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On Berman's phenomenon in interpolation theory

W. Lyle Cook and T.M. Mills

In 1965, D.L. Berman established an interesting divergence theorem concerning Hermite-Fejér interpolation on the extended Chebyshev nodes. In this paper it is shown that this phenomenon is not an isolated incident. A similar divergence theorem is proved for a higher order interpolation process. The paper closes with a list of several related open problems.

1. Introduction

Let $+1 = x_{0,n} > x_{1,n} > \ldots > x_{n,n} > x_{n+1,n} = -1$, $n = 1, 2, \ldots$ be the triangular matrix of nodes or abscissas defined by

$$x_{k,n} = \cos\left((2k-1)\pi/2n\right) ,$$

for $k = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$.

The abscissas $\{x_{k,n};\ k = 1, 2, \ldots, n\}$ are sometimes called the Chebyshev nodes and they play an important rôle in the theory of interpolation polynomials.

Fejér [3] first showed the importance of these nodes when he considered the following method of approximation. Let $f(x)$ be a real valued function defined on the interval $[-1, 1]$. For each n , we define $H_n(f, x)$ to be the unique polynomial of degree $2n - 1$ such that

$$H_n\left(f, x_{k,n}\right) = f\left(x_{k,n}\right) , \quad k = 1, 2, \ldots, n$$

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and

$$H_n'(f, x_{k,n}) = 0 , \quad k = 1, 2, \ldots, n ,$$

where the dash (') represents differentiation with respect to x. Fejér then provided the first interpolatory proof of Weierstrass' approximation theorem:

THEOREM 1 (Fejér). *If* $f(x)$ *is continuous on the interval* $[-1, 1]$ *then*

$$\lim_{n\to\infty} \|H_n(f)-f\| = 0$$

where $\|\cdot\|$ *denotes the uniform norm on* $[-1, 1]$.

Much later, Berman [1] of Leningrad showed the effect of prescribing end point conditions on the interpolatory polynomials. For each n, define $K_n(f, x)$ to be the unique polynomial of degree $2n + 3$ such that

$$K_n(f, x_{k,n}) = f(x_{k,n}) , \quad k = 0, 1, \ldots, n, n+1$$

and

$$K_n'(f, x_{k,n}) = 0 , \quad k = 0, 1, \ldots, n, n+1 .$$

Concerning this process, he proved the following startling result:

THEOREM 2 (Berman). *If* $g(x) = |x|$ *then the sequence* $\{K_n(g, 0); n = 1, 2, \ldots\}$ *diverges.*

Loosely speaking, Berman showed that by prescribing too many conditions at the end points, one causes the interpolation polynomials to buckle in the middle of the interval. We call this Berman's phenomenon.

In 1922, the Soviet mathematicians Krylov and Steuermann [5] considered an extension of Féjer's approximation process. Define $L_n(f, x)$ to be the unique polynomial of degree $4n - 1$ such that

$$L_n(f, x_{k,n}) = f(x_{k,n}) , \quad k = 1, 2, \ldots, n ,$$

and

$$L_n^{(j)}(f, x_{k,n}) = 0 , \quad k = 1, 2, \ldots, n ,$$

and $j = 1, 2, 3$.

They asserted that L_n enjoys the same approximation properties as H_n:

THEOREM 3 (Krylov and Steuermann). *If $f(x)$ is continuous on the interval $[-1, 1]$ then*

$$\lim_{n\to\infty} \|L_n(f)-f\| = 0 .$$

The aim of this paper is to show that Berman's phenomenon occurs in this extended setting as well. In particular we shall define $P_n(f, x)$ to be the unique polynomial of degree $4n + 7$ such that

(1) $$P_n(f, x_{k,n}) = f(x_{k,n}) , \quad k = 0, 1, \ldots, n, n+1 ,$$

and

(2) $$P_n^{(j)}(f, x_{k,n}) = 0 , \quad k = 0, 1, \ldots, n, n+1 ,$$

and $j = 1, 2, 3$.

The main result of this work is the following:

THEOREM 4. *If $h(x) = (1-x^2)^3$ then the sequence*

$$\{P_n(h, 0); n = 1, 2, \ldots\}$$

diverges.

In Section 2 we establish certain formulae which are necessary for the proof of the theorem. Then Section 3 contains the proof of the theorem. Finally, in Section 4, we conclude with a few remarks on the existing literature in this area and some suggestions for further research problems.

2. Formulae

To compute the polynomial $P_n(f, x)$ we shall need certain differential equations which are satisfied by

$$w_n(x) = (1-x^2)T_n(x) .$$

Here $T_n(x) = \cos(n \text{ arc cos } x)$ is the Chebyshev polynomial of degree n.

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It is well known (see Rivlin, [8], p. 33) that $T_n(x)$ satisfies the equation

$$\left(1-x^2\right)T_n^{(k+1)}(x) - (2k-1)xT_n^{(k)}(x) + \left(n^2-(k-1)^2\right)T_n^{(k-1)}(x) = 0 ,$$

for $-1 \le x \le +1$ and any natural number k .

Then, it is straight forward, but tedious, to show that $w_n(x)$ satisfies the following three equations for $-1 \le x \le +1$:

$$(3) \qquad \left(1-x^2\right)^2 w_n''(x) + 3x\left(1-x^2\right)w_n'(x) + \left(6x^2+\left(n^2+2\right)\left(1-x^2\right)\right)w_n(x) = 0 ;$$

$$(4) \quad \left(1-x^2\right)^2 w_n^{(3)}(x) - x\left(1-x^2\right)w_n''(x) + \left(1-x^2\right)\left(5+n^2\right)w_n'(x) + 2x\left(4-n^2\right)w_n(x) = 0 ;$$

and

$$(5) \quad \left(1-x^2\right)^2 w_n^{(4)}(x) - 5x\left(1-x^2\right)w_n^{(3)}(x) + \left(\left(1-x^2\right)\left(5+n^2\right)-\left(1-3x^2\right)\right)w_n''(x) - 2x\left(1+2n^2\right)w_n'(x) + 2\left(4-n^2\right)w_n(x) = 0 .$$

We now recall that $\{x_{k,n};\ k = 0, 1, \ldots, n, n+1\}$ are the simple zeros of $w_n(x)$. (We shall write x_k for $x_{k,n}$ whenever there is no danger of confusion.) Hence from (3), (4), and (5) one can derive the following formulae for $1 \le k \le n$:

$$(6) \qquad \frac{w''\left(x_k\right)}{w'\left(x_k\right)} = \frac{-3x_k}{1-x_k^2} ;$$

$$(7) \qquad \frac{w^{(3)}\left(x_k\right)}{w'\left(x_k\right)} = -\left\{\frac{3x_k^2}{\left(1-x_k^2\right)^2} + \frac{5+n^2}{\left(1-x_k^2\right)}\right\} ;$$

$$(8) \qquad \frac{w^{(4)}\left(x_k\right)}{w'\left(x_k\right)} = \frac{\left[-6x_k^3-3x_k\right]}{\left(1-x_k^2\right)^3} + \frac{\left(2n^2-8\right)x_k}{\left(1-x_k^2\right)^2} .$$

Now we have enough information to compute $P_n(f, x)$.

In order to examine the effect of large n on $P_n(f)$ we shall need some identities concerning the nodes x_k for $1 \le k \le n$. Since $\{x_k; 1 \le k \le n\}$ is the complete set of zeros of $T_n(x)$ it follows that

(9) $$\frac{T_n'(x)}{T_n(x)} = \sum_{k=1}^{n} \frac{1}{(x-x_k)} .$$

Using (9) and assuming that $n = 4p$ (p an integer) one can prove the following identities:

(10) $$\sum_{k=1}^{n} 1/(1+x_k) = \sum_{k=1}^{n} 1/(1-x_k) = n^2 ,$$

(11) $$\sum_{k=1}^{n} 1/\left(1-x_k^2\right) = n^2 ,$$

(12) $$\sum_{k=1}^{n} 1/x_k^2 = n^2 ,$$

(13) $$\sum_{k=1}^{n} 1/(1+x_k)^2 = \sum_{k=1}^{n} 1/(1-x_k)^2 = \left(2n^4+n^2\right)/3 ,$$

(14) $$\sum_{k=1}^{n} 1/\left(1-x_k^2\right)^2 = \left(n^4+2n^2\right)/3 ,$$

(15) $$\sum_{k=1}^{n} x_k^2/\left(1-x_k^2\right)^2 = \left(n^4-n^2\right)/3 ,$$

(16) $$\sum_{k=1}^{n} 1/x_k^4 = \left(n^4+2n^2\right)/3 .$$

The proofs of these identities are not hard. For example in deriving (16) one merely differentiates both sides of (9) thrice with respect to x and then sets $x = 0$. One may look up relevant facts in the excellent text by Rivlin [8]. To show (15) one breaks the summand into partial fractions and employs some of the other identities. In fact most of these identities are well known and may be found elsewhere (see, for example, the papers of Berman [2] and Saxena [9]). However we mention them here for the sake of completeness.

3. Proof of Theorem 4

A paper by Laden [6] gives the following formula for $P_n(f, x)$ for any matrix of nodes:

$$P_n(f, x) = \sum_{k=0}^{n+1} f(x_k)u_k(x)(l_k(x))^4 \tag{17}$$

where

$$u_k(x) = 1 - 2(x-x_k)\frac{w_n''(x_k)}{w_n'(x_k)} + \frac{(x-x_k)^2}{2}\left\{5\left[\frac{w_n''(x_k)}{w_n'(x_k)}\right]^2 - \frac{4}{3}\frac{w_n^{(3)}(x_k)}{w_n'(x_k)}\right\}$$
$$+ \frac{(x-x_k)^3}{6}\left\{-15\left[\frac{w_n''(x_k)}{w_n'(x_k)}\right]^3 + 10\frac{w_n''(x_k)w_n^{(3)}(x_k)}{(w_n'(x_k))^2} - \frac{w_n^{(4)}(x_k)}{w_n'(x_k)}\right\}$$

and

$$l_k(x) = \frac{w_n(x)}{w_n'(x_k)(x-x_k)} .$$

Now we want to consider the sequence $P_n(h, 0)$ where $h(x) = (1-x^2)^3$. Note that in this case, $h(x_0) = h(1) = 0$ and $h(x_{n+1}) = h(-1) = 0$. For our choice of $w_n(x) = (1-x^2)T_n(x)$,

$$(l_k(0))^4 = 1/\left(n^4x_k^4\left(1-x_k^2\right)^2\right) .$$

The quantity $u_k(0)$ may be calculated with the help of formulae (6) to (8). We thus eventually obtain

$$P_n(h, 0) = n^{-4}\sum_{k=1}^{n}\left\{1/x_k^4 + (2n^2+5)/\left(3x_k^2\right)\right.$$
$$\left. - (28n^2+11)/\left(6\left(1-x_k^2\right)\right) - 1/\left(2\left(1-x_k^2\right)^2\right) - 167x_k^2/\left(2\left(1-x_k^2\right)^2\right)\right\} .$$

Now use the formulae (10) to (16). Thus, if $n = 4p$, we have

$$P_n(h, 0) = -95/3 + O(1/n^2) .$$

Therefore,

(18) $$\lim_{n\to\infty} P_{4n}(h,\ 0) = -\ 95/3 \ .$$

However if n is an odd integer then $n = 2m + 1$. In this case

$$x_{m+1,n} = \cos \pi/2 = 0 \ .$$

So, by (1), $P_n(h,\ 0) = h(0) = 1$. Consequently,

(19) $$\lim_{n\to\infty} P_{2n+1}(h,\ 0) = 1 \ .$$

The combination of (18) and (19) yields the desired conclusion; namely, that the sequence

$$\{P_n(h,\ 0);\ n = 1,\ 2,\ \ldots\}$$

is divergent.

4. Related theorems and problems

In light of the results in Section 1, one naturally asks the following question. Let p be a fixed positive integer. For each n , define $T_n(p,\ f,\ x)$ to be the unique polynomial of degree $2p(n+2) - 1$ such that

$$T_n(p,\ f,\ x_k) = f(x_k) \ , \quad k = 0,\ 1,\ 2,\ \ldots,\ n,\ n+1$$

and

$$T_n^{(j)}(p,\ f,\ x_k) = 0 \ , \quad k = 0,\ 1,\ 2,\ \ldots,\ n,\ n+1 \ ,$$

where $j = 1,\ 2,\ \ldots,\ 2p-1$.

Does the sequence $\{T_n(p,\ f,\ x) : n = 1,\ 2,\ \ldots\}$ exhibit Berman's phenomenon? Note that $T_n(1,\ f,\ x) = K_n(f,\ x)$ and $T_n(2,\ f,\ x) = P_n(f,\ x)$.

Berman [1], [2] has shown examples of very simple functions $f(x)$ such that the sequence $\{K_n(f,\ x) : n = 1,\ 2,\ \ldots\}$ diverges. It would be very interesting to determine the class of functions $f(x)$ such that

$$\lim_{n\to\infty} \|K_n(f)-f\| = 0 .$$

We suspect that this class is the class of constant functions. Similar questions and suspicions can be raised about $P_n(f, x)$.

It should be mentioned that the proof of Theorem 2 which was published by Krylov and Steuermann is incorrect. A correct proof can be found in Laden's paper [6]. This last paper, although quite thorough, raises many problems. Here Laden concerns himself with operators like L_n which are based on the zeros of the Jacobi polynomials as the nodes of interpolation. Now there have been many recent results concerning operators like $H_n(f)$ based on these Jacobi abscissas. (See for example, Szabados [11], Szász [12], and Vértesi [13].) Can these results be carried over to operators like $L_n(f)$? Laden's paper suggests that the answer to such questions depends on the parameters α, β involved in the particular Jacobi polynomial $P_n^{(\alpha,\beta)}$ under consideration.

Finally we note that the sequence of operators L_n has been studied more closely in references [4], [7], and [10].

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Department of Mathematics,
Eastern Montana College,
Billings,
Montana,
USA;

Department of Mathematics,
Bendigo Institute of Technology,
Bendigo,
Victoria.