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EXISTENCE OF NEAREST POINTS IN BANACH SPACES

JONATHAN M. BORWEIN AND SIMON FITZPATRICK

1. Introduction. This paper makes a unified development of what the authors know about the existence of nearest points to closed subsets of (real) Banach spaces. Our work is made simpler by the methodical use of subderivatives. The results of Section 3 and Section 7 in particular are, to the best of our knowledge, new. In Section 5 and Section 6 we provide refined proofs of the Lau-Konjagin nearest point characterizations of reflexive Kadec spaces (Theorem 5.11, Theorem 6.6) and give a substantial extension (Theorem 5.12). The main open question is: are nearest points dense in the boundary of every closed subset of every reflexive space? Indeed can a proper closed set in a reflexive space fail to have any nearest points? In Section 7 we show that there are some non-Kadec reflexive spaces in which nearest points are dense in the boundary of every closed set.

If E is a real Banach space and C is a closed non-empty subset of E then the distance function d_C is defined by

$$d_C(x) := \inf\{\|x - z\| : z \in C\},$$

and any z in C with $d_C(x) = \|x - z\|$ is a *nearest point* in C to x. If $z \in C$ and there is some $x \in E\backslash C$ with z as its nearest point we call z a *nearest point*. Also $B[x, \alpha]$ and $B(x, \alpha)$ denote respectively the closed and open balls around x of radius $\alpha \geqq 0$.

Definition 1.1. (a) If every $x \in E\backslash C$ has a nearest point in C, we call C *proximinal*. (b) If the set of points in $E\backslash C$ possessing nearest points in C is generic (contains a dense G_δ) we call C *almost proximinal*. (c) A sequence $\{z_n\}$ of elements in C is called a *minimizing sequence* in C for x if

$$d_C(x) = \lim_{n \to \infty} \|x - z_n\|.$$

Definition 1.2. Let f be an extended real valued function f defined on a Banach space with $f(x)$ finite. Then f is *Fréchet subdifferentiable* at x with $x^* \in E^*$ belonging to the *Fréchet subdifferential* at $x, \partial^F f(x)$, provided that

$$\liminf_{y \to 0} \frac{f(x+y) - f(x) - \langle x^*, y\rangle}{\|y\|} \geqq 0.$$

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THEOREM 1.3. [**3**] *Let f be a lower semicontinuous function on a Banach space with equivalent Fréchet differentiable norm (in particular, E reflexive will do). Then f is Fréchet subdifferentiable on a dense subset of its graph.*

For distance functions, Fréchet subdifferentiability has the following important consequences.

PROPOSITION 1.4. *Suppose that C is a closed non-empty subset of a Banach space and that $x^* \in \partial^F d_C(x)$ for $x \in E/C$. Then $\|x^*\| = 1$, and for each minimizing sequence $\{z_n\}$ in C for x*

$$d_C(x) = \lim_{n\to\infty} \langle x^*, x - z_n \rangle.$$

Proof. Suppose $\{z_n\}$ is a minimizing sequence in C for x while $0 < t < 1$. We have

$$\begin{aligned} d_C(x + t(z_n - x)) - d_C(x) &\leqq \|x + t(z_n - x) - z_n\| - d_C(x) \\ &\leqq \|x + t(z_n - x) - z_n\| - \|x - z_n\| + [\|x - z_n\| - d_C(x)] \\ &= -t\|x - z_n\| + [\|x - z_n\| - d_C(x)], \end{aligned}$$

and, letting

$$t_n := 2^{-n} + [\|x - z_n\| - d_C(x)]^{1/2},$$

we have from Fréchet subdifferentiability that

$$\liminf_{n\to\infty} \frac{d_C(x + t_n(z_n - x)) - d_C(x)}{t_n} - \langle x^*, z_n - x \rangle \geqq 0$$

so that

$$\liminf[-\|x - z_n\| + \langle x^*, z_n - x \rangle + t_n] \geqq 0,$$

and

$$d_C(x) = \lim \|x - z_n\| \leqq \liminf \langle x^*, x - z_n \rangle.$$

Now $\|x^*\| \leqq 1$ since d_C is 1-Lipschitz. It follows that

$$d_C(x) = \lim \|x - z_n\| \geqq \limsup \langle x^*, x - z_n \rangle.$$

Comparison of these last two inequalities shows that $\|x^*\| = 1$ and that

$$d_C(x) = \lim_{n\to\infty} \langle x^*, x - z_n \rangle.$$

2. Special classes of sets: weak compactness. The first class of closed sets which have many nearest points are those with weak compactness properties.

LEMMA 2.1. *Suppose that C is a closed subset of a Banach space E while $x \in E \backslash C$. If some minimizing sequence $\{z_n\}$ in C for x has a weak cluster point z which lies in C then z is a nearest point to x in C.*

Proof. By the weak lower semicontinuity of the norm we have

$$d_C(x) \leqq \|x - z\| \leqq \liminf \|x - z_n\| \leqq d_C(x),$$

so that z is a nearest point to x.

We say that C is *boundedly weakly compact* provided that $C \cap B[0, r]$ is weakly compact for every $r \geqq 0$.

PROPOSITION 2.2. *If C is non-empty and boundedly weakly compact then C is proximinal.*

Proof. Suppose that $x \in E \backslash C$ and let $\{z_n\}$ be a minimizing sequence in C for x. Then $\{z_n\}$ lies $C \cap B[0, r]$ for some positive r, and so has a weak cluster point z belonging to C. By Lemma 2.1 z is a nearest point to x.

As a consequence we have the following.

PROPOSITION 2.3 *Closed non-empty convex subsets of relexive Banach spaces are proximal.*

Proof. $B[0, r]$ is weakly compact and closed convex sets are weakly closed.

3. Special classes of sets: "Swiss cheese" in reflexive spaces. In this section we show that the complements of open convex sets in reflexive Banach spaces are not badly behaved, despite being far from weakly closed. The first lemma should be known but we include a proof.

LEMMA 3.1. *If C is a closed non-empty subset of a Banach space E such that $E \backslash C$ is convex then d_C is concave on $E \backslash C$.*

Proof. Let x and y belong to $E \backslash C$ and take $0 < t < 1$. If $x_t := tx + (1 - t)y$ and v lies in the open unit ball $B(0, 1)$ then $a := x + d_C(x)v$ and $b := y + d_C(y)v$ lie in $E \backslash C$. By convexity $ta + (1 - t)b \in E \backslash C$. That is,

$$x_t + [td_C(x) + (1 - t)d_C(y)]v \in E \backslash C.$$

Since v is arbitrary in $B(0, 1)$,

$$d_C(x_t) \geqq td_C(x) + (1 - t)d_C(y),$$

as required.

THEOREM 3.2. *If C is a closed non-empty subset of a reflexive Banach space E such that $E\backslash C$ is convex then C is almost proximinal.*

Proof. The lemma shows d_C is concave on $E\backslash C$. Since E is an Asplund space [**1**, **6**] the continuous convex function $-d_C$ is Fréchet differentiable on a dense G_δ subset G of $E\backslash C$. We show that each $x \in G$ has a nearest point in C. Let x^* be the Fréchet (sub-)derivative of d_C at $x \in G$ and let $\{z_n\}$ be any minimizing sequence in C for x. By reflexivity, we may take a weakly convergent subsequence with limit z. If z is in C then z is a nearest point to x by Lemma 2.1. Otherwise, by concavity of d_C on $E\backslash C$

$$d_C(z) - d_C(x) \leqq \langle x^*, z - x\rangle \leqq \limsup \langle x^*, z_n - x\rangle = -d_C(x)$$

where the last equality follows from Proposition 1.4. This shows that $d_C(z) \leqq 0$ and that z is in C after all.

COROLLARY (SWISS CHEESE LEMMA) 3.3. *Let $\{U_\alpha : \alpha \in A\}$ be a collection of mutually disjoint open convex subsets of a reflexive Banach space. Then*

$$C := E\backslash \cup \{U_\alpha : \alpha \in A\} \text{ is almost proximinal if it is non-empty.}$$

Proof. Using Theorem 3.2 it suffices to show that if $x \in U_\beta$ has a nearest point y in the closed set $e\backslash U_\beta$ (which contains C) then $y \in C$.

Failing that, $y \in U_\alpha$ with $\alpha \neq \beta$. Since U_α and U_β are disjoint and U_α is open, for small positive t the point $z := tx + (1-t)y$ lies in $U_\alpha\backslash U_\beta$ and so in $E\backslash U_\beta$. But $\|x - z\| < \|x - y\|$, so y was not a nearest point to x in $E\backslash U_\beta$.

REMARKS 3.4. (i) A closed set is convex if and only if d_C is convex, while an open set C is convex if and only if $d_{X\backslash C}$ is concave on C.

(i) By James' theorem [**6**, p. 63], in any non-reflexive space there are closed hyperplanes H so that no point of $E\backslash H$ has a nearest point in H. (See Theorem 5.10.) This shows that Proposition 2.3 characterizes reflexive spaces. Also the Swiss cheese lemma characterizes reflexive spaces, letting U_1 and U_2 be the open half spaces determined by H.

4. Special classes of Banach spaces: finite dimensional spaces. For any closed non-empty subset C of a finite dimensional Banach space E and any point $x \in E\backslash C$ there is a nearest point in C to x (by Proposition 2.2). Furthermore this characterizes finite dimensional Banach spaces.

THEOREM 4.1. (a) *In any infinite dimensional Banach space there is a closed non-empty set C and a point $x \in E\backslash C$ so that x has no nearest point in C.* (b) *Consequently, a Banach space is finite dimensional if and only if every non-empty closed subset is proximinal.*

Proof. (a) Since the space is infinite dimensional we can find a sequence $\{x_n\}$ of norm one elements with $\|x_n - x_m\| > 1/2$ for $n \neq m$ [**12**]. Let

$$C := \{(1 + 2^{-n})x_n : n \in \mathbf{Z}^+\}.$$

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Then C is closed and

$$d_C(0) = 1 < \|0 - (1 + 2^{-n})x_n\| \quad \text{for each } n \in \mathbf{Z}^+.$$

Part (b) now follows.

5. Reflexive Kadec spaces. We say that a Banach space E is *(sequentially) Kadec* provided that for each sequence $\{x_n\}$ in E which converges weakly to x with $\lim_{n\to\infty} \|x_n\| = \|x\|$ we have

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

[Each L_p space $(1 < p < \infty)$ has this property, as does any $l_1(S)$ and any locally uniformly convex Banach space.]

Lau [**13**] showed that nonempty closed subsets in reflexive Kadec spaces are almost proximinal. Konjagin [**14**] showed that in any non Kadec space there is a non-empty bounded closed set C such that points in $E\backslash C$ with nearest points in C are not dense in $E\backslash C$. We will develop both of these results in detail.

Definition 5.1. We modify the sets used by Lau so that it is easier to see they are open. This is helpful since we have access to Theorem 1.3. If C is a closed non-empty subset of a Banach space E and $n \in \mathbf{Z}^+$ we define

$$\begin{aligned} L_n(C) := \{x \in E\backslash C\colon & \text{ for some } \delta > 0 \text{ and some } x^* \in E^* \text{ with } \|x^*\| = 1, \\ & \inf\{\langle x^*, x - z\rangle\colon z \in C \cap B(x, d_C(x) + \delta)\} > (1 - 2^{-n})d_C(x)\}. \end{aligned}$$

Also let

$$L(C) := \cap_n L_n(C)$$

and let

$$\begin{aligned} \Omega(C) := \{x \in E\backslash C\colon & \text{ there exists } x^* \in E^* \text{ with } \|x^*\| = 1, \\ & \text{such that for each } \epsilon > 0 \text{ there is } \delta > 0 \text{ so that} \\ & \inf\{\langle x^*, x - z\rangle\colon z \in C \cap B(x, d_C(x) + \delta)\} > (1 - \epsilon)d_C(x)\}. \end{aligned}$$

LEMMA 5.2. *Each $L_n(C)$ is open in E.*

Proof. Let $x \in L_n(C)$. Then there are $x^* \in E^*$ with $\|x^*\| = 1$ and $\delta > 0$ so that

$$0 < \tau := \inf\{\langle x^*, x - z\rangle\colon z \in C \cap B(x, d_C(x) + \delta)\} - (1 - 2^{-n})d_C(x).$$

Let $\lambda > 0$ be such that $\lambda < \delta/2$ and $\lambda < \tau/2$ and fix y with $\|y - x\| < \lambda$. For $\delta^* := \delta - 2\lambda$ we have

$$C \cap B(x, d_C(x) + \delta) \supseteq A := C \cap B(y, d_C(y) + \delta^*)$$

since d_C is non-expansive. Hence if $z \in A$ then

$$\langle x^*, x - z\rangle \geqq \tau + (1 - 2^{-n})d_C(x),$$

and

$$\begin{aligned}\langle x^*, y - z\rangle &\geqq \tau + (1 - 2^{-n})d_C(y)\\ &\quad + \langle x^*, y - x\rangle + (1 - 2^{-n})[d_C(x) - d_C(y)]\\ &\geqq (1 - 2^{-n})d_C(y) + \tau - 2\|x - y\|\\ &\geqq (1 - 2^{-n})d_C(y) + \tau - 2\lambda.\end{aligned}$$

Thus

$$\inf\{\langle x^*, y - z\rangle : z \in A\} > (1 - 2^{-n})d_C(y)$$

and $B(x, \lambda)\backslash C$ lies in $L_n(C)$, which shows $L_n(C)$ is open.

LEMMA 5.3. *If $x \in E\backslash C$ and $\partial^F d_C(x) \neq \emptyset$ then $x \in \Omega(C)$.*

Proof. Let $x^* \in \partial^F d_C(x)$. By Proposition 1.4, $\|x^*\| = 1$ and for each minimizing sequence $\{z_n\}$ for x we have $\langle x^*, x - z_n\rangle \longrightarrow d_C(x)$. Thus for each $\epsilon > 0$ there is $\delta > 0$ so that whenever

$$z \in C \cap B(x, d_C(x) + \delta)$$

It follows that

$$\langle x^*, x - z\rangle > (1 - \epsilon/2)d_C(x).$$

it follows that

$$\inf\{\langle x^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta)\} > (1 - \epsilon)d_C(x)$$

as required.

Next we have:

LEMMA 5.4. *In any Banach space E the set $\Omega(C)$ always lies in $L(C)$.*

Proof. This follows directly from the definitions of the two sets.

LEMMA 5.5. *If E has an equivalent Fréchet differentiable renorm then $\Omega(C)$ is dense in $E\backslash C$.*

Proof. By Theorem 1.3 the Lipschitz function $d_C(x)$ is Fréchet subdifferentiable on a dense subset of $E\backslash C$. Now Lemma 5.3 completes the proof.

LEMMA 5.6. *When E is reflexive $\Omega(C) = L(C)$.*

Proof. By Lemma 5.4 we need to show that $L(C)$ is contained in $\Omega(C)$. Let $x \in L(C) = \cap_n L_n(C)$. Select x_n^* with $\|x_n^*\| = 1$ and $\delta_n > 0$ so that

$$\inf\{\langle x_n^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta_n)\} > (1 - 2^{-n})d_C(x)$$

and let x^* be any weak* cluster point of $\{x_n^*\}$. Let

$$K_n := \text{weak-cl}[C \cap B(x, d_C(x) + \delta_n)]$$

and observe that each K_n is weakly compact. Thus $K := \cap_n K_n$ is non-empty. For each z in K we have

$$\langle x_n^*, x - z\rangle \geqq (1 - 2^{-n})d_C(x)$$

so that $\langle x^*, x - z\rangle \geqq d_C(x)$. Since $\|x^*\| \leqq 1$ and $\|x - z\| \leqq d_C(x)$ we see that $\|x^*\| = 1$ and

$$\langle x^*, x - z\rangle = d_C(x) = \|x - z\|.$$

Now if $\epsilon > 0$ then K is contained in the weakly open set

$$U(\epsilon) := \{z : \langle x^*, x - z\rangle > (1 - \epsilon/2)d_C(x)\}$$

and as the K_n are nested and weakly compact some K_n lies in $U(\epsilon)$. This implies that

$$\inf\{\langle x^*, x - z\rangle : z \in C \cap B(x, d_C(x) + \delta_n)\} > (1 - \epsilon)d_C(x)$$

and x^* is as required.

We have now completed the proof of the following result.

THEOREM 5.7. *If C is a closed non-empty subset of a reflexive Banach space E then $\Omega(C) = L(C)$ is a dense G_δ subset of $E\backslash C$.*

COROLLARY 5.8. (Lau) *If E is a reflexive Kadec space then for each closed non-empty set C in E the set of points of $E\backslash C$ with nearest points in C contains the dense G_δ subset $\Omega(C)$ of $E\backslash C$.*

Proof. If $x \in \Omega(C)$ and $\{z_n\}$ is a minimizing sequence in C for x then (by extracting a subsequence if necessary) we may assume that weak-$\lim_{n\to\infty} z_n = z$ exists. If x^* is the norm-1 functional guaranteed by the the definition of $\Omega(C)$ then

$$\|x - z\| \geqq \langle x^*, x - z\rangle = \lim\langle x^*, x - z_n\rangle \geqq d_C(x) = \lim \|x - z_n\|.$$

By weak lower semicontinuity of the norm,

$$\lim \|x - z_n\| \geqq \|x - z\|.$$

It follows that $\|x - z\| = \lim \|x - z_n\|$. Since $x - z_n$ converges weakly to $x - z$ we may deduce from the Kadec property that z_n converges in norm to z; which must then lie in C. Thus z is a nearest point in C for x (by Lemma 2.1).

We turn next to describe Konjagin's construction.

LEMMA 5.9. (i) *If E is not a Kadec space one can find $x_n \in E$ and $x^* \in E^*$ such that*

$$\text{(a)} \quad x^*(x_n) = \|x^*\| = 1 = \lim_{n\to\infty} \|x_n\|, \quad \textit{and}$$

$$\text{(b)} \quad \inf_{n \neq m} \|x_n - x_m\| > 0.$$

(ii) *If* (a) *and* (b) *hold and E is reflexive then E is not Kadec.*

Proof. Suppose E is not Kadec. Select y_n converging weakly to y in E with $\|y_n\| = \|y\| = 1$, but with $y_n - y$ not norm convergent to zero. Relabling if needed we may take $\|y_n - y\| > \epsilon$ for all n. Let x^* be a (norm-1) support functional for the unit ball at y and let

$$z_n := y_n/\langle x^*, y_n\rangle$$

(which may be assumed finite). Then z_n tends weakly to y and

$$x^*(z_n) = 1 = \|x^*\| = \lim_{n\to\infty} \|z_n\|,$$

while

$$\liminf_{m\to\infty} \|z_n - y\| > \epsilon \quad \text{for some } \epsilon > 0.$$

Relabling again if needed we may assume $\|z_n - y\| > \epsilon$ for all n. Now for each n, we have

$$\liminf_{n\to\infty} \|z_n - z_m\| \geqq \|z_n - y\| > \epsilon$$

by weak lower semicontinuity of the norm. Thus for each n there is an integer $m(n) > n$ such that

$$\|z_n - z_m\| > \epsilon \quad \text{for } m \geqq m(n).$$

Set $n(1) := 1$ and $n(k+1) := m(n(k))$ for each k. Then

$$\|z_{n(k)} - z_{n(j)}\| > \epsilon \quad \text{if } j > k.$$

Then x^* and $x_k := z_{n(k)}$ satisfy (a) and (b).

Conversely if E is reflexive and (a) and (b) hold then there is a weakly convergent subsequence of $\{x_n\}$ with limit x. Now (a) shows that we have

$$\|x\| \geqq x^*(x) = 1 = \lim_{n\to\infty} \|x_n\|,$$

while (b) now contradicts the Kadec property.

THEOREM 5.10. (Konjagin) *Suppose that E is a Banach space which is not both reflexive and Kadec. Then there is a closed bounded non-empty set C in E and an open non-empty subset U of $E\backslash C$ such that*

(i) *for each $x \in U$ there is no nearest point in C,*

(ii) *d_C is affine on U;*

in particular

(iii) *d_c is Fréchet differentiable on U.*

Proof. Case 1. E is not reflexive. By James' theorem [**11**] there is x^* in E^* with $1 = \|x^*\| > \langle x^*, y\rangle$ for each y in the closed unit ball. Let

$$C := B[0,1] \cap \{x \in E: \langle x^*, x\rangle = 0\}$$

and

$$U := B[0,1/3] \cap \{x \in E: \langle x^*, x\rangle > 0\}.$$

Then $d_C(x) = \langle x^*, x\rangle$ for each $x \in U$. Suppose a point $x \in U$ had a nearest point $z \in C$. Then, since $0 \in C$,

$$d_C(x) = \|x - z\| \leqq \|x - 0\| \leqq 1/3 \quad \text{and} \quad \|z\| \leqq 2/3.$$

In particular z would be a nearest point to x in $\ker x^*$, contradicting the fact that x^* does not attain its norm.

Case 2. E is not Kadec. By (i) of the last lemma we can select $x^* \in E^*$ and $y_n \in E$ so that $\|y_n\| \leqq 2$ and for some $0 < \delta < 1$

$$x^*(y_n) = 1 = \|x^*\| = \lim_{n\to\infty} \|y_n\|, \quad \text{and} \quad \inf_{n\neq m} \|y_n - y_m\| \geqq \delta.$$

Set $z_n := (1 + 2^{-n})y_n$ and define

$$C := \cup_n M_n \quad \text{where} \quad M_n := z_n + (B[0,\delta/3] \cap \{x \in E: \langle x^*, x\rangle = 0\}).$$

Then C is our desired set. First, C is norm closed: if $n \neq m$ and $z \in M_n$, $w \in M_m$ we have

$$\begin{aligned}\|z - w\| &\geqq \|y_n - y_m\| - \|y_m - z_m\| \\ &\quad - \|y_n - z_n\| - \|z_m - z\| - \|z_n - w\| \\ &\geqq \delta - 2^{1-n} - 2^{1-m} - \delta/3 - \delta/3 > \delta/9\end{aligned}$$

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for $m \geqq p$ and $n \geqq p$, p sufficiently large. Since each M_n is closed and since

$$\mathrm{cl}\left(\bigcup_{n\geqq p} M_n\right) = \bigcup_{n\geqq p} M_n,$$

C is norm closed as the finite union of closed sets. Next let $U := B(0,\delta/9)$. For x in U, we will show that $d_C(x) = 1 - \langle x^*, x\rangle$ but x has no nearest point in C. This will conclude the proof. If $x \in U$ set

$$w_n := x + z_n - \langle x^*, x\rangle y_n.$$

Then

$$\|w_n - z_n\| \leqq \|x\| + 2\|x\| < \delta/3$$

while

$$\langle x^*, w_n - z_n\rangle = 0.$$

Thus $w_n \in M_n$ and

$$\begin{aligned} d_C(x) &\leqq \liminf_{n\to\infty} \|w_n - x\| \\ &= \liminf_{n\to\infty} \|z_n - \langle x^*, x\rangle y_n\| \\ &= \liminf_{n\to\infty} [(1+2^{-n}) - \langle x^*, x\rangle]\|y_n\| \\ &= 1 - \langle x^*, x\rangle \end{aligned}$$

since $\langle x^*, x\rangle < 1$. If, however, $z \in C$ then $z \in M_n$ for some n and

$$\langle x^*, z\rangle = \langle x^*, z_n\rangle = (1+2^{-n}) > 1.$$

Thence

$$\|z - x\| = \|x^*\|\,\|z - x\| \geqq \langle x^*, z\rangle - \langle x^*, x\rangle > 1 - \langle x^*, x\rangle$$

and $d_C(x) = 1 - \langle x^*, x\rangle$ but no nearest point exists in C for U.

Let us observe that, in the non-Kadec case, by translation we can arrange for d_C to be linear on U. Also, observe that by taking only the tail of C, C may be supposed locally convex being made up of discrete translates of a fixed convex set. We gather up results as follows.

THEOREM 5.11. (Lau-Konjagin) *In any Banach space E the following conditions are equivalent.*

(A) *E is reflexive and Kadec.*

(B) *For each closed non-empty subset C of E, the set of points in $E \backslash C$ with nearest points in C is dense in $E \backslash C$.*

(C) *For each closed non-empty subset C of E, the set of points in $E \backslash C$ with nearest points in C is generic in $E \backslash C$ (i.e., C is almost proximinal).*

One consequence of Theorem 5.11 is that in any reflexive Kadec space there is a workable "proximal normal formula" [**2**]. It is also possible to generalize Lau's result to some sets in non-reflexive Kadec spaces. (See also [**3**].) Recall that a set C in a Banach space E is *boundedly relatively weakly compact* if $B[0, r] \cap C$ has a weakly compact closure for each positive r. It is equivalent to require that each bounded sequence in C possesses a weakly convergent subsequence with limit in E. (This is not entirely obvious.) Clearly every subset of a reflexive space and every subset of a weakly compact set possess this property. The next result is therefore a complete extension of Theorem 5.7.

THEOREM 5.12. *If C is a closed, boundedly relatively weakly compact, non-empty subset of a Banach space E then $\Omega(C) = L(C)$ is a dense G_δ subset of $E \backslash C$.*

From this exactly as in the proof of Corollary 5.8 we obtain a generalization of Lau's theorem.

COROLLARY 5.13. *Every closed, boundedly relatively weakly compact, non-empty subset of a Kadec Banach space E is almost proximinal. Indeed $\Omega(C)$ is a dense G_δ subset of $E \backslash C$ with nearest points in C.*

To prove Theorem 5.12 we need a replacement for Lemma 5.5 (and the results it depended on). The factorization theorem of Davis, Figiel, Johnson, and Pelczynski provides an avenue. We will use it in the following form.

THEOREM 5.14. [**7**] *Let K be a weakly compact subset of a Banach space Y with $Y =$ closed-span (K). Then there is a reflexive Banach space R and a one to one continuous linear mapping $T: R \longrightarrow Y$ such that $T(B[0, 1]) \supseteq K$.*

Now we can show density of $\Omega(C)$.

LEMMA 5.15. *If C is a closed, boundedly relatively weakly compact, non-empty subset of a Banach space E then $\Omega(C)$ is dense in $E \backslash C$.*

Proof. Let $x_0 \in E \backslash C$ and suppose $d_C(x_0) > \epsilon > 0$. Fix $N > \|x_0\| + d_C(x_0) + 2\epsilon$ and let

$$K := \text{weak-cl}[\{(B[0, N] \cap C\} \cup \{x_0\}].$$

Then K is weakly compact and if Y is the closed span of K, we can apply Theorem 5.14 to obtain a reflexive Banach space R and a one to one continuous linear mapping $T: R \longrightarrow Y$ such that $T(B[0, 1]) \supseteq K$. Define $f_C: R \longrightarrow [0, \infty)$ by $f_C(u) := d_C(Tu)$ for each u in R.

By Theorem 1.3 the Lipschitz function f_C is Fréchet subdifferentiable on a dense subset on R. Thus there is a point of subdifferentiability $v \in R$ with $y := Tv \in B(x_0, \epsilon)$. Note that y is in $E \backslash C$. Let $v^* \in \partial^F f_C(v)$ so that

$$\liminf_{h \to 0} \frac{d_C(T(v+h)) - d_C(Tv) - \langle v^*, h\rangle}{\|h\|} \geqq 0$$

and hence

$$\liminf_{h \to 0} \frac{d_C(y+Th) - d_C(y) - \langle v^*, h\rangle}{\|h\|} \geqq 0.$$

Next for $u \in R$, we have

$$\langle v^*, u\rangle \leqq \|Tu\|$$

on substituting tu for h in the previous expression and using the non-expansiveness of d_C. This shows $v^* = T^*y^*$ for some $y^* \in Y^*$ (by the Hahn-Banach theorem). In particular $\langle y^*, Tu\rangle \leqq \|Tu\|$ for each $u \in R$. Since T has dense range this shows that $\|y^*\| \leqq 1$. We extend y^* to $x^* \in E^*$ with $\|x^*\| \leqq 1$ and observe that

$$\liminf_{t \to 0_+, \|h\| \leqq 1+\|v\|} \frac{d_C(y+tTh) - d_C(y) - t\langle x^*, Th\rangle}{t} \geqq 0$$

so that

$$\liminf_{t \to 0_+, k \in K} \frac{d_C(y+t(k-y)) - d_C(y) - t\langle x^*, k-y\rangle}{t} \geqq 0.$$

(Since $T(B[0,1]-v) \supseteq K - y$.) Suppose now that $\{z_n\}$ is a minimizing sequence in C for y. By the construction of N, $z_n \in K$ for large n. Also we may suppose that

$$\|y - z_n\| < d_C(y) + 4^{-n}.$$

Then

$$\begin{aligned} 0 &\leqq \liminf_{n \to \infty} [d_C(y + 2^{-n}(z_n - y)) - d_C(y)]2^n - \langle x^*, z_n - y\rangle \\ &\leqq \liminf_{n \to \infty} [\|y + 2^{-n}(z_n - y) - z_n\| - \|y - z_n\| - 4^{-n}]2^n - \langle x^*, z_n - y\rangle \\ &= \liminf_{n \to \infty} [-\|z_n - y\| - \langle x^*, z_n - y\rangle]. \end{aligned}$$

Thus

$$\liminf_{n \to \infty} \langle x^*, y - z_n\rangle \geqq \lim_{n \to \infty} \|z_n - y\| = d_C(y),$$

which again shows $\|x^*\| \geqq 1$. Thus

$$\|x^*\| = 1 \quad \text{and} \quad \lim_{n\to\infty} \langle x^*, y - z_n \rangle = d_C(y).$$

As in Lemma 5.3, $y \in \Omega(C)$. Since $\|y - x_0\| < \epsilon$ this establishes our density assertion.

Proof. (of Theorem 5.12) By Lemmas 5.2 and 5.4, $\Omega(C)$ is always contained in the G_δ set $L(C)$. We note that the proof of Lemma 5.6 holds unchanged for C boundedly relatively weakly compact. Thus $\Omega(C) = L(C)$ is a G_δ set in $E\backslash C$. Finally $\Omega(C)$ is dense in $E\backslash C$ by the last lemma.

6. Uniqueness of nearest points. Having constructed the set $\Omega(C)$ we can also use it to prove uniqueness results. The first is a reasonable new partial answer to Stechkin's question whether in every strictly convex Banach space the nearest points to a closed set are generically not multiple. (See also [**3**] and [**10**].)

THEOREM 6.1. *Let E be a strictly convex Banach space and let C be a non-empty, boundedly relatively weakly compact, closed subset of E. Then each point of the dense G_δ subset $\Omega(C)$ of $E\backslash C$ has at most one nearest point.*

Proof. If $x \in \Omega(C)$ and $y, z \in C$ with $\|x - y\| = \|x - z\| = d_C(x) > 0$ then the functional x^* guaranteed by the definition of $\Omega(C)$ has $\|x^*\| = 1$ and

$$x^*(x - y) = x^*(x - z) = d_C(x)$$

and

$$\begin{aligned}\|(x - y) + (x - z)\| &\geqq x^*(x - y) + x^*(x - z)\\ &= 2d_C(x) = \|x - y\| + \|x - z\|.\end{aligned}$$

By strict convexity $y = z$ as required. By Theorem 5.12, $\Omega(C)$ is a dense G_δ subset.

Definition 6.2. A subset C of a Banach space E is *almost Chebyshev* provided there is a generic subset of $E\backslash C$ with unique nearest points in C.

COROLLARY 6.3. *Let E be a Kadec strictly convex Banach space and let C be a non-empty, boundedly relatively weakly compact, closed subset of E. Then each point of the dense G_δ subset $\Omega(C)$ of $E\backslash C$ has exactly one nearest point, and C is almost Chebyshev.*

Proof. Combine Corollary 5.13 and Theorem 6.1.

Definition 6.4. A Banach space E is *strongly convex* provided it is reflexive, Kadec, and strictly convex.

COROLLARY 6.5. *Every closed subset of a strongly convex Banach space is almost Chebyshev.*

It is of interest to note that Corollary 6.5 can be turned into various characterizations of strongly convex spaces; many due to Konjagin.

THEOREM 6.6. *Let E be a Banach space. The following statements are equivalent.*

(1) *E is strongly convex.*

(2) *The norm on E^* is Fréchet differentiable.*

(3) *Every closed non-empty subset of E is almost Chebyshev.*

(4) *For every closed non-empty subset C of E there is a dense set of points in $E\backslash C$ possessing unique nearest points.*

Proof. (1) $\Rightarrow$ (3) by Corollary 6.5, while (3) $\Rightarrow$ (4) is immediate.

(4) $\Rightarrow$ (1). If E is not strongly convex then either E is not both reflexive and Kadec, or E is not strictly convex. In the first case Theorem 5.11 applies. In the second case, let $[a, b]$ be a closed non-trivial interval in the unit sphere of E. Take $x^* \in E^*$ with $\|x^*\| = 1$ and $\langle x^*, (a+b)\rangle = 2$, so that $\langle x^*, a\rangle = \langle x^*, b\rangle = 1$. Then for $C := \ker x^*$ and $x \in E\backslash C$ there are always multiple nearest points. [Indeed y is a nearest point to x if and only if $\langle x^*, y\rangle = 0$ and $\|x-y\| = |\langle x^*, x\rangle| = d_C(x)$, which holds for $x - \langle x^*, x\rangle c$ whenever $c \in [a, b]$.]

(1) $\Rightarrow$ (2). Since E is reflexive and strictly convex, E^* is smooth. Let x_n^* and $x^* \in E^*\backslash\{0\}$ with $x_n^* \longrightarrow x^*$. Then the corresponding Gateaux derivatives x_n and $x \in E$ of the norm on E^* satisfy $x_n \longrightarrow x$ weakly. $\|x_n\| = \|x\|$ and E is Kadec $x_n \longrightarrow x$ in norm. Thus the norm on E^* is Fréchet differentiable at x^*.

(2) $\Rightarrow$ (1). Here we use the fact that the norm on a Banach space X is Fréchet differentiable at $x \in X$ with derivative x^* if and only if x strongly exposes the unit ball of X^* at x^* [**6**]. (See Definition 8.1.) Now suppose the norm on E^* is Fréchet differentiable. Let F be a norm one support functional so $\langle F, x^*\rangle = \|x^*\| = 1$ for some $x^* \in X^*$. By smoothness F is the Fréchet derivative of the norm at x^*. But then x^* strongly exposes the unit ball of E^{**} at F. Let $\{x_\alpha\}$ be a net converging weak* to F with $x_\alpha \in E$, $\|x_\alpha\| = 1$. Thus

$$\langle x^*, x_\alpha\rangle \longrightarrow \langle F, x^*\rangle = 1 = \|x^*\|,$$

and in consequence x_α converges to F in norm. Thus F lies in E. The Bishop-Phelps theorem shows that the norm one support functionals are dense in the unit sphere. Hence E is reflexive.

Next the smoothness of E^* implies that E is strictly convex. Finally, to settle the Kadec property, let x_n and $x \in E$ satisfy $\|x_n\| = \|x\| = 1$ while $n_n \longrightarrow x$ weakly. There is $x^* \in E^*$, $\|x^*\| = \|x\| = 1 = \langle x^*, x\rangle$. Again x must be the Fréchet derivative of the norm at x^*. But then x^* strongly exposes the unit ball of E at x. Since $\langle x^*, x_n\rangle \longrightarrow \langle x^*, x\rangle = 1$, this forces $x_n \longrightarrow x$ in norm as required.

This completes the proof that (2) implies (1) and so the theorem.

Remark 6.7. It is clear that every reflexive locally uniformly convex space is strongly convex. The converse fails since the following renorm $l_2(\mathbf{Z}^+)$ is strongly convex but not locally uniformly convex, as observed by Mark Smith [**16**]. Let $\|\cdot\|$ be the original norm on l_2. Define $|||\cdot|||$ by

$$|||x|||^2 := \|Tx\|^2 + (|x_1| + \|Px\|)^2$$

where

$$Tx := (0, x_2/2, x_3/3, \ldots, x_n/n, \ldots) \quad \text{and} \quad Px := (0, x_2, x_3, \ldots, x_n, \ldots).$$

It is easy to verify that $|||\ |||$ is strongly convex. It is not locally uniformly convex since

$$|||e_n||| \longrightarrow |||e_1||| = 1 \quad \text{and} \quad |||e_1 + e_n||| \longrightarrow 2,$$

but $|||e_1 - e_n||| \longrightarrow 2$ not zero.

7. Spaces where nearest points are dense. In this section we show that there are reflexive Banach spaces E which do not have the Kadec property but such that, nevertheless, for each closed non-empty subset C of E the set of nearest points in C to points of $E\backslash C$ is dense in the boundary of C. It is an open question as to whether all reflexive Banach spaces have the latter property.

THEOREM 7.1. *Let X be a reflexive Kadec space, Y a finite dimensional normed space and $|||\cdot|||$ a Riesz (lattice) norm on R^2. Let $E := X \oplus Y$ in the norm*

$$\|(x, y)\| := |||(\|x\|, \|y\|)|||.$$

For each closed non-empty subset C of E the set of nearest points in C to points not in C is dense in the boundary of C.

We will need the following lemma.

LEMMA 7.2. *Suppose E, X, Y, and C are as above. Suppose d_C is Fréchet differentiable at $u \in E\backslash C$ but u has no nearest point in C. Then*

$$\{0\} \oplus Y^* \supseteq \partial^F d_C(u).$$

Proof. Let u be as hypothesised. If $(x^*, y^*) \in \partial^F d_C(u)$ then, by Theorem 1.4, $|||(x^*, y^*)||| = 1$ and for every minimizing sequence $z_n := (x_n, y_n)$ in C for $u = (x, y)$ we have

$$\langle (x^*, y^*), (x_n - x, y_n - y) \rangle \longrightarrow -d_C(u).$$

Thus $|||(\|x^*\|, \|y^*\|)|||^* = 1$ where $||| \cdot |||^*$ is the dual norm on R^2, and

$$\begin{aligned} d_C(u) &= \lim_{n\to\infty} |||(\|x_n - x\|, \|y_n - y\|)||| \\ &= \lim_{n\to\infty} (\langle x^*, x - x_n\rangle + \langle y^*, y - y_n\rangle). \end{aligned}$$

Extracting a subsequence we may and do assume that the sequences $\{\langle x^*, x - x_n\rangle\}$, $\{\|x_n - x\|\}$, and $\{y_n\}$ all converge. Then

$$\begin{aligned} &\lim_{n\to\infty} \langle x^*, x - x_n\rangle + \lim_{n\to\infty} \langle y^*, y - y_n\rangle = d_C(u) \\ &= |||(\lim_{n\to\infty} \|x_n - x\|, \lim_{n\to\infty} \|y_n - y\|)||| \\ &= |||(\|x^*\|, \|y^*\|)|||^* \, ||| \left(\lim_{n\to\infty} \|x_n - x\|, \lim_{n\to\infty} \|y_n - y\|\right) ||| \\ &\geqq \|x^*\| \lim_{n\to\infty} \|x_n - x\| + \|y^*\| \lim_{n\to\infty} \|y_n - y\| \end{aligned}$$

so that

$$\lim_{n\to\infty} \langle x^*, x - x_n\rangle = \|x^*\| \lim_{n\to\infty} \|x_n - x\|$$

and

$$\lim_{n\to\infty} \langle y^*, y - y_n\rangle = \|y^*\| \lim_{n\to\infty} \|y_n - y\|.$$

If $x^* \neq 0$ the Kadec property and reflexivity determine a norm convergent subsequence of $\{x_n\}$ with $\lim x^\#$. Since $\{y_n\}$ converges to some $y^\#$, $(x^\#, y^\#)$ lies in C and is a nearest point to u. This contradiction shows $x^* = 0$ and the conclusion.

Proof. (of Theorem 7.1) Suppose $z_0 := (x_0, y_0)$ is in the boundary of C and that $\epsilon > 0$ is such that $U := B(z_0, \epsilon)\backslash C$ contains no points with nearest points in C; this will happen if z_0 is a boundary point not in the closure of nearest points. By Lemma 7.2 we have $\{0\} \oplus Y^* \supseteq \partial^F d_C(u)$ for every u in U (of course $\partial^F d_C(u) = \phi$ is possible). In addition we have by **[3]**, or **[15]** that

$$\{0\} \oplus Y^* \supseteq \text{weak*cl-conv}\{z^* : z^* \in \partial^F d_C(u), u \in U\} \supseteq \partial d_C(u).$$

Now let (x_2, y) and (x_1, y) lie in $B(z_0, \epsilon)$ with

$$u_t := (tx_1 + (1-t)x_2, y) \in U \quad \text{for all } 0 < t < 1.$$

By Lebourg's Mean-value theorem **[5]**

$$d_C(x_1, y) - d_C(x_2, y) \in \langle \partial d_C(u_t), (x_1 - x_2, 0)\rangle.$$

But $\partial d_C(u_t)$ annihilates $(x_1 - x_2, 0)$ so that $d_C(x_1, y) = d_C(x_2, y)$.

After some consideration of the case where $(x_1, y) \in C$, it follows that on $B(z_0, \epsilon)$ the distance $d_C(x, y)$ depends only on y. In particular, if $(x, y) \in B(z_0, \epsilon)$ then $d_C(x, y) = d_C(x_0, y)$ where $z_0 = (x_0, y_0)$. Now let $(x, y) \in B(z_0, \epsilon/2)$ have minimizing sequence $\{(x_n, y_n)\}$ from C. Then $z_0 \in C$ so we can assume

$$\|(x_n, y_n) - (x, y)\| \leqq \|z_0 - (x, y)\| < \epsilon/2$$

and $(x_n, y_n) \in B(z_0, \epsilon)$. Thus $0 = d_C(x_n, y_n) = d_C(x_0, y_n)$ so $(x_0, y_n) \in C$ and

$$\begin{aligned} d_C(x_0, y) = d_C(x, y) &= \lim_{n \to \infty} \|(x_n, y_n) - (x, y)\| \\ &\geqq \limsup_{n \to \infty} \|y_n - y\| \geqq \|y^\# - y\| \end{aligned}$$

where $y^\#$ is any cluster point of $\{y_n\}$. Since $(x_0, y^\#) \in C$ we have

$$d_C(x_0, y) \leqq \|(x_0, y^\#) - (x_0, y^\#)\| = \|y^\# - y\| \leqq d_C(x_0, y)$$

and $(x_0, y^\#)$ is a nearest point to (x_0, y), contrary to our assumption. Hence nearest points are dense in the boundary of C.

REMARKS 7.3. (a) Choosing

$$|||(s, t)||| := \max\{|s|, |t|\}, Y := R,$$

and any infinite dimensional reflexive Kadec space for X, we obtain a non-Kadec reflexive space E to which Theorem 7.2 applies. If, specifically, $X := l_2(\mathbf{Z}^+)$ it is easy to construct an explicit example of the set promised by the non-Kadec construction of Theorem 5.10.

(b) Choosing $X := l_2(\mathbf{Z}^+)$, $Y := R$ and $|||\cdot|||$ such that the unit ball is

$$B_{|||\cdot|||}[0, 1] := \{(s, t) : |t| \leqq 1, |s| \leqq 1 + (1 - t^2)^{1/2}\}$$

we obtain a uniformly smooth non-Kadec space to which Theorem 7.2 applies.

8. Spaces with the Radon-Nikodym property. We refer the reader to [**4**] for the vast amount known about spaces with the Radon-Nikodym property. All we need here is one definition and one characterization.

Definition 8.1. A functional $x^* \in E^*$ *strongly exposes* a subset C of E at $x \in \operatorname{cl} C$ if $\sup_{z \in C} \langle x^*, z \rangle = \langle x^*, x \rangle$ and

$$\lim_{\alpha \to 0^+} \operatorname{diam}\{y \in C : \langle x^*, y \rangle > \sup_{z \in C} \langle x^*; z \rangle - \alpha\} = 0.$$

A functional $x^* \in E^*$ *strongly exposes* a set C if it strongly exposes some point of the closure of C. This is equivalent to saying that

$$\lim_{\alpha \to 0^+} \operatorname{diam}\{y \in C : \langle x^*, y \rangle > \sup_{z \in C} \langle x^*, z \rangle - \alpha\} = 0.$$

THEOREM 8.2. *A Banach space E has the Radon-Nikodym property* (RNP) *if and only if for every bounded non-empty subset C of E the set $SE(C)$ of strongly exposing functionals for C is dense in E^*. In particular, reflexive spaces and duals of Asplund spaces have the RNP.*

For unbounded subsets of non-reflexive subspaces there are no general results on nearest points, as shown by the example of the closed hyperplane determined by a non-norm attaining functional (Remark 3.4). For bounded closed sets in spaces with the RNP we have a positive result.

THEOREM 8.3. *Let E be a Banach space with the Radon-Nikodym property and let C be a closed bounded non-empty subset of E. Then C is contained in the closed convex hull of its nearest points to points in $E \backslash C$. In particular C possesses nearest points.*

Proof. If $x \in C$ does not lie in the convex hull of its nearest points we may separate by $x^* \in E^*$ to obtain

$$\langle x^*, x\rangle > \sup\{\langle x^*, y\rangle : y \text{ is a nearest point in } C\}.$$

Let $K := C + B[0,1]$ and by Theorem 8.2 find $y^* \in SE(K)$ with $\|y^*\| = 1$ such that

$$\langle y^*, x\rangle > \sup\{\langle y^*, y\rangle : y \text{ is a nearest point in } C\}.$$

Then, a completeness argument shows that y^* actually both strongly exposes C at $z \in C$ and strongly exposes $B[0,1]$ at u with $\|u\| \leqq 1$. Hence we have

$$\langle y^*, z\rangle = \sup\{\langle y^*, y\rangle : y \in C\} \quad \text{and} \quad \langle y^*, u\rangle = \|y^*\| = 1.$$

Now $z + u$ has a nearest point $z \in C$. Indeed, for $c \in C$

$$\begin{aligned}\|(z+u) - c\| &\geqq \langle y^*, z + u - c\rangle \geqq \langle y^*, u\rangle \\ &= 1 = \|u\| = \|(z+u) - u\|.\end{aligned}$$

However this contradicts

$$\langle y^*, z\rangle \geqq \langle y^*, x\rangle > \sup\{\langle y^*, y\rangle : y \text{ is a nearest point in } C\}.$$

For convex sets we state a deeper result of Edelstein [**8**].

THEOREM 8.4. *Let E be a Banach space with the Radon-Nikodym property and let C be a non-empty closed, convex, bounded subset of E. Then the points in $E \backslash C$ which have nearest points in C are weakly dense in $E \backslash C$.*

REMARKS 8.5. (i) We observe that outside of a space with the Radon-Nikodym property, Theorem 8.4 can go badly wrong. An example of Edelstein and Thompson [**9**] shows that in $c_0(\mathbf{Z}^+)$ with the supremum norm there is an equivalent ball

B such B has no nearest points in $\| \|_\infty$. The sets C and B are called *companion (anti-proximinal) bodies*. The only known examples are in c_0 and its isomorphs. Does the non-existence of companion bodies characterize *RNP* spaces?

(ii) Let E be a Banach space with the Radon-Nikodym property and let C be an arbitrary non-empty closed bounded subset of E. Are the points in $E\backslash C$ which have nearest points in C weakly dense in $E\backslash C$?

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Dalhousie University,
Halifax, Nova Scotia;
University of Auckland,
Auckland, New Zealand