ is a solution of this equation in $Q = \Omega \times (0, L]$ and f satisfies a suitable condition, then $qf(u)$ takes its maximum value on the parabolic boundary of Q. We also prove a maximum principle for $qf(v)$ where q and f are the same as before, whereas v is a solution of an associated parabolic equation. As an application we find a new estimate for the gradient of a solution to the classical heat equation.

In order to investigate the convexity of the solutions of certain parabolic equations, the following concavity function is introduced in [5]:

$$(1.1) \qquad C(x, y, t) = u(z, t) - \alpha u(x, t) - \beta u(y, t), \qquad z = \alpha x + \beta y,$$

where $x, y, z \in \Omega$, $t \in (0, L]$, α and β are positive real numbers satisfying $\alpha + \beta = 1$. If Ω is convex and $C(x, y, t) \le 0$ for all $x, y \in \Omega$ and $t \in (0, L]$ then the function $u(x, t)$ is convex with respect to x. In [6] it is proved that if $u(x, t)$ is a smooth solution of the parabolic equation

$$(1.2) \qquad a^{ij}(Du)u_{ij} - b(x, t, u, Du) = u_t,$$

where $\partial b/\partial u \geq 0$ and b is jointly concave with respect to (x, u) then, if $C(x, y, t)$ is anywhere positive, its maximum value is attained at some point (x, y, t) satisfying: $t = 0$ or one of $x, y, z \in \partial\Omega$ (the boundary of Ω). Consequently, if $C(x, y, t) \leq 0$ when $x, y, z \in \Omega$, and $t = 0$ and when $t \in (0, L]$, and one of $x, y, z \in \partial\Omega$ then necessarily $C(x, y, t) \leq 0$ everywhere.

In Section 4 of the present paper we consider the equation (1.2) where b is jointly concave with respect to (x, t, u), and, for a solution $u(x, t)$ of this equation we define the (more general) concavity function

$$\begin{aligned} C(x, y, t, \tau) &= u(z, \theta) - \alpha u(x, t) - \beta u(y, \tau), \\ z &= \alpha x + \beta y, \\ \theta &= \alpha t + \beta \tau. \end{aligned} \tag{1.3}$$

We prove that if $C(x, y, t, \tau) \leq 0$ when $x, y, z \in \Omega$, and one of $t, \tau = 0$, and when $t, \tau \in (0, L]$, and one of $x, y, z, \in \partial\Omega$, then necessarily $C(x, y, t, \tau) \leq 0$ everywhere. We remark that no use of the condition $\partial b/\partial u \geq 0$ is made.

We close this introduction with some examples of problems described by the equations involved in the present paper. Equation $u_t = F(u, Du, D^2u)$ is a general autonomous evolution equation, and $u_t = \Delta u + h(q^2)u_i u_j u_{ij}$ is a special case of it. When $F(u, Du, D^2u) = K\Delta u$ (K a positive constant) we obtain the classical heat equation for a homogeneous medium in the absence of heat sources [3, p. 41]. If $F(u, Du, D^2u) = \Delta(u^m)$, $m > 1$, the corresponding equation describes the flow through a porous media [4, p. 121]. If $F(u, Du, D^2u) = \Delta u + ug(u^2)$, where g is a suitable smooth function, we find a parabolic dissipative equation [2, p. 26]. Equation (1.2) includes also some non-autonomous equations. This is the case of the heat equation in presence of heat sources [3, p. 41]. The population genetic equation is a special case of (1.2) [3, p. 43]. More examples may be found in [3] and [2].

2. Notation and preliminary results

Throughout this paper we denote by Ω a bounded domain of $\mathbb{R}^n$, by $\partial\Omega$ its boundary, by Q the domain $Q = \Omega \times (0, L]$ and by (x, t) a point in Q. Subscripts denote partial derivatives; for example, u_i is the derivative of u with respect to x_i. We use the convention that the sum from 1 to n is understood over the repeated indices i, j and k. For the indices r and s we use a slightly different convention: u_r means the derivative of u with respect to x_r if $r = 1, \ldots, n$,

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and u_t if $r = n+1$. The same convention is used for the index s. The sum over repeated indices r and s is extended from 1 to $n+1$. As usual we denote $Du = (u_1, \dots, u_n)$, $\nabla u = (u_1, \dots, u_n, u_t)$.

Now we state three lemmas for later use.

LEMMA 2.1. *If $[a^{ij}]$ is an $n \times n$ symmetric positive definite matrix and $[b^{rs}]$ is an $(n+1) \times (n+1)$ negative semidefinite matrix then, for any $(n+1) \times n$ matrix $[\xi^{ri}]$ we have*

$$\xi^{ri} a^{ij} \xi^{sj} b^{rs} \leq 0.$$

PROOF. The result is well-known. It can be proved easily be diagonalizing the matrix $[a^{ij}]$.

LEMMA 2.2. *Let $[a^{ij}]$ be an $n \times n$ symmetric positive definite matrix, and let u be a $C^2(Q)$ function. If $q^2 = u_i u_i$, then*

$$a^{ij} u_{ki} u_{kj} \geq a^{ij} q_i q_j. \tag{2.1}$$

Furthermore, if $\rho^2 = u_r u_r$ then

$$a^{ij} u_{ri} u_{rj} \geq a^{ij} \rho_i \rho_j. \tag{2.2}$$

PROOF. Let us recall that the sum with respect to i, j and k is from 1 to n, whereas, the sum with respect to r is from 1 to $n+1$ (consequently, $\rho^2 = q^2 + u_t u_t$). This lemma is also well-known. For completeness let us prove (2.2) (the proof of (2.1) being similar). Let A denote the matrix $[a^{ij}]$, let H be the $(n+1) \times n$ Hessian matrix $[u_{ri}]$ and let H^T be the transposed matrix of H. Since A is symmetric and positive definite, the matrix HAH^T is symmetric and positive semidefinite, and its trace is the left hand side of (2.2). The right hand side of (2.2) can be written as $a^{ij} u_{ri} u_{sj} (u_r/\rho)(u_s/\rho)$; but this is the quadratic form associated to HAH^T and computed at the unit vector $\nabla u/\rho$. Hence (2.2) is true.

LEMMA 2.3. *Let $[a^{ij}]$ be an $n \times n$ positive semidefinite matrix. Then the $2n \times 2n$ matrix*

$$B = \begin{vmatrix} [a^{ij}] & [a^{ij}] \\ [a^{ij}] & [a^{ij}] \end{vmatrix} \tag{2.3}$$

is positive semidefinite.

PROOF. For a given vector $\zeta \in \mathbb{R}^{2n}$ let ξ be the vector of $\mathbb{R}^n$ defined by the first n coordinates of ζ, and let η be the vector defined by the last n coordinates of ζ. Then, the quadratic form $(B\zeta, \zeta)$ can be written as

$$(B\zeta, \zeta) = a^{ij}\xi_i\xi_j + a^{ij}\xi_i\eta_j + a^{ij}\eta_i\xi_j + a^{ij}\eta_i\eta_j = a^{ij}(\xi_i + \eta_i)(\xi_j + \eta_j).$$

The lemma follows.

3. Maximum principles involving the gradient

THEOREM 3.1. *Let $u(x,t)$ be a smooth solution of the parabolic equation*

$$F(u, Du, D^2u) = u_t \tag{3.1}$$

in Q, where F is a smooth function, Du is the vector of the spatial derivatives $[u_i]$ and $D^2u = [u_{ij}]$. If a^{ij} denote the partial derivatives of F with respect to u_{ij} we assume that $a^{ij} = a^{ji}$ and that for every $\xi \in \mathbb{R}^n$ and some $\nu > 0$ the condition.

$$a^{ij}\xi_i\xi_j \geq \nu\xi_i\xi_i. \tag{3.2}$$

holds. If $g(\tau)$ is a smooth positive decreasing function and $\rho^2 = u_r u_r$ we define $w(x,t)$ by

$$g(w)\rho = G(u_1, \dots, u_n, u_t), \tag{3.3}$$

where $G(\eta_1, \dots, \eta_{n+1})$ is a smooth concave function satisfying

$$\eta_r \partial G/\partial\eta_r = G. \tag{3.4}$$

If $G(u_1, \dots, u_n, u_t)$ and ρ are positive corresponding to the solution $u(x,t)$ then the function $w(x,t)$ defined by (3.3), takes its maximum value on the parabolic boundary of Q, that is on $\Omega \times [0] \cup \partial\Omega \times [0, L]$.

PROOF. Recall that we sum from 1 to n with respect to the repeated indices i, j, k and from 1 to $n+1$ with respect to r, s. By (3.3) we derive

$$\dot{g}\rho w_i + g\rho_i = \partial G/\partial\eta_r u_{ri}, \tag{3.5}$$

$$\dot{g}\rho w_t + g\rho_t = \partial G/\partial\eta_r u_{rt}, \tag{3.6}$$

where $\dot{g}$ is the derivative of g. By (3.5) we obtain

$$\dot{g}\rho w_{ij} + (\dot{g}\rho)_j w_i + \dot{g}\rho_i w_j + g\rho_{ij} = b^{rs}u_{ri}u_{sj} + \partial G/\partial\eta_r u_{rij}, \tag{3.7}$$

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where $[b^{rs}]$ is the $(n+1)\times(n+1)$ Hessian matrix of G. Since G is concave, the matrix $[b^{rs}]$ is negative semidefinite. By (3.2) the $n \times n$ matrix $[a^{ij}]$ is positive definite; hence by Lemma 2.1 we have $a^{ij}b^{rs}u_{ri}u_{sj} \leq 0$ and (3.7) implies

$$\dot{g}\rho a^{ij}w_{ij} + b^i w_i + g a^{ij}\rho_{ij} \leq \partial G/\partial\eta_r a^{ij}u_{rij}, \tag{3.8}$$

where $b^i = a^{ij}\left((\dot{g}\rho)_j + \dot{g}\rho_j\right)$.

Since $\rho^2 = u_r u_r$ we have

$$\rho\rho_i = u_r u_{ri}, \qquad \rho\rho_t = u_r u_{rt}, \tag{3.9}$$

$$\rho\rho_{ij} = u_{ri}u_{rj} - \rho_i\rho_j + u_r u_{rij}. \tag{3.10}$$

By (2.2) $a^{ij}u_{ri}u_{rj} - a^{ij}\rho_i\rho_j \geq 0$. Hence (3.10) implies

$$\rho a^{ij}\rho_{ij} \geq u_r a^{ij}u_{rij}. \tag{3.11}$$

This inequality and (3.8) imply

$$\dot{g}\rho a^{ij}w_{ij} + b^i w_i \leq (\partial G/\partial\eta_r - u_r g/\rho)a^{ij}u_{rij}. \tag{3.12}$$

By equation (3.1) we derive, for $r = 1, \dots, n+1$,

$$a^{ij}u_{ijr} = u_{tr} - F_u u_r - u_{ir}\partial F/\partial u_i.$$

By inserting the last equation into (3.12) we obtain

$$-\dot{g}\rho a^{ij}w_{ij} - b^i w_i \geq g\rho_t - g\rho F_u - g\rho_i\frac{\partial F}{\partial u_i} - u_{tr}\frac{\partial G}{\partial\eta_r} + GF_u + u_{ir}\frac{\partial F}{\partial u_i}\frac{\partial G}{\partial\eta_r}$$

where (3.9) and (3.4) have been used. Finally, (3.3), (3.5), (3.6) and the last inequality lead to

$$a^{ij}w_{ij} + d^i w_i \geq w_t, \tag{3.13}$$

where $d^i = b^i/(\dot{g}\rho) + \partial F/\partial u_i$. Now the theorem follows by the well-known maximum principle for parabolic inequalities [11, p. 173].

REMARK 3.1. If $F_u = 0$ in equation (3.1), then condition (3.4) can be omitted in Theorem 3.1.

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THEOREM 3.2. *Let $u(x,t)$ be a smooth solution of the equation*

$$\Delta u + 2h(q^2)u_i u_j u_{ij} = u_t \tag{3.14}$$

in $Q = \Omega \times (0, L]$, where Δ is the Laplace operator, $q^2 = u_i u_i$ and $h(\tau)$ is a smooth function satisfying $1 + 2q^2 h(q^2) > 0$. We define $T(q^2) = 1 + 2q^2 h(q^2)$ and assume there is a finite γ such that

$$1 + q^2 \dot{T}/T \le \gamma, \tag{3.15}$$

where $\dot{T}$ is the derivative of T with respect to q^2. If $f(\tau)$ is a smooth positive function satisfying

$$(\dot{f} f^{-2\gamma})^{\cdot} \ge 0, \tag{3.16}$$

then the function

$$\Phi(x,t) = qf(u) \tag{3.17}$$

takes its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$.

PROOF. By (3.17) we derive:

$$\Phi_i = q_i f + q\dot{f}u_i, \qquad \Phi_t = q_t f + q\dot{f}u_t, \tag{3.18}$$

$$\Phi_{ij} = q_{ij} f + q_i \dot{f} u_j + q_j \dot{f} u_i + q\ddot{f}u_i u_j + q\dot{f}u_{ij}. \tag{3.19}$$

We set $a^{ij} = \delta^{ij} + 2h(q^2)u_i u_j$, where δ^{ij} is the Kronecker delta. The matrix $[a^{ij}]$ is symmetric and its eigenvalues are 1 and $1 + 2q^2 h(q^2)$. Hence it is positive definite by virtue of the assumption $1 + 2q^2 h(q^2) > 0$. With this notation, equation (3.14) reads $a^{ij}u_{ij} = u_t$, and (3.19) implies

$$a^{ij}\Phi_{ij} = f a^{ij} q_{ij} + 2\dot{f} a^{ij} q_i u_j + q\ddot{f} a^{ij} u_i u_j + q\dot{f}u_t. \tag{3.20}$$

By the definition of q we derive

$$qq_i = u_k u_{ki}, \quad qq_t = u_k u_{kt} \tag{3.21}$$

and

$$qq_{ij} = u_{ki}u_{kj}, q_i q_j + u_k u_{kij}.$$

The last equation and inequality (2.1) of Lemma 2.2 imply

$$qa^{ij}q_{ij} \ge u_k a^{ij} u_{kij}. \tag{3.22}$$

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By the equation $a^{ij}u_{ij} = u_t$ we derive

$$u_k a^{ij} u_{ijk} = u_k u_{tk} - u_k a_k^{ij} u_{ij} = u_k u_{tk} - 2u_k(2\dot{h}qq_k u_i u_j + hu_{ik}u_j + hu_i u_{jk})u_{ij}.$$

The latter and equations (3.21) give

$$u_k a^{ij} u_{ijk} = qq_t - 2q^2(2\dot{h}(q_k u_k)^2 + 2hq_j q_j). \tag{3.23}$$

By using (3.23), (3.22) and the last of equations (3.18), (3.20) gives

$$a^{ij}\Phi_{ij} \geq \Phi_t - 2fq\left(2\dot{h}(q_i u_i)^2 + 2hq_i q_i\right) + 2\dot{f}a^{ij}q_i u_j + q\ddot{f}a^{ij}u_i u_j. \tag{3.24}$$

By (3.18) it follows that

$$q_i u_i = f^{-1}\Phi_i u_i - q^3 f^{-1}\dot{f}, \tag{3.25}$$

$$\begin{aligned} q_i q_i &= f^{-1}\Phi_i q_i - qf^{-1}\dot{f}q_i u_i \\ &= f^{-1}\Phi_i q_i - qf^{-2}\dot{f}\Phi_i u_i + q^4 f^{-2}\dot{f}^2, \end{aligned} \tag{3.26}$$

$$\begin{aligned} a^{ij}q_i u_j &= a^{ij}\left(f^{-1}\Phi_i - f^{-1}\dot{f}qu_i\right)u_j \\ &= f^{-1}a^{ij}\Phi_i u_j - f^{-1}\dot{f}qa^{ij}u_i u_j. \end{aligned} \tag{3.27}$$

Insertion of (3.25), (3.26) and (3.27) into (3.24) leads to

$$a^{ij}\Phi_{ij} + b^i\Phi_i \geq \Phi_t - 2f^{-1}\dot{f}^2 q^5\left(2\dot{h}q^2 + 2h\right) - 2f^{-1}\dot{f}^2 qa^{ij}u_i u_j + q\ddot{f}a^{ij}u_i u_j, \tag{3.28}$$

where b^i are expressions depending on f, h, a^{ij}, u_i, q_i and Φ_i (but not on Φ_t).

Since $2\dot{h}q^2 + 2h = \dot{T}$ and $a^{ij}u_i u_j = q^2 T$, (3.28) can be rewritten as

$$a^{ij}\Phi_{ij} + b^i\Phi_i \geq \Phi_t + q^3 T\left(\ddot{f} - 2f^{-1}\dot{f}^2(1 + q^2\dot{T}/T)\right). \tag{3.29}$$

Since, by assumption, $1 + q^2\dot{T}/T \leq \gamma$, inequality (3.29) implies

$$a^{ij}\Phi_{ij} + b^i\Phi_i - \Phi_t \geq q^3 T\left(\ddot{f} - 2\gamma f^{-1}\dot{f}^2\right) = q^3 T f^{2\gamma}\left(\dot{f}f^{-2\gamma}\right)^{\cdot} \geq 0,$$

where assumption (3.16) has been used in the last step. The theorem follows by the maximum principle for parabolic equations.

THEOREM 3.3. *Under the same notation and assumptions of Theorem* 3.2, *if* $h \leq 0$ *and* $\gamma = 1$, *then the function*

$$\Phi(x, t) = qf(v) \tag{3.30}$$

attains its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$. *Here* q *and* f *are the same as in Theorem* 3.2, *whereas* v *is any solution of the equation.*

$$\Delta v + 2h(q^2)u_i u_j v_{ij} = v_t. \tag{3.31}$$

PROOF. Arguing as in the proof of Theorem 3.2 we obtain

$$(3.32)\quad a^{ij}\Phi_{ij} \geq \Phi_t - 2fq\left[2\dot{h}(q_iu_i)^2 + 2hq_iq_i\right] + 2\dot{f}a^{ij}q_iv_j + q\ddot{f}a^{ij}v_iv_j.$$

Our assumption $\gamma = 1$ implies $2\dot{h}q^2 + 2h = \dot{T} \leq 0$. This inequality, together with the Schwarz inequality $(q_iu_i)^2 \leq q_iq_iu_ku_k$ and the assumption $h \leq 0$, makes the quantity between square brackets in (3.32) non-positive, hence it can be deleted.

On the other side, by (3.30) we derive

$$q_i = f^{-1}\Phi_i - qf^{-1}\dot{f}v_i,$$

hence (3.32) implies

$$a^{ij}\Phi_{ij} + b^i\Phi_i - \Phi_t \geq qa^{ij}v_iv_j\left(\ddot{f} - 2f^{-1}\dot{f}^2\right) = qa^{ij}v_iv_jf^2\left(\dot{f}f^{-2}\right)^{\cdot} \geq 0,$$

where $b^i = -2f^{-1}\dot{f}a^{ij}v_j$ and condition (3.15) (with $\gamma = 1$) has been used in the last step. The theorem follows.

COROLLARY 3.1. *Let u satisfy $u_t = \Delta u$ in $Q = \Omega \times [0, L]$ with $q(x,t) = 0$ on $\partial\Omega \times [0, L]$, and let v be a positive solution of $v_t = \Delta v$ in Q with $v(x,t) = 1$ on $\Omega \times [0]$. Then we have*

$$q(x,t) \leq q(\bar{x}, 0)v(x,t)$$

for all $(x,t) \in Q$ and some $\bar{x} \in \Omega$.

PROOF. The function $f(v) = v^{-1}$ satisfies (3.16) with $\gamma = 1$, hence, by Theorem 3.3, the expression qv^{-1} attains its maximum value on $\Omega \times [0] \cup \partial\Omega \times [0, L]$. But $q = 0$ on $\partial\Omega \times [0, L]$ and $v = 1$ on $\Omega \times [0]$ by assumption. The corollary follows.

4. A maximum principle for the concavity function

If α and β are positive real numbers satisfying $\alpha + \beta = 1$, if λ is a real number, and if $u(x,t)$ is a function defined in Q, let us define

$$(4.1)\quad \begin{aligned} \Phi(x, y, t, \tau) &= e^{\lambda\theta}\left[u(z,\theta) - \alpha u(x,t) - \beta u(y,\tau)\right], \\ z &= \alpha x + \beta y, \\ \theta &= \alpha t + \beta\tau, \end{aligned}$$

for all x, y such that $x, y, z \in \Omega$ and for all $t, \tau \in (0, L]$.

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THEOREM 4.1. *Let $u(x,t)$ be a smooth function defined in Q and satisfying equation* (1.2). *Suppose that the $n \times n$ matrix $[a^{ij}(Du)]$ of* (1.2) *is positive semidefinite for any vector Du, that the function $b(x,t,u,Du)$ is jointly concave with respect to (x,t,u), and that the derivative $\partial b/\partial u$ is bounded from below. Then the function ϕ defined in* (4.1) *with λ satisfying*

$$\partial b/\partial u - \lambda > 0 \tag{4.2}$$

has no positive maximum for $x, y, z \in \Omega$ and $t, \tau \in (0, L]$.

PROOF. Assume to the contrary that (x, y, t, τ) is a positive maximum for Φ with $x, y, z \in \Omega$ and $t, \tau \in (0, L]$. At this point we have

$$\Phi_{x_i} = \alpha e^{\lambda\theta}\left[u_{z_i}(z,\theta) - u_{x_i}(x,t)\right] = 0, \tag{4.3}$$

$$\Phi_{y_i} = \beta e^{\lambda\theta}\left[u_{z_i}(z,\theta) - u_{y_i}(y,\tau)\right] = 0. \tag{4.4}$$

Equations (4.3) - (4.4) imply that the spatial gradient Du is the same at the three points (z,θ), (x,t), (y,τ). We also have

$$\Phi_t = \lambda\alpha\Phi + e^{\lambda\theta}\left[\alpha u_\theta(z,\theta) - \alpha u_t(x,t)\right] \geq 0,$$
$$\Phi_\tau = \lambda\beta\Phi + e^{\lambda\theta}\left[\beta u_\theta(z,\theta) - \beta u_\tau(y,\tau)\right] \geq 0,$$

from which it follows that

$$\lambda\Phi + e^{\lambda\theta}\left[u_\theta(z,\theta) - \alpha u_t(x,t) - \beta u_\tau(y,\tau)\right] \geq 0. \tag{4.5}$$

At the maximum point (x, y, t, τ), the Hessian matrix

$$H = \begin{vmatrix} \left[\Phi_{x_ix_j}\right] & \left[\Phi_{x_iy_j}\right] \\ \left[\Phi_{y_ix_j}\right] & \left[\Phi_{y_iy_j}\right] \end{vmatrix}$$

is negative semidefinite. By Lemma 2.3, the matrix B defined in (2.3) by using the matrix $[a^{ij}(Du)]$ of the equation (1.2) (computed at any of the points (z,θ), (x,t), (y,τ)) is positive semidefinite. Consequently, the matrix BH is negative semidefinite and its trace is non positive. Hence

$$0 \geq a^{ij}\Phi_{x_ix_j} + a^{ij}\Phi_{x_iy_j} + a^{ij}\Phi_{y_ix_j} + a^{ij}\Phi_{y_iy_j}. \tag{4.6}$$

By (4.1) we derive

$$\Phi_{x_ix_j} = e^{\lambda\theta}\left[\alpha^2 u_{z_iz_j}(z,\theta) - \alpha u_{x_ix_j}(x,t)\right],$$
$$\Phi_{x_iy_j} = \Phi_{y_ix_j} = e^{\lambda\theta}\alpha\beta u_{z_iz_j}(z,\theta),$$
$$\Phi_{y_iy_j} = e^{\lambda\theta}\left[\beta^2 u_{z_iz_j}(z,\theta) - \beta u_{y_iy_j}(y,\tau)\right].$$

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Insertion of the last equations into (4.6) leads to

$$0 \geq e^{\lambda\theta}\left[(\alpha+\beta)^2 a^{ij} u_{z_i z_j}(z,\theta) - \alpha a^{ij} u_{x_i x_j}(x,t) - \beta a^{ij} u_{y_i y_j}(y,\tau)\right].$$

The latter inequality and the equation (1.2) give

$$\begin{aligned} (4.7) \quad 0 \geq e^{\lambda\theta}\big[b(z,\theta,u) + u_\theta(z,\theta) - \alpha b(x,t,u) - \alpha u_t(x,t) \\ - \beta b(y,\tau,u) - \beta u_\tau(y,\tau)\big], \end{aligned}$$

where $Du(z,\theta) = Du(x,t) = Du(y,\tau)$ are suppressed in the expression of b. Using the concavity of b and inequality (4.5), (4.7) implies

$$\begin{aligned} (4.8) \qquad 0 &\geq e^{\lambda\theta}\big[b(z,\theta,u(z,\theta)) - b(z,\theta,\alpha u(x,t) + \beta u(y,\tau))\big] - \lambda\Phi \\ &= (\partial b/\partial u - \lambda)\Phi. \end{aligned}$$

Since, by assumption, $\partial b/\partial u - \lambda > 0$ and the value of Φ at the point of maximum (x, y, t, τ) is positive, (4.8) is a contradiction. The theorem is proved.

COROLLARY 4.1. *With the notation and assumptions of Theorem* 4.1, *if the concavity function* $C(x, y, t, \tau)$ *defined in* (1.3) *is non-positive when* $x, y, z, \in \Omega$, *and one of* $t, \tau = 0$, *and when* $t, \tau \in (0, L]$, *and one of* $x, y, z \in \partial\Omega$, *then* $C(x, y, t, \tau) \leq 0$ *for all* x, y *in* Ω *and* t, τ *in* $(0, L]$.

PROOF. Since the inequality $C(x, y, t, \tau) \leq 0$ implies $\Phi(x, y, t, \tau) \leq 0$ and vice versa, the corollary follows by Theorem 4.1.

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