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On the Classification of Rational Quantum Tori and the Structure of Their Automorphism Groups

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Abstract. An n-dimensional quantum torus is a twisted group algebra of the group $\mathbb{Z}^n$. It is called rational if all invertible commutators are roots of unity. In the present note we describe a normal form for rational n-dimensional quantum tori over any field. Moreover, we show that for $n = 2$ the natural exact sequence describing the automorphism group of the quantum torus splits over any field.

1 Introduction

Let $\mathbb{K}$ be a field and Γ an abelian group. A Γ-*quantum torus* is a Γ-graded $\mathbb{K}$-algebra $A = \bigoplus_{\gamma\in\Gamma} A_\gamma$, for which all grading spaces are one-dimensional and all non-zero elements in these spaces are invertible. For any basis $(\delta_\gamma)_{\gamma\in\Gamma}$ of such an algebra with $\delta_\gamma \in A_\gamma$, we have $\delta_\gamma\delta_{\gamma'} = f(\gamma,\gamma')\delta_{\gamma+\gamma'}$, where $f\colon \Gamma\times\Gamma \to \mathbb{K}^\times$ is a group cocycle. In this sense Γ-quantum tori are the same as *twisted group algebras* in the terminology of [20]. Quantum tori arise very naturally in non-commutative geometry as non-commutative algebras which are still very close to commutative ones, [12] and they also show up in topology [4, §3].

For $\Gamma = \mathbb{Z}^n$, we also speak of n-dimensional quantum tori, also called *skew Laurent polynomial rings* if the image of f lies in a cyclic subgroup of $\mathbb{K}^\times$ [8]. Important special examples arise for $n = 2$ and $f(\gamma,\gamma') = q^{\gamma_1\gamma_2'}$, which leads to an algebra A_q with generators $u_1 = \delta_{(1,0)}$ and $u_2 = \delta_{(0,1)}$, satisfying $u_1u_2 = qu_2u_1$, and their inverses. Finite-dimensional quantum tori and their Jordan analogs also play a key role in the structure theory of infinite-dimensional Lie algebras because they are the natural coordinate structures of extended affine Lie algebras [2, 3].

The first problem we address in this note is the normal form of the finite-dimensional *rational quantum tori*, *i.e.,* quantum tori with grading group $\Gamma = \mathbb{Z}^n$, for which f takes values in the torsion group of $\mathbb{K}^\times$. Let $P \subseteq \mathbb{K}^\times$ be a subset containing a single representative for each finite element order arising in the multiplicative group $\mathbb{K}^\times$. We then show in Section 4 that any rational n-dimensional quantum torus A is isomorphic to a tensor product

$$A \cong A_{q_1} \otimes \cdots \otimes A_{q_{s-1}} \otimes A_{q_s^m} \otimes \mathbb{K}[\mathbb{Z}^{n-2s}], \tag{1.1}$$

where $q_1,\ldots,q_s \in P$ satisfy $1 < \operatorname{ord}(q_s) | \operatorname{ord}(q_{s-1}) | \cdots | \operatorname{ord}(q_1)$ and $\operatorname{ord}(q_s^m) = \operatorname{ord}(q_s)$ (with $m = 1$ if $2s < n$). The existence of such a decomposition is not new.

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Under the assumption that the field $\mathbb{K}$ is algebraically closed of characteristic zero, (1.1) can be found in [1], and a version for skew Laurent polynomial rings is stated in [8, Remark in 7.2]. Our main new point in Section 4 is criteria for two such rational quantum tori as in (1.1) to be isomorphic.

For any $\mathbb{Z}^n$-quantum torus A, its group of automorphisms is an abelian extension described by a short exact sequence

$$\mathbf{1} \to \mathrm{Hom}(\mathbb{Z}^n, \mathbb{K}^\times) \to \mathrm{Aut}(A) \to \mathrm{Aut}(\mathbb{Z}^n, \lambda) \to \mathbf{1}, \tag{1.2}$$

where $\lambda \colon \mathbb{Z}^n \times \mathbb{Z}^n \to \mathbb{K}^\times, (\gamma, \gamma') \mapsto \delta_\gamma \delta_{\gamma'} \delta_\gamma^{-1} \delta_{\gamma'}^{-1}$ is the alternating biadditive map determined by the commutator map of the unit group $A^\times$, and

$$\mathrm{Aut}(\mathbb{Z}^n, \lambda) \subseteq \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\mathbb{Z}^n)$$

is the subgroup preserving λ. The second main result of this note is that for $n = 2$ the sequence (1.2) always splits. In this case $A \cong A_q$ for some $q \in \mathbb{K}^\times$, and $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \mathrm{GL}_2(\mathbb{Z})$ if $q^2 = 1$ and $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \mathrm{SL}_2(\mathbb{Z})$ otherwise. The statement of this result (in case q is not a root of unity) can also be found in [17, Theorem 1.5], but without any argument for the splitting of the exact sequence (1.2). In the context of the irrational rotation algebra, a C^*-completion of A_q for $q \in \mathbb{C}^\times$ of infinite order with absolute value 1, the corresponding result is due to Brenken [5]. According to [20, p. 429], the determination of the automorphism groups of general quantum tori seems to be a hopeless problem, but we think that our splitting result stimulates some hope that more explicit descriptions might be possible if the range of the commutator map is sufficiently well behaved.

Rational quantum tori also show up naturally in the theory of forms of algebras over Laurent polynomial rings because they are finite rank modules over their centers, which are algebras of Laurent polynomials. In this context one must distinguish among the automorphism group over $\mathbb{K}$, the considerably smaller automorphism group over the center, and the group of graded automorphisms. For more details and a development of the general theory we refer to [11, Example 4.11].

1.1 Notation

Throughout this paper $\mathbb{K}$ denotes an arbitrary field. We write $A^\times$ for the unit group of a unital $\mathbb{K}$-algebra A.

Let Γ and Z be abelian groups, both written additively. A function $f \colon \Gamma \times \Gamma \to Z$ is called a 2-*cocycle* if $f(\gamma, \gamma') + f(\gamma + \gamma', \gamma'') = f(\gamma, \gamma' + \gamma'') + f(\gamma', \gamma'')$ holds for $\gamma, \gamma', \gamma'' \in \Gamma$. The set of all 2-cocycles is an additive group $Z^2(\Gamma, Z)$ with respect to pointwise addition. The functions of the form $h(\gamma) - h(\gamma + \gamma') + h(\gamma')$ are called *coboundaries*. They form a subgroup $B^2(\Gamma, Z) \subseteq Z^2(\Gamma, Z)$, and the quotient group $H^2(\Gamma, Z) := Z^2(\Gamma, Z)/B^2(\Gamma, Z)$ is called the second cohomology group of Γ with values in Z. It classifies central extensions of Γ by Z up to equivalence. Here we assign to $f \in Z^2(\Gamma, Z)$ the central extension $Z \times_f \Gamma$, which is the set $Z \times \Gamma$, endowed with the group multiplication

$$(z, \gamma)(z', \gamma') = (z + z' + f(\gamma, \gamma'), \gamma + \gamma') \quad z, z' \in Z,\ \gamma, \gamma' \in \Gamma. \tag{1.3}$$

We also write $\mathrm{Ext}(\Gamma, Z) \cong H^2(\Gamma, Z)$ for the group of all central extensions of Γ by Z, and $\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z)$ for the subgroup corresponding to the abelian extensions of the group Γ by Z, which correspond to symmetric 2-cocycles.

We call a biadditive map $\Gamma \times \Gamma \to Z$ vanishing on the diagonal *alternating*, and denote the set of these maps by $\mathrm{Alt}^2(\Gamma, Z)$. A function $q\colon \Gamma \to Z$ is called a *quadratic form* if the map

$$\beta_q\colon \Gamma \times \Gamma \to Z, \quad (\gamma, \gamma') \mapsto q(\gamma + \gamma') - q(\gamma) - q(\gamma')$$

is biadditive. Note that we do not require here that $q(n\gamma) = n^2 q(\gamma)$ holds for $n \in \mathbb{Z}$ and $\gamma \in \Gamma$.

For $n \in \mathbb{N} := \{1, 2, 3, \dots\}$, we write $Z[n] := \{z \in Z : nz = 0\}$ for the n-torsion subgroup of Z. We also write $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.

2 The Correspondence between Quantum Tori and Central Extensions

Definition 2.1 Let Γ be an abelian group. A unital associative $\mathbb{K}$-algebra A is said to be a Γ-*quantum torus* if it is Γ-graded, $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$, with one-dimensional grading spaces A_γ, and each non-zero element of A_γ is invertible.[1]

For $\Gamma \cong \mathbb{Z}^d$ we call a Γ-quantum torus also a *d-dimensional quantum torus*.

Remark 2.2 In each Γ-quantum torus A the set $A_h^\times := \bigcup_{\gamma \in \Gamma} \mathbb{K}^\times \delta_\gamma$ of homogeneous units (called *trivial units* in [20]) is a subgroup containing $\mathbb{K}^\times \mathbf{1} \cong \mathbb{K}^\times$ in its center. We thus obtain a central extension

$$\mathbf{1} \to \mathbb{K}^\times \to A_h^\times \to \Gamma \to \mathbf{1}$$

of abelian groups.

It is instructive to see how this can be made more explicit in terms of cocycles, and to show in particular that each central extension of Γ by $\mathbb{K}^\times$ arises as $A_h^\times$ for some Γ-quantum torus A. Let A be a Γ-quantum torus and pick non-zero elements $\delta_\gamma \in A_\gamma$, so that $(\delta_\gamma)_{\gamma \in \Gamma}$ is a basis of A. Then each δ_γ is an invertible element of A, so that we get

$$\delta_\gamma \delta_{\gamma'} = f(\gamma, \gamma') \delta_{\gamma + \gamma'} \quad \text{for } \gamma, \gamma' \in \Gamma, \tag{2.1}$$

where $f \in Z^2(\Gamma, \mathbb{K}^\times)$ is a 2-cocycle for which $A_h^\times \cong \mathbb{K}^\times \times_f \Gamma$ (1.3).

Conversely, starting with a cocycle $f \in Z^2(\Gamma, \mathbb{K}^\times)$, we define a multiplication on the vector space $A := \bigoplus_{\gamma \in \Gamma} \mathbb{K}\delta_\gamma$ with basis $(\delta_\gamma)_{\gamma \in \Gamma}$ by $\delta_\gamma \delta_{\gamma'} := f(\gamma, \gamma') \delta_{\gamma + \gamma'}$. Then the cocycle property implies that we get a unital associative algebra, and it is clear from the construction that it is a Γ-quantum torus.

Definition 2.3 There are two natural equivalence relations between quantum tori. The finest one is the notion of graded equivalence: two Γ-quantum tori A and B are

[1] In [20], these algebras are called *twisted group algebras.*

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called *graded equivalent* if there is an algebra isomorphism $\varphi\colon A \to B$ with $\varphi(A_\gamma) = B_\gamma$ for all $\gamma \in \Gamma$.

A slightly weaker notion is graded isomorphy: two Γ-quantum tori A and B are called *graded isomorphic* if there is an isomorphism $\varphi\colon A \to B$ and an automorphism $\varphi_\Gamma \in \operatorname{Aut}(\Gamma)$ with $\varphi(A_\gamma) = B_{\varphi_\Gamma(\gamma)}$ for all $\gamma \in \Gamma$.

The following theorem reduces the corresponding classification problems to purely group theoretic ones.

Theorem 2.4 *The graded equivalence classes of Γ-quantum tori are in one-to-one correspondence with the central extensions of the group Γ by the multiplicative group $\mathbb{K}^\times$, hence parametrized by the cohomology group $H^2(\Gamma, \mathbb{K}^\times)$.*

The graded isomorphy classes of Γ-quantum tori are parametrized by the set

$$H^2(\Gamma, \mathbb{K}^\times)/\operatorname{Aut}(\Gamma)$$

of orbits of the group $\operatorname{Aut}(\Gamma)$ in the cohomology group $H^2(\Gamma, \mathbb{K}^\times)$, where the action is given on the level of cocycles by $\psi.f := (\psi^{-1})^* f = f \circ (\psi^{-1} \times \psi^{-1})$.

Proof If $\varphi\colon A \to B$ is a graded equivalence of Γ-quantum tori, then the restriction to the group $A_h^\times$ of homogeneous units leads to the commutative diagram

$$\begin{tikzcd}
\mathbb{K}^\times \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{K}^\times}"] & A_h^\times \arrow[r] \arrow[d, "\varphi"] & \Gamma \arrow[d, "\mathrm{id}_\Gamma"] \\
\mathbb{K}^\times \arrow[r] & B_h^\times \arrow[r] & \Gamma
\end{tikzcd}$$

This means that the central extensions $A_h^\times$ and $B_h^\times$ of Γ by $\mathbb{K}^\times$ are equivalent. If, conversely, these extensions are equivalent, then any equivalence $\varphi\colon A_h^\times \to B_h^\times$ extends linearly to a graded equivalence $A \to B$. Now the observation from Remark 2.2 implies that the graded equivalence classes of Γ-quantum tori are parametrized by the cohomology group $H^2(\Gamma, \mathbb{K}^\times) \cong \operatorname{Ext}(\Gamma, \mathbb{K}^\times)$.

If $\varphi\colon A \to B$ is a graded isomorphism of Γ-quantum tori, then the diagram

$$\begin{tikzcd}
\mathbb{K}^\times \arrow[r] \arrow[d, "\mathrm{id}_{\mathbb{K}^\times}"] & A_h^\times \arrow[r] \arrow[d, "\varphi"] & \Gamma \arrow[d, "\varphi_\Gamma"] \\
\mathbb{K}^\times \arrow[r] & B_h^\times \arrow[r] & \Gamma
\end{tikzcd}$$

commutes, which means that the corresponding central extensions $A_h^\times$ and $B_h^\times$ are contained in the same orbit of $\operatorname{Aut}(\Gamma)$ on $\operatorname{Ext}(\Gamma, \mathbb{K}^\times) \cong H^2(\Gamma, \mathbb{K}^\times)$ (we leave the easy verification to the reader). Conversely, any isomorphism $\varphi\colon A_h^\times \to B_h^\times$ of central extensions extends linearly to an isomorphism of algebras $A \to B$. ∎

3 Central Extensions of Abelian Groups

In this section Γ and Z are abelian groups, written additively. We shall derive some general facts on the set of equivalence classes $\mathrm{Ext}(\Gamma, Z) \cong H^2(\Gamma, Z)$ of central extensions of Γ by Z. In Sections 4 and 5 below we shall apply these to the special case $Z = \mathbb{K}^\times$ for a field $\mathbb{K}$.

Remark 3.1 Let $Z \hookrightarrow \widehat{\Gamma} \xrightarrow{q} \Gamma$ be a central extension of the abelian group Γ by the abelian group Z and

$$\widehat{\lambda}\colon \widehat{\Gamma} \times \widehat{\Gamma} \to Z, \quad (x, y) \mapsto [x, y] := xyx^{-1}y^{-1},$$

the commutator map of $\widehat{\Gamma}$. Its values lie in Z because Γ is abelian. Obviously, $\widehat{\lambda}(x, x) = 0$, and $\widehat{\lambda}$ is an alternating biadditive map [20, p. 418]. Moreover, the commutator map is constant on the fibers of the map q, hence factors through a biadditive map $\lambda \in \mathrm{Alt}^2(\Gamma, Z)$.

Next we write $\widehat{\Gamma}$ as $Z \times_f \Gamma$ with a 2-cocycle $f \in Z^2(\Gamma, Z)$. For the map $\sigma\colon \Gamma \to \widehat{\Gamma}$, $\gamma \mapsto (0, \gamma)$ we then have $\sigma(\gamma)\sigma(\gamma') = \sigma(\gamma + \gamma')f(\gamma, \gamma')$, which leads to

$$\begin{aligned}\lambda(\gamma, \gamma') = \widehat{\lambda}\big(\sigma(\gamma), \sigma(\gamma')\big) &= \sigma(\gamma)\sigma(\gamma')\big(\sigma(\gamma')\sigma(\gamma)\big)^{-1}\\ &= \sigma(\gamma + \gamma')f(\gamma, \gamma')\big(\sigma(\gamma + \gamma')f(\gamma', \gamma)\big)^{-1}\\ &= f(\gamma, \gamma')f(\gamma', \gamma)^{-1} = f(\gamma, \gamma') - f(\gamma', \gamma).\end{aligned}$$

Therefore the map $\lambda_f \in \mathrm{Alt}^2(\Gamma, Z)$ defined by $\lambda_f(\gamma, \gamma') := f(\gamma, \gamma') - f(\gamma', \gamma)$ can be identified with the commutator map of $\widehat{\Gamma}$.

Note that the commutator map λ_f only depends on the cohomology class $[f] \in H^2(\Gamma, Z)$. We thus obtain a group homomorphism

$$\Phi\colon H^2(\Gamma, Z) \to \mathrm{Alt}^2(\Gamma, Z), \quad [f] \mapsto \lambda_f.$$

Remark 3.2 Each biadditive map $f\colon \Gamma \times \Gamma \to Z$ is a cocycle, but it is not true that each cohomology class in $H^2(\Gamma, Z)$ has a biadditive representative. A typical example is the class corresponding to the exact sequence $\mathbf{0} \to m\mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to \mathbf{0}$.

Proposition 3.3 *For abelian groups Γ and Z we have a split short exact sequence*

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \to \mathrm{Ext}(\Gamma, Z) \cong H^2(\Gamma, Z) \xrightarrow{\Phi} \mathrm{Alt}^2(\Gamma, Z) \to \mathbf{0},$$

describing the kernel of the map Φ.

Proof For the exactness in $\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z)$ and $\mathrm{Ext}(\Gamma, Z)$, we only have to observe that an extension $\widehat{\Gamma}$ of Γ by Z is an abelian group if and only if the commutator map of $\widehat{\Gamma}$ is trivial (Remark 3.1).

The remaining assertions can be found as Exercise 5 in [6, §V.6]. The main point of the argument is to use the short exact universal coefficient sequence

$$\mathbf{0} \to \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \to H^2(\Gamma, Z) \xrightarrow{\Psi} \mathrm{Hom}(H_2(\Gamma), Z) \to \mathbf{0},$$

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then show that $H_2(\Gamma) \cong \Lambda^2(\Gamma)$, which leads to an isomorphism $\mathrm{Hom}(H_2(\Gamma), Z) \cong \mathrm{Alt}^2(\Gamma, Z)$ [6, Theorem 6.4], and then to verify that Φ corresponds to Ψ under this identification.

For the sake of completeness, we also give a direct argument for the surjectivity of Φ, based on transfinite induction [14, Lemma 1]. Given an alternating bilinear map $f \in \mathrm{Alt}^2(\Gamma, Z)$, we are looking for a central extension $\widehat{\Gamma}$ of Γ by Z for which f is the commutator map. The key point in the transfinite induction is to show that if $q_1 : \widehat{\Gamma}_1 \to \Gamma_1$ is a central Z-extension of the subgroup Γ_1 of Γ whose commutator map is $f|_{\Gamma_1 \times \Gamma_1}$ and $\Gamma_2 \supseteq \Gamma_1$ is a subgroup for which Γ_2/Γ_1 is cyclic, then we find a central extension $\widehat{\Gamma}_2$ containing $\widehat{\Gamma}_1$ whose commutator map is $f|_{\Gamma_2 \times \Gamma_2}$. Pick $z \in \Gamma_2$ with $\Gamma_2 = \Gamma_1 + \mathbb{Z}z$. Then we consider the group

$$\widetilde{\Gamma}_2 := \widehat{\Gamma}_1 \rtimes_\alpha \mathbb{Z}z, \quad \alpha(mz)(\widehat{\gamma}) := f(q_1(\gamma), mz)\widehat{\gamma}.$$

If $\mathbb{Z}z \cap \Gamma_1 = \{0\}$, then we put $\widehat{\Gamma}_2 := \widetilde{\Gamma}_2$ and note that $\widehat{\Gamma}_2/Z \cong \Gamma_1 \times \mathbb{Z}z \cong \Gamma_2$. If $\mathbb{Z}z \cap \Gamma_1$ is non-trivial and generated by kz, then we put $\widehat{\Gamma}_2 := \widetilde{\Gamma}_2/S$, where S is the cyclic subgroup generated by $(\widehat{\gamma}, kz)$ and $\widehat{\gamma} \in \widehat{\Gamma}_1$ satisfies $q_1(\widehat{\gamma}) = -kz$. Clearly, S is central in $\widehat{\Gamma}_2$, hence normal, and it is a direct consequence of the construction that the commutator map of $\widehat{\Gamma}_2$ is given by f. ■

In [6], the proof of the surjectivity of Φ is based on the observation that each abelian group is a direct limit of its finitely generated subgroups which in turn are products of cyclic groups. Below we give a direct argument for the surjectivity of Φ if Γ is a direct sum of cyclic groups (the only case relevant in the following). We thus obtain an explicit description of $H^2(\Gamma, Z)$.

For the following proposition we recall that, as a consequence of the well-ordering theorem, each set I carries a total order. We also recall the notation $Z[n] = \{z \in Z : nz = 0\}$.

Proposition 3.4 *Let $\Gamma = \bigoplus_{i \in I} \Gamma_i$ be a direct sum of cyclic groups $\Gamma_i \cong \mathbb{Z}/m_i\mathbb{Z}$, $m_i \in \mathbb{N}_0$. Further let $\leq$ be a total order on I. Then*

$$H^2(\Gamma, Z) \cong \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \oplus \mathrm{Alt}^2(\Gamma, Z) \cong \prod_{m_i \neq 0} Z/m_iZ \oplus \prod_{i<j} Z[\gcd(m_i, m_j)], \tag{3.1}$$

where we put $\gcd(m, 0) := m$ for $m \in \mathbb{N}_0$. If, in addition, Γ is free, then Φ is an isomorphism, $H^2(\Gamma, Z) \cong Z^{\{(i,j) \in I^2 : i<j\}}$, and each cohomology class has a biadditive representative.

Proof To see that Φ is surjective, let $\eta \in \mathrm{Alt}^2(\Gamma, Z)$. If γ_i is a generator of Γ_i, we have $\eta(n\gamma_i, m\gamma_i) = nm\eta(\gamma_i, \gamma_i) = 0$ for $n, m \in \mathbb{Z}$, so that η vanishes on $\Gamma_i \times \Gamma_i$. We define a biadditive map $f_\eta : \Gamma \times \Gamma \to Z$ by

$$f_\eta(\gamma_i, \gamma_j) := \begin{cases} \eta(\gamma_i, \gamma_j) & \text{for } i > j, \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j, \\ 0 & \text{for } i \leq j, \gamma_i \in \Gamma_i, \gamma_j \in \Gamma_j. \end{cases}$$

Then f_η is biadditive, hence a 2-cocycle (Remark 3.2), and $\Phi(f_\eta) = \eta$.

Clearly, the assignment $\eta \mapsto f_\eta$ defines an injective homomorphism $\mathrm{Alt}^2(\Gamma, Z) \to H^2(\Gamma, Z)$, splitting Φ. We know from Proposition 3.3 that $\ker \Phi = \mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z)$.

We next observe that

$$\mathrm{Alt}(\Gamma, Z) \cong \prod_{i<j} \mathrm{Hom}(\Gamma_i \otimes \Gamma_j, Z) \quad \text{and} \quad \Gamma_i \otimes \Gamma_j \cong \mathbb{Z}/\gcd(m_i, m_j)\mathbb{Z},$$

which leads to $\mathrm{Hom}(\Gamma_i \otimes \Gamma_j, Z) \cong Z[\gcd(m_i, m_j)]$. On the other hand,

$$\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) \cong \prod_{i \in I} \mathrm{Ext}_{\mathrm{ab}}(\Gamma_i, Z) \cong \prod_{m_i \neq 0} Z/m_i Z$$

(see [10, §52]), which leads to (3.1). If, in addition, Γ is free, then $m_i = 0$ for each $i \in I$, and the assertion follows from $\mathrm{Ext}_{\mathrm{ab}}(\Gamma, Z) = \mathbf{0}$. ∎

4 The Normal Form of Rational Quantum Tori

In this section we write $\Gamma := \mathbb{Z}^n$ for the free abelian group of rank n. For an abelian group Z we write $\mathrm{Alt}_n(Z)$ for the set of *alternating* $(n \times n)$*-matrices with entries in* Z, *i.e.*, $a_{ii} = 0$ for each i and $a_{ij} = -a_{ji}$ for $i \neq j$. This is an abelian group with respect to matrix addition.

Clearly the map $\mathrm{Alt}^2(\Gamma, Z) \to \mathrm{Alt}_n(Z)$, $f \mapsto (f(e_i, e_j))_{i,j=1,\ldots,n}$ is an isomorphism of abelian groups, so that $\mathrm{Alt}_n(Z) \cong H^2(\Gamma, Z)$ by Proposition 3.4. Writing $\lambda_A \in \mathrm{Alt}^2(\Gamma, Z)$ for the alternating form $\lambda_A(\alpha, \beta) := \beta^\top A\alpha$ determined by the alternating matrix A, we have for $g \in \mathrm{GL}_n(\mathbb{Z}) \cong \mathrm{Aut}(\Gamma)$ the relation $\lambda_A(g.\alpha, g.\beta) = \beta g^\top A g \alpha$, so that the orbits of the natural action of $\mathrm{Aut}(\Gamma) \cong \mathrm{GL}_n(\mathbb{Z})$ on the set of alternating forms correspond to the orbits of the action of $\mathrm{GL}_n(\mathbb{Z})$ on $\mathrm{Alt}_n(Z)$ by

$$g.A := gAg^\top, \tag{4.1}$$

where we multiply matrices in $M_n(\mathbb{Z})$ with matrices in $M_n(Z)$ in the obvious fashion. We conclude that

$$H^2(\Gamma, Z)/\mathrm{Aut}(\Gamma) \cong \mathrm{Alt}_n(Z)/\mathrm{GL}_n(\mathbb{Z}), \tag{4.2}$$

the set of $\mathrm{GL}_n(\mathbb{Z})$-orbits in $\mathrm{Alt}_n(Z)$.

If $n = n_1 + \cdots + n_r$ is a partition of n and $A_i \in M_{n_i}(Z)$, then we write

$$A_1 \oplus A_2 \oplus \cdots \oplus A_r := \mathrm{diag}(A_1, \ldots, A_r),$$

for the block diagonal matrix with entries $A_1, \ldots, A_r$. For $h_1, \ldots, h_s \in Z$ we further write

$$N(h_1, \ldots, h_s) := \begin{pmatrix} 0 & h_1 \\ -h_1 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & h_2 \\ -h_2 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & h_s \\ -h_s & 0 \end{pmatrix} \oplus \mathbf{0}_{n-2s} \in \mathrm{Alt}_n(Z).$$

In the following we shall assume that Z is a cyclic group, hence of the form $\mathbb{Z}/(m)$ for some $m \in \mathbb{N}_0$. If $m = 0$, then $Z = \mathbb{Z}$ is a principal ideal domain. This is not the

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case for $m > 0$, but Z still carries a natural ring structure given by $\overline{x} \cdot \overline{y} := \overline{xy}$ for $x, y \in \mathbb{Z}, \overline{x} := x + m\mathbb{Z}$, turning it into a principal ideal ring. We write $Z^\times$ for the set of units in Z and note that if $m = p_1^{\ell_1} \cdots p_k^{\ell_k}$ is the prime factorization of m, then the set

$$P := \{\overline{p}_1^{j_1} \cdots \overline{p}_k^{j_k} : 0 \leq j_i \leq \ell_i, i = 1, \dots, k\} \subseteq Z$$

is a multiplicatively closed set of representatives for the multiplicative cosets of the unit group $Z^\times$. We say that *a divides b* in Z, written $a|b$, if $bZ \subseteq aZ$. Since each subgroup of Z is cyclic and determined by its order, we have

$$a|b \iff \operatorname{ord}(b) | \operatorname{ord}(a) \quad \text{and} \quad aZ = bZ \iff b \in aZ^\times.$$

If $h_1, h_2 \in P$ are non-zero and $h_1|h_2$, then the explicit description of the set P shows that there exists a unique element $h \in P$ with $h_2 = h_1 h$. We then write $h_2/h_1 := h$.

Although Z is not a principal ideal domain for $m > 0$, we define for a matrix $A \in M_n(Z)$ the *determinantal divisor* $d_i(A) \in P$, $i = 1, \dots, n$, as the unique element in P generating the additive subgroup of Z generated by all j-minors of the matrix A. As a consequence of the Cauchy–Binet formula [19, II.12],

$$d_i(AB) = d_i(BA) = d_i(A) \quad \text{for } A \in M_n(Z), B \in \mathrm{GL}_n(Z).$$

We thus obtain a set of n P-valued invariants for the action of $\mathrm{GL}_n(Z)$ on $\mathrm{Alt}_n(Z)$ satisfying for $N := N(h_1, \dots, h_s)$ with $h_i \in P$ and $h_1|h_2|\cdots|h_s$:

$$d_1(N) = h_1, \quad d_2(N) = h_1^2, \quad d_3(N) = h_1^2 h_2, \quad \dots$$
$$d_{2s}(N) = h_1^2 \cdots h_s^2, \quad d_j(N) = 0, \; j > 2s.$$

Unfortunately, these invariants do not separate the orbits for a *finite* cyclic group Z, but they do for $Z = \mathbb{Z}$ (Theorem 4.3 below and [19, Theorem II.9]).

Theorem 4.1 **(Smith normal form over cyclic rings)** *We consider the action of the group* $\mathrm{GL}_n(Z) \times \mathrm{GL}_n(Z)$ *on* $M_n(Z)$ *by* $(g, h).A := gAh^{-1}$.

(i) *Each* $\mathrm{GL}_n(Z)^2$*-orbit contains a unique matrix of the form*

$$\operatorname{diag}(h_1, \dots, h_n) \quad \textit{with } h_i \in P, \; h_1|h_2|\cdots|h_n.$$

(ii) *Each* $\mathrm{SL}_n(Z)^2$*-orbit contains a unique matrix of the form*

$$\operatorname{diag}(h_1, \dots, zh_n) \quad \textit{with } h_i \in P, \; h_1|h_2|\cdots|h_n, z \in Z^\times.$$

(iii) *For* $h_i \in P$ *with* $h_1|h_2|\cdots|h_n$, *we consider the multiplicative subgroup*

$$D_{(h_1,\dots,h_s)} := \{\det(g) : g \in \mathrm{GL}_n(Z), gN(h_1, \dots, h_s)g^\top = N(h_1, \dots, h_s)\} \leq Z^\times.$$

Then $D_{(h_1,\dots,h_s)} = Z^\times$ *for* $2s < n$, *and if* $2s = n$, *then*

$$\{z \in Z^\times : zh_s = h_s\} \subseteq D_{(h_1,\dots,h_s)} \subseteq \{z \in Z^\times : z^2 h_s = h_s\}.$$

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Proof (i) See [7, Theorem 15.24].

(ii) For $z \in Z$ we write $\sigma(z) := \operatorname{diag}(1, \dots, 1, z)$ and observe that $\sigma\colon Z^\times \to \mathrm{GL}_n(Z)$ is an embedding, which leads to a semidirect product decomposition $\mathrm{GL}_n(Z) = \mathrm{SL}_n(Z)\sigma(Z) \cong \mathrm{SL}_n(Z) \rtimes Z^\times$.

Existence: For each $A \in M_n(Z)$, (i) implies the existence of $g, h \in \mathrm{GL}_n(Z)$ such that $N := gAh^{-1} = \operatorname{diag}(h_1, \dots, h_n)$. Writing $g = \sigma(z)g_1$ and $h = \sigma(w)h_1$ with $g_1, h_1 \in \mathrm{SL}_n(Z)$, it follows that

$$g_1Ah_1^{-1} = \sigma(z)^{-1}N\sigma(w) = \sigma(z^{-1}w)N = \operatorname{diag}(d_1, \dots, d_{n-1}, z^{-1}wd_n).$$

Uniqueness: Suppose first that $Z = \mathbb{Z}$ is infinite. If $d_n = 0$, then there is nothing to show. If $d_n \neq 0$, then the fact that the determinant function is constant on the orbits of $\mathrm{SL}_n(Z)^2$ implies the assertion.

We may therefore assume that $Z = \mathbb{Z}/(m)$ for some $m > 0$. Writing $m = p_1^{m_1} \cdots p_k^{m_k}$ for its prime factorization, we obtain a direct product of rings

$$\mathbb{Z}/(m) \cong \mathbb{Z}/(p_1^{m_1}) \times \cdots \times \mathbb{Z}/(p_k^{m_k}).$$

For $Z_i := \mathbb{Z}/(p_i^{m_i})$, we accordingly have $\mathrm{SL}_n(Z) \cong \prod_{i=1}^k \mathrm{SL}_n(Z_i)$ and $M_n(Z) \cong \prod_{i=1}^k M_n(Z_i)$, as direct products of groups, resp., rings. Therefore it suffices to prove the assertion for the case $Z = \mathbb{Z}/(p^m)$, where $m \in \mathbb{N}$ and p is a prime. Each element $z \in Z$ can be uniquely written as

$$z = a_0 + a_1p + \cdots + a_{m-1}p^{m-1} \quad \text{with } 0 \leq a_i < p.$$

It is a unit if and only if $a_0 \neq 0$, *i.e.*, $p \nmid z$. If p^k, $0 \leq k \leq m-1$, is the maximal power of p dividing z, then

$$z = \sum_{j=k}^{m-1} a_jp^j = p^k \sum_{j=k}^{m-1} a_jp^{j-k},$$

where the second factor is a unit. Therefore $\{1, p, p^2, \dots, p^{m-1}, p^m = 0\}$ is a system of representatives of the multiplicative cosets of $Z^\times$ in Z.

Step 1: We must show that if two matrices $D(z_1)$ and $D(z_2)$ of the form

$$D(z) := \operatorname{diag}(p^{k_1}, \dots, zp^{k_n}) = \sigma(z)D(1), \quad 0 \leq k_1 \leq \cdots \leq k_n < m,$$

lie in the same orbit of $\mathrm{SL}_n(Z)^2$, then $D(z_1) = D(z_2)$. Since the orbit of $D(z) = \sigma(z)D(1)$ under $\mathrm{SL}_n(Z)^2$ coincides with the set $\sigma(z)(\mathrm{SL}_n(Z)^2.D(1))$, it suffices to consider the case $z_1 = 1$ and $z_2 = z \in Z^\times$.

Step 2: We proceed by induction on the size n of the matrices. For $n = 1$ the group $\mathrm{SL}_n(Z)$ is trivial, which immediately implies the assertion.

Step 3: We reduce the assertion to the special case $k_1 = 0$. So let us assume that the assertion is correct if $k_1 = 0$ and assume that there are $g, h \in \mathrm{SL}_n(Z)$ with $gD(1)h =$

$D(z)$. Writing $D(z) = p^{k_1}D'(z)$ with $D'(z) = \operatorname{diag}(1, p^{k_2-k_1}, \dots, zp^{k_n-k_1})$, this means that $p^{k_1}(gD'(1)h - D'(z)) = 0$, *i.e.*, that p^{m-k_1} divides each entry of the matrix $gD'(1)h - D'(z)$. Over the quotient ring $Z' := \mathbb{Z}/(p^{m-k_1})$ we then have $gD'(1)h = D'(z)$ with $k_1 = 0$. Since we assume that the theorem holds in this situation, we derive that $zp^{k_n-k_1} \equiv p^{k_n-k_1} \mod p^{m-k_1}$. This means that $p^{m-k_1} | (z-1)p^{k_n-k_1}$, and hence that $p^{k_1}(z-1)p^{k_n-k_1} = (z-1)p^{k_n} = 0$ in Z.

Step 4: Now we consider the special case $k_1 = 0$. Let n_1 be maximal with $k_{n_1} = 0$, $n_2 := n - n_1$, and write elements of $M_n(Z)$ accordingly as (2×2)-block matrices. We further put $k := k_{n_1+1} > 0$. Suppose that $gD(1)h = D(z)$ for

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad h = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \in \mathrm{SL}_n(Z).$$

We write

$$D(z) = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & p^k D'(z) \end{pmatrix}, \quad \text{where } D'(z) = \operatorname{diag}(1, p^{k_{n_1+2}-k}, \dots, zp^{k_n-k}).$$

If $n_1 = n$, then $D(1) = \mathbf{1}$ is the identity matrix, and $z = \det(D(z)) = \det(gD(1)h) = 1$ proves the assertion in this case. We may therefore assume that $1 \le n_1 < n$. We now have

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & p^k D'(z) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & p^k D'(1) \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
$$= \begin{pmatrix} aa' + p^k bD'(1)c' & ab' + p^k bD'(1)d' \\ ca' + p^k dD'(1)c' & cb' + p^k dD'(1)d' \end{pmatrix}.$$

From $aa' + p^k bD'(1)c' = \mathbf{1}$, it follows that $aa' \equiv \mathbf{1} \mod p$, hence that $\det(a) \in Z^\times$, which means that $a \in \mathrm{GL}_{n_1}(Z)$. Multiplication of g from the right with the matrix

$$g' := \begin{pmatrix} a^{-1} & -a^{-1}b\sigma(\det a) \\ 0 & \sigma(\det a) \end{pmatrix} \in \mathrm{SL}_n(Z),$$

leads to the relations

$$gg' = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^{-1} & -a^{-1}b\sigma(\det a) \\ 0 & \sigma(\det a) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ * & * \end{pmatrix},$$
$$(g')^{-1}D(1) = \begin{pmatrix} a & b \\ 0 & \sigma(\det a)^{-1} \end{pmatrix} D(1) = \begin{pmatrix} a & p^k bD'(1) \\ 0 & p^k \sigma(\det a)^{-1} D'(1) \end{pmatrix}$$
$$= \begin{pmatrix} a & p^k bD'(1) \\ 0 & p^k D'(1)\sigma(\det a)^{-1} \end{pmatrix} = D(1) \begin{pmatrix} a & p^k bD'(1) \\ 0 & \sigma(\det a)^{-1} \end{pmatrix}.$$

With

$$h' := \begin{pmatrix} a & p^k bD'(1) \\ 0 & \sigma(\det a)^{-1} \end{pmatrix},$$

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we thus arrive at

$$D(z) = gD(1)h = gg'(g')^{-1}D(1)h = (gg')D(1)(h'h).$$

We may now replace g by gg' and h by $h'h$, so that we may assume that $a = \mathbf{1}$ and $b = 0$. Now

$$\begin{pmatrix} \mathbf{1} & 0 \\ 0 & p^k D'(z) \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{1} & 0 \\ 0 & p^k D'(1) \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$
$$= \begin{pmatrix} a' & b' \\ ca' + p^k dc' & cb' + p^k dD'(1)d' \end{pmatrix}$$

leads to $a' = \mathbf{1}$ and $b' = 0$, which in turn implies $p^k dD'(1)d' = p^k D'(z)$. In view of $\det(g) = \det(d) = 1$ and $\det(h) = \det(d') = 1$, we may now use our induction hypothesis that the theorem holds for matrices of smaller size. Since we have $dD'(1)d' = D'(z)$ in the ring $\mathbb{Z}/(p^{m-k})$, we thus obtain $p^{k_n - k}(1 - z) = 0$ modulo p^{m-k}. This leads to $0 = p^k p^{k_n - k}(1 - z) = p^{k_n}(1 - z)$ modulo p^m, and from that we derive $D(z) = D(1)$.

(iii) If $2s < n$ and $N(z) := N(h_1, \dots, zh_s)$, then $\sigma(z).N(1) = N(z) = N(1)$ for each $z \in Z^\times$, so that $D_{(h_1,\dots,h_s)} = Z^\times$.

Assume $2s = n$. If $zh_s = h_s$, then $\sigma(z).N(1) = N(1)$ implies that $z = \det(\sigma(z)) \in D_{(h_1,\dots,h_s)}$. If, conversely, $z \in D_{(h_1,\dots,h_s)}$, then we pick $g \in \mathrm{GL}_n(Z)$ with $\det(g) = z$ and $g.N(1) = N(1)$. Then $\sigma(z)^{-1}g \in \mathrm{SL}_n(Z)$ implies that $\sigma(z)^{-1}g.tN(1) = N(z^{-1})$ lies in the $\mathrm{SL}_n(Z)^2$-orbit of $\mathrm{diag}(h_1, h_1, \dots, h_s, z^{-2}h_s)$, and the assertion follows from (ii). For this last argument we use that for $w \in Z^\times$,

$$\begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix} \in \mathrm{SL}_2(Z) \text{ satisfies } \begin{pmatrix} 0 & -w^{-1} \\ w & 0 \end{pmatrix} \begin{pmatrix} 0 & wh_s \\ -wh_s & 0 \end{pmatrix} = \begin{pmatrix} h_s & 0 \\ 0 & w^2 h_s \end{pmatrix}. \quad \blacksquare$$

Conjecture 4.2 We believe that if $2s = n$, then $zh_s = h_s$ for each $z \in D_{(h_1,\dots,h_s)}$, so that we have equality in Theorem 4.1(iii), whose present version only implies that $D_{(h_1,\dots,h_s)} \cdot h_s$ can be identified with an elementary abelian 2-group, hence is of cardinality 2^k for some k.

The conjecture is true if all h_i coincide. In fact, for $h_1 = \cdots = h_s$ and $N := N(h_1, \dots, h_s)$ we write $N = h_s N'$, so that the relation $g^\top N g = N$ implies that $h_s \cdot (g^\top N' g - N') = 0$. We conclude that $g^\top N' g \equiv N' \bmod \mathrm{ord}(h_s)$, so that [19, Theorem VII.21] implies the existence of some $\widetilde{g} \in \mathrm{Sp}_{2n}(\mathbb{Z})$ with $g \equiv \widetilde{g} \bmod \mathrm{ord}(h_s)$. Therefore $\det \widetilde{g} = 1$ implies $\det g \equiv 1 \bmod \mathrm{ord}(h_s)$, *i.e.*, $\det(g) \cdot h_s = h_s$.

The following theorem provides a normal form for the orbits of $\mathrm{GL}_n(\mathbb{Z})$ in $\mathrm{Alt}_n(Z)$ for any cyclic group Z. For $Z = \mathbb{Z}$ it follows from Theorem 2.19 [21].

Theorem 4.3 *For any cyclic group Z the following assertions hold.*

(i) *Each $\mathrm{GL}_n(\mathbb{Z})$-orbit in $\mathrm{Alt}_n(Z)$ contains a matrix of the form*

$$N(h_1, \dots, h_s), 2s < n \quad \textit{or} \quad N(h_1, \dots, zh_s), 2s = n,$$

with $z \in Z^\times$ and $0 \neq h_i \in P$ satisfying $h_1 | h_2 | \cdots | h_s$.

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(ii) *If the matrices* $N(h_1,\dots,zh_s)$ *and* $N(h'_1,\dots,z'h'_s)$ *lie in the same* $\mathrm{GL}_n(\mathbb{Z})$*-orbit, then* $s=s'$ *and* $h'_i=h_i$ *for each* i.

(iii) *If* $2s<n$ *or* $Z\cong\mathbb{Z}$*, then any corresponding* $\mathrm{GL}_n(\mathbb{Z})$*-orbit contains a unique matrix of the form* $N(h_1,\dots,h_s)$*. If* $2s=n$*, then two matrices* $N(h_1,\dots,h_{s-1},zh_s)$ *and* $N(h_1,\dots,h_{s-1},wh_s)$ *lie in the same orbit if and only if*

$$d:=zw^{-1}\in\pm D_{(h_1,\dots,h_s)}.$$

In this case, $d^2h_s=h_s$

Proof (i) Let $q\colon\mathbb{Z}\to Z$ be a surjective homomorphism and $q_n\colon M_n(\mathbb{Z})\to M_n(Z)$ the induced homomorphism which is equivariant with respect to the action (4.1) of $\mathrm{GL}_n(\mathbb{Z})$ on both groups. If $A\in\mathrm{Alt}_n(Z)$, then its diagonal vanishes and $a_{ij}=-a_{ji}$, and there exists a matrix $\widetilde{A}\in\mathrm{Alt}_n(\mathbb{Z})$ with $q_n(\widetilde{A})=A$. As $\mathbb{Z}$ is a principal ideal domain, the theorem on the skew normal form [19, Theorems IV.1–IV.2] implies the existence of $g\in\mathrm{GL}_n(\mathbb{Z})$ with $g.\widetilde{A}=N(\widetilde{h}_1,\dots,\widetilde{h}_t)$ and $\widetilde{h}_1|\widetilde{h}_2|\cdots|\widetilde{h}_t$. We then have $g.tA=q_n(g.\widetilde{A})=N(z_1h_1,\dots,z_sh_s)$, where $q(\widetilde{h}_j)=z_jh_j$ with $z_j\in Z^\times$, $h_j\in P$, and s is maximal with $h_s\neq 0$. We further get $h_1|h_2|\cdots|h_s$.

Next we recall [19, Theorem VII.6] that $q_n(\mathrm{SL}_n(\mathbb{Z}))=\mathrm{SL}_n(Z)$, which implies that

$$q_n(\mathrm{GL}_n(\mathbb{Z}))=\{g\in\mathrm{GL}_n(Z):\det g\in\{\pm 1\}\}. \tag{4.3}$$

For $2s=n$ the matrix

$$d:=\mathrm{diag}(z_1^{-1},1,z_2^{-1},1,\dots,z_{s-1}^{-1},1,\dots,1,z_1\cdots z_{s-1})\in\mathrm{SL}_n(Z)$$

now satisfies $d.N(z_1h_1,\dots,z_sh_s)=N(h_1,\dots,h_{s-1},z_1\cdots z_sh_s)$, and for $2s<n$, the matrix

$$d:=\mathrm{diag}(z_1^{-1},1,z_2^{-1},1,\dots,z_s^{-1},1,\dots,1,z_1\cdots z_s)\in\mathrm{SL}_n(Z)$$

satisfies $d.N(z_1h_1,\dots,z_sh_s)=N(h_1,\dots,h_s)$. Since $d\in q(\mathrm{GL}_n(\mathbb{Z}))$, this implies (i).

(ii) The Smith normal form of $N(h_1,\dots,zh_s)$ is $\mathrm{diag}(h_1,h_1,\dots,h_s,h_s,0,\dots,0)$ and for $N(h'_1,\dots,z'h'_s)$ it is $\mathrm{diag}(h'_1,h'_1,\dots,h'_{s'},h'_{s'},0,\dots,0)$. Therefore Theorem 4.1 implies (ii).

(iii) In view of (ii), the number s and $h_1,\dots,h_s$ are uniquely determined by the $\mathrm{GL}_n(\mathbb{Z})$-orbit. If $2s<n$, then it follows already that the corresponding orbit contains a unique matrix of the form $N(h_1,\dots,h_s)$. If $Z=\mathbb{Z}$, then the uniqueness assertion follows from the uniqueness of the skew normal form [19, Theorems IV.1–IV.2], which follows from the fact that the determinantal divisors of $N=N(h_1,\dots,h_s)$ satisfy

$$h_1=d_1(N)=d_2(N)/d_1(N),\dots,h_s=d_{2s-1}(N)/d_{2s-2}(N)=d_{2s}(N)/d_{2s-1}(N)$$

and $d_j(N)=0$ for $j>2s$.

It remains to consider the case $2s=n$. For $\sigma(z):=\mathrm{diag}(1,\dots,1,z)$ and $N(z):=N(h_1,\dots,zh_s)$, we get $N(z)=\sigma(z).N(1)$, and if there exists a $g\in\mathrm{GL}_n(\mathbb{Z})$ with

$g.(\sigma(z).N(1)) = \sigma(w).N(1)$, then $\det(\sigma(w)^{-1}g\sigma(z)) = w^{-1}z\det(g) \in D_{(h_1,\dots,h_s)}$. In view of $\det(g) \in \{\pm 1\}$, this implies that $w^{-1}z \in \pm D_{(h_1,\dots,h_s)}$.

If, conversely, $w^{-1}z \in \pm D_{(h_1,\dots,h_s)}$, then there exists a matrix $g \in \mathrm{GL}_n(Z)$ fixing $N(1)$ with $\det(g) \in \{\pm zw^{-1}\}$. Hence $\det(\sigma(w)g\sigma(z)^{-1}) \in \{\pm 1\}$, and (4.3) imply the existence of $g_1 \in \mathrm{GL}_n(\mathbb{Z})$ with $q_n(g_1) = \sigma(w)g\sigma(z)^{-1}$. We now have

$$\begin{aligned} g_1.N(z) &= g_1\sigma(z).N(1) = \sigma(w)g\sigma(z)^{-1}\sigma(z).N(1) \\ &= \sigma(w)g.N(1) = \sigma(w)N(1) = N(w). \end{aligned} \qquad \blacksquare$$

Definition 4.4 (i) We call a Γ-quantum torus *rational* if the commutator group C_A of $A^\times = A_h^\times$ (see Proposition A.1) consists of roots of unity in $\mathbb{K}$. We call it *of cyclic type* if C_A is a cyclic subgroup of $\mathbb{K}^\times$.

(ii) For each $q \in \mathbb{K}^\times$ we write A_q for the $\mathbb{Z}^2$-quantum torus corresponding to the biadditive cocycle $f\colon \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{K}^\times$ determined by

$$f(e_1, e_1) = f(e_2, e_2) = f(e_2, e_1) = 1 \quad \text{and} \quad f(e_1, e_2) = q.$$

Then the algebra A_q is generated by $u_1 = \delta_{e_1}$, $u_2 = \delta_{e_2}$ satisfying $u_1u_2 = qu_2u_1$, and their inverses. Then $C_{A_q} = \langle q \rangle$, so that the quantum torus A_q is rational if and only if q is a root of unity.

Theorem 4.5 **(Normal form of rational quantum tori)** *Let $\mathbb{K}$ be any field.*

(i) *For any rational n-dimensional quantum torus A over $\mathbb{K}$, the commutator group $C_A \subseteq \mathbb{K}^\times$ is cyclic. Let q be a generator of C_A and choose $P \subseteq \mathbb{Z}/(m)$ for $m = |C_A| =$ $\mathrm{ord}(q)$ as above. Then there exists an $s \in \mathbb{N}_0$ with $2s \leq n$ and $h_2 | \dots | h_s$ in $P \setminus \{0\}$ such that*

$$A \cong A_q \otimes A_{q^{h_2}} \otimes \cdots \otimes A_{q^{h_s}} \otimes \mathbb{K}[\mathbb{Z}^{n-2s}] \quad \textit{and} \quad 2s < n \tag{4.4}$$

or

$$A \cong A_q \otimes A_{q^{h_2}} \otimes \cdots \otimes A_{q^{h_{s-1}}} \otimes A_{q^{zh_s}} \quad \textit{and} \quad 2s = n \tag{4.5}$$

for some $z \in \mathbb{N}$ with $\mathrm{ord}(q^{zh_s}) = \mathrm{ord}(q^{h_s})$.

(ii) *If two n-dimensional rational quantum tori A and A' are (graded) isomorphic, then $C_A = C_{A'}$. Both can be described by some data $(h_2, \dots, zh_s)$ and $(h_2', \dots, z'h_{s'}')$ as in* (i)*, related to the same choice of generator q of $C_A = C_{A'}$.*

(iii) *Two n-dimensional rational quantum tori A and A' given by such data are (graded) isomorphic if and only if $s = s'$, $h_i = h_i'$ for $i = 2, \dots, s$, and*

$$z' \in \pm z \cdot D_{(1,h_2,\dots,h_s)},$$

where

$$D_{(h_1,\dots,h_s)} = \{\det(g) : g \in \mathrm{GL}_n(Z), gN(h_1, \dots, h_s)g^\top = N(h_1, \dots, h_s)\} \leq Z^\times$$

for the ring $Z := \mathbb{Z}/\,$ $\mathrm{ord}(q)$*. In this case $z^2h_s = (z')^2h_s$ holds in Z.*

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Proof (i) We know from Theorem 2.4 and (4.2) that the Γ-quantum tori over $\mathbb{K}$ are classified by the orbits of $\mathrm{Aut}(\Gamma) \cong \mathrm{GL}_n(\mathbb{Z})$ in $H^2(\Gamma, \mathbb{K}^\times) \cong \mathrm{Alt}^2(\Gamma, \mathbb{K}^\times)$. In this picture, the rational quantum tori correspond to alternating forms $f \in \mathrm{Alt}^2(\Gamma, \mathbb{K}^\times)$ on Γ whose values are roots of unity. Since the group C_A generated by the image of f is generated by the finite set $f(e_i, e_j)$, $i, j = 1, \ldots, n$, it is a finite subgroup of $\mathbb{K}^\times$, hence cyclic [18, Theorem IV.1.9].

Therefore Theorem 4.3 applies, and we see that for $s < 2n$ the quantum torus A is isomorphic to one defined by a biadditive cocycle $f \colon \Gamma \times \Gamma \to C_A \subseteq \mathbb{K}^\times$, satisfying $(\lambda_f(e_i, e_j))_{i,j} = N(q, q^{h_2}, \ldots, q^{h_s})$, where $h_2 | \cdots | h_s$. Here $h_1 = 1$ follows from the fact that the commutator subgroup of $A^\times$ is generated by q. The quantum torus $A_f \cong A$ defined by f then satisfies (4.4). In the other case we have $2s = n$ and (4.5) holds.

(ii) That (graded) isomorphic quantum tori have the same commutator group is clear. Therefore (ii) follows from (i).

(iii) The remaining assertion now follows from Theorem 2.4, combined with Theorem 4.3. ■

Remark 4.6 If A is a $\mathbb{Z}^n$-quantum torus of cyclic type and the group of commutators in $A^\times$ is generated by $q \in \mathbb{K}^\times$, then the skew normal form over $\mathbb{Z}$ and the argument from the proof of Theorem 4.1 imply the existence of $h_2 | \cdots | h_s \in \mathbb{N}$ and $s \in \mathbb{N}_0$ such that

$$A \cong A_q \otimes A_{q^{h_2}} \otimes \cdots \otimes A_{q^{h_s}} \otimes \mathbb{K}[\mathbb{Z}^{n-2s}].$$

See [8, §7.2 Remark]. If $\mathrm{ord}(q) = \infty$, two such decompositions describe isomorphic algebras if and only if $s = s'$ and $h_i = h_i'$ for all i [21, Theorem 2.19] or Theorem 4.3. The main point of the preceding theorem is that it gives more precise information on the isomorphism classes in the rational case.

5 Graded Automorphisms of Quantum Tori

In this section we briefly discuss the group of automorphisms of a general quantum torus, but our main result only concerns the 2-dimensional case. For $A = A_q$ and the corresponding alternating form λ on $\mathbb{Z}^2$, the group $\mathrm{Aut}(A)$ is a semi-direct product $\mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times) \rtimes \mathrm{Aut}(\mathbb{Z}^2, \lambda)$. Here the remarkable point is that this holds without any assumptions on the field $\mathbb{K}$ and in particular without assuming that q is a square.

Definition 5.1 Let A be a Γ-quantum torus. We write $\mathrm{Aut}_{\mathrm{gr}}(A)$ for the group of *graded automorphisms of A, i.e.,* all those automorphisms $\varphi \in \mathrm{Aut}(A)$ for which there exists an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$ with $\varphi(A_\gamma) = A_{\varphi_\Gamma(\gamma)}$ for all $\gamma \in \Gamma$.

Note that Proposition A.1 in the appendix implies that if Γ is torsion free, then all units are homogeneous, which implies that each automorphism of A is graded.

Remark 5.2 We fix a basis $(\delta_\gamma)_{\gamma\in\Gamma}$ of A and suppose that $f \in Z^2(\Gamma, Z)$ is the corresponding cocycle determined by (2.1). Then for each graded automorphism φ

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of A there is an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$ and a function $\chi\colon \Gamma \to \mathbb{K}^\times$ such that

$$\varphi(\delta_\gamma) = \chi(\gamma)\delta_{\varphi_\Gamma(\gamma)}, \quad \gamma \in \Gamma.$$

Conversely, for a pair (χ, φ_Γ) of a function $\chi\colon \Gamma \to \mathbb{K}^\times$ and an automorphism $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$, the prescription $\varphi(\delta_\gamma) := \chi(\gamma)\delta_{\varphi_\Gamma(\gamma)}$ defines an automorphism of A if and only if

$$\text{(5.1)} \qquad \frac{(\varphi_\Gamma^* f)(\gamma, \gamma')}{f(\gamma, \gamma')} = \frac{\chi(\gamma + \gamma')}{\chi(\gamma)\chi(\gamma')} \quad \text{for all } \gamma, \gamma' \in \Gamma.$$

Note that if f is biadditive, then $\varphi_\Gamma^* f/f$ is biadditive, so that χ is a corresponding $\mathbb{K}^\times$-valued quadratic form. If f and φ_Γ are given, then a χ satisfying (5.1) exists if and only if $[\varphi_\Gamma^* f] = [f]$ holds in $H^2(\Gamma, Z)$.

Lemma 5.3 *The image of the map* $Q\colon \mathrm{Aut}_{\mathrm{gr}}(A) \to \mathrm{Aut}(\Gamma), \varphi \mapsto \varphi_\Gamma$ *is the group*

$$\mathrm{Aut}(\Gamma)_{[f]} := \{\psi \in \mathrm{Aut}(\Gamma) : [\psi^* f] = [f]\},$$

which is contained in $\mathrm{Aut}(\Gamma, \lambda_f) := \{\psi \in \mathrm{Aut}(\Gamma) : \psi^* \lambda_f = \lambda_f\}$, *where* $\lambda_f(\gamma, \gamma') = \frac{f(\gamma, \gamma')}{f(\gamma', \gamma)}$. *If, in addition,* Γ *is free, then* $\mathrm{Aut}(\Gamma)_{[f]} = \mathrm{Aut}(\Gamma, \lambda_f)$.

Proof Let $\varphi_\Gamma \in \mathrm{Aut}(\Gamma)$. In view of Remark 5.2, the existence of $\varphi \in \mathrm{Aut}_{\mathrm{gr}}(A)$ with $Q(\varphi) = \varphi_\Gamma$ is equivalent to the existence of χ satisfying (5.1), which is equivalent to $[\varphi_\Gamma^* f] = [f]$ in $H^2(\Gamma, \mathbb{K}^\times)$. Since (5.1) implies that $\varphi_\Gamma^* f/f$ is symmetric, we have $\varphi_\Gamma^* \lambda_f = \lambda_{\varphi_\Gamma^* f} = \lambda_f$.

If, in addition, Γ is free, then Proposition 3.4 entails that $\varphi_\Gamma^* \lambda_f = \lambda_f$ is equivalent to $[\varphi_\Gamma^* f] = [f]$ in $H^2(\Gamma, \mathbb{K}^\times)$ [20, Lemma 3.3(iii)]. ∎

From (5.1) we derive in particular that $(\chi, \mathbf{1})$ defines an automorphism of A if and only if $\chi \in \mathrm{Hom}(\Gamma, \mathbb{K}^\times)$, so that we obtain the exact sequence

$$\text{(5.2)} \qquad \mathbf{1} \to \mathrm{Hom}(\Gamma, \mathbb{K}^\times) \to \mathrm{Aut}_{\mathrm{gr}}(A) \to \mathrm{Aut}(\Gamma)_{[f]} \to \mathbf{1}.$$

(See [20, Lemma 3.3(iii)].) We call the automorphisms of the form $(\chi, \mathbf{1})$ *scalar*.

Remark 5.4 If the map Φ from Proposition 3.4 is not injective, then the groups $\mathrm{Aut}(\Gamma, \lambda_f)$ and $\mathrm{Aut}(\Gamma)_{[f]}$ need not coincide, but with Proposition 3.3 we obtain a 1-cocycle

$$I\colon \mathrm{Aut}(\Gamma, \lambda_f) \to \mathrm{Ext}_{\mathrm{ab}}(\Gamma, \mathbb{K}^\times), \quad \psi \mapsto [\psi^* f - f]$$

with respect to the right action of $\mathrm{Aut}(\Gamma, \lambda_f)$ on $\mathrm{Ext}(\Gamma, \mathbb{K}^\times) \cong H^2(\Gamma, \mathbb{K}^\times)$ by $\psi.[f] := [\psi^* f]$. We then have $\mathrm{Aut}(\Gamma)_{[f]} = I^{-1}(0)$.

In the remainder of this section we restrict our attention to the case, where $\Gamma = \mathbb{Z}^n$ is a free abelian group of rank n, which implies that $\mathrm{Aut}(\Gamma)_{[f]} = \mathrm{Aut}(\Gamma, \lambda_f)$ and that $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$ (Corollary A.2).

Remark 5.5 (i) For $n = 1$, each alternating biadditive map λ on Γ vanishes, so that $\mathrm{Aut}(\Gamma, \lambda) = \mathrm{Aut}(\Gamma) \cong \{\pm \mathrm{id}_\Gamma\}$.

(ii) For each alternating form $\lambda \colon \Gamma \times \Gamma \to \mathbb{K}^\times$ we have $-\mathrm{id}_\Gamma \in \mathrm{Aut}(\Gamma, \lambda)$.

(iii) In [20], it is shown that if $n \geq 3$ and the subgroup $\langle \mathrm{im}(\lambda) \rangle$ of $\mathbb{K}^\times$ generated by the image of λ is free of rank $\binom{n}{2}$, then $\mathrm{Aut}(\Gamma, \lambda_f) = \{\pm \mathrm{id}_\Gamma\}$.

Moreover, for $n = 3$ and $\langle \mathrm{im}(\lambda) \rangle$ free of rank 2, [20, Proposition 3.7] implies the existence of a basis $\gamma_1, \gamma_2, \gamma_3 \in \Gamma$ with $\lambda(\gamma_1, \gamma_2) = 1$ and

$$\begin{aligned} \mathrm{Aut}(\Gamma, \lambda) &\cong \big\{ \sigma \in \mathrm{Aut}(\Gamma) : (\exists a, b \in \mathbb{Z}, \varepsilon \in \{\pm 1\}) \\ &\qquad\qquad \sigma(\gamma_1) = \gamma_1^\varepsilon,\ \sigma(\gamma_2) = \gamma_2^\varepsilon,\ \sigma(\gamma_3) = \gamma_1^a \gamma_2^b \gamma_3^\varepsilon \,\big\} \\ &\cong \mathbb{Z}^2 \rtimes \{\pm \mathrm{id}_{\mathbb{Z}^2}\}. \end{aligned}$$

We now take a closer look at the case $n = 2$. An alternating form $\lambda \in \mathrm{Alt}^2(\mathbb{Z}^2, \mathbb{K}^\times)$ is uniquely determined by $q := \lambda(e_1, e_2)$, which implies $\lambda(\gamma, \gamma') = q^{\gamma_1\gamma_2' - \gamma_2\gamma_1'}$. We may therefore assume that a corresponding bimultiplicative cocycle f satisfies $f(\gamma, \gamma') = q^{\gamma_1\gamma_2'}$, which leads to the quantum torus A_q with two generators $u_i = \delta_{e_i}$ and their inverses, satisfying $u_1 u_2 = q u_2 u_1$, as defined in the introduction.

We start with two simple observations.

Lemma 5.6

$$\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \begin{cases} \mathrm{SL}_2(\mathbb{Z}) & \textit{for } q^2 \neq 1, \\ \mathrm{GL}_2(\mathbb{Z}) & \textit{for } q^2 = 1. \end{cases}$$

Proof Clearly $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{Aut}(\mathbb{Z}^2, \lambda) \subseteq \mathrm{GL}_2(\mathbb{Z})$. The map $g_0(\gamma) = (\gamma_2, \gamma_1)$ satisfies $\mathrm{GL}_2(\mathbb{Z}) \cong \mathrm{SL}_2(\mathbb{Z}) \rtimes \langle g_0 \rangle$, and we have

$$\frac{g_0^* \lambda(e_1, e_2)}{\lambda(e_1, e_2)} = \frac{\lambda(e_2, e_1)}{\lambda(e_1, e_2)} = q^{-2}. \qquad \blacksquare$$

Example 5.7 (i) On $\mathbb{Z}^2$ the map $\chi(\gamma) := \gamma_1\gamma_2$ is a quadratic form with

$$\chi(\gamma + \gamma') - \chi(\gamma) - \chi(\gamma') = \gamma_1\gamma_2' + \gamma_2\gamma_1'.$$

(ii) On $\mathbb{Z}$ the map $\chi(n) := \binom{n}{2}$ is a quadratic form with

$$\begin{aligned} \chi(n + n') - \chi(n) - \chi(n') &= \frac{(n + n')(n + n' - 1) - n(n - 1) - n'(n' - 1)}{2} \\ &= \frac{nn' + n'n}{2} = nn'. \end{aligned}$$

From $\mathrm{SL}_2(\mathbb{Z}) \subseteq \mathrm{Aut}(\mathbb{Z}^2, \lambda)$, it follows in particular that each matrix $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ can be lifted to an automorphism of A_q. To determine a corresponding quadratic form $\chi \colon \mathbb{Z}^2 \to \mathbb{K}^\times$, we must solve the equation (5.1):

$$\frac{(g^* f)(\gamma, \gamma')}{f(\gamma, \gamma')} = \frac{\chi(\gamma + \gamma')}{\chi(\gamma)\chi(\gamma')}.$$

The form $g^* f/f$ is determined by its values on the pairs (e_1, e_1), (e_1, e_2) and (e_2, e_2):

$$(g^* f/f)(e_1, e_1) = f(g.e_1, g.e_1) = q^{ac}, \quad (g^* f/f)(e_1, e_2) = f(g.e_1, g.e_2)q^{-1} = q^{ad-1},$$
$$(g^* f/f)(e_2, e_2) = f(g.e_2, g.e_2) = q^{bd}.$$

This means that

$$(g^* f/f)(\gamma, \gamma') = q^{ac\gamma_1\gamma_1' + (ad-1)(\gamma_1\gamma_2' + \gamma_1'\gamma_2) + bd\gamma_2\gamma_2'}.$$

Before we turn to lifting the full groups $\mathrm{Aut}(\mathbb{Z}^2, \lambda)$ to an automorphism group of A, we discuss certain specific elements of finite order separately.

Remark 5.8 (i) For the central element $z = -\mathbf{1} \in \mathrm{SL}_2(\mathbb{Z})$, any lift $\widehat{z} \in \mathrm{Aut}(A_q)$ is of the form

$$\widehat{z}.\delta_\gamma = r^{\gamma_1} s^{\gamma_2} \cdot \delta_{-\gamma} \quad \text{for some } r, s \in \mathbb{K}^\times,$$

and any such element satisfies $\widehat{z}^2.\delta_\gamma = r^{\gamma_1} s^{\gamma_2} \cdot \widehat{z}.\delta_{-\gamma} = r^{\gamma_1 - \gamma_1} s^{\gamma_2 - \gamma_2} \cdot \delta_\gamma = \delta_\gamma$. Hence each lift $\widehat{z}$ of z is an element of order 2.

(ii) The matrices

$$g_1 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{and} \quad g_2 := \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$$

satisfy $g_1^2 = z = g_2^3$, which leads to $\mathrm{ord}(g_1) = 4$ and $\mathrm{ord}(g_2) = 6$. From the preceding paragraph we conclude that for any lift $\widehat{g}_j$ of g_j, $j = 1, 2$, we have $\widehat{g}_1^4 = \mathbf{1} = \widehat{g}_2^6$.

In view of

$$(g_1^* f/f)(\gamma, \gamma') = q^{-(\gamma_1\gamma_2' + \gamma_1'\gamma_2)},$$

a lift $\widetilde{g}_1$ of g_1 is given by $\widetilde{g}_1.\delta_\gamma = q^{-\gamma_1\gamma_2}\delta_{g_1.\gamma}$ (Example 5.7(i)). We then have

$$\widetilde{g}_1^2.\delta_\gamma = q^{-\gamma_1\gamma_2}\widetilde{g}_1.\delta_{(\gamma_2, -\gamma_1)} = q^{-\gamma_1\gamma_2} q^{\gamma_2\gamma_1}.\delta_{-\gamma} = \delta_{-\gamma}.$$

Any other lift $\widehat{g}_1$ of g_1 is of the form

$$\widehat{g}_1.\delta_g = r_1^{\gamma_1} s_1^{\gamma_2} q^{-\gamma_1\gamma_2} \delta_{g_1.\gamma}$$

for two elements $r_1, s_1 \in \mathbb{K}^\times$. The square of this element is given by

$$\widehat{g}_1^2.\delta_g = r_1^{\gamma_1} s_1^{\gamma_2} \widehat{g}_1 \widetilde{g}_1.\delta_\gamma = r_1^{\gamma_1+\gamma_2} s_1^{\gamma_2-\gamma_1} \widetilde{g}_1^2.\delta_\gamma = \left(\frac{r_1}{s_1}\right)^{\gamma_1} (r_{s_1})^{\gamma_2} \cdot \delta_{-\gamma}. \tag{5.3}$$

For the matrix g_2 we have $(g_2^* f/f)(\gamma, \gamma') = q^{-\gamma_1\gamma_1' - (\gamma_1\gamma_2' + \gamma_1'\gamma_2)}$, so that we obtain a lift $\widetilde{g}_2$ of g_2 by $\widetilde{g}_2.\delta_\gamma = q^{-\binom{\gamma_1}{2} - \gamma_1\gamma_2}\delta_{(\gamma_1+\gamma_2, -\gamma_1)}$ (Example 5.7(ii)). Hence each lift $\widehat{g}_2$ of g_2 is of the form

$$\widehat{g}_2.\delta_g = r_2^{\gamma_1} s_2^{\gamma_2} q^{-\binom{\gamma_1}{2} - \gamma_1\gamma_2} \delta_{(\gamma_1+\gamma_2, -\gamma_1)},$$

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for some $r_2, s_2 \in \mathbb{K}^\times$. In view of $g_2^2 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & -1 \end{smallmatrix}\right)$, we get with Example 5.7(ii):

$$\begin{aligned}
\widetilde{g}_2^3.\delta_\gamma &= q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}\widetilde{g}_2^2.\delta_{\gamma_1+\gamma_2,-\gamma_1} = q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}q^{-\binom{\gamma_1+\gamma_2}{2}+(\gamma_1+\gamma_2)\gamma_1}\widetilde{g}_2.\delta_{\gamma_2,-\gamma_1-\gamma_2} \\
&= q^{-2\binom{\gamma_1}{2}-\frac{\gamma_2}{2}-\gamma_1\gamma_2+\gamma_1^2}q^{-\binom{\gamma_2}{2}+(\gamma_1+\gamma_2)\gamma_2}\delta_{-\gamma} \\
&= q^{-\gamma_1(\gamma_1-1)-\gamma_2(\gamma_2-1)+\gamma_1^2+\gamma_2^2}\delta_{-\gamma} = q^{\gamma_1+\gamma_2}\delta_{-\gamma}.
\end{aligned}$$

This further leads to

$$\begin{aligned}
\widehat{g}_2^3.\delta_\gamma &= r_2^{\gamma_1}s_2^{\gamma_2}\widehat{g}_2^2\widetilde{g}_2.\delta_\gamma = r_2^{2\gamma_1+\gamma_2}s_2^{-\gamma_1+\gamma_2}\widehat{g}_2\widetilde{g}_2^2.\delta_\gamma = r_2^{2\gamma_1+2\gamma_2}s_2^{-2\gamma_1}\widetilde{g}_2^3.\delta_\gamma \\
&= r_2^{2(\gamma_1+\gamma_2)}s_2^{-2\gamma_1}q^{\gamma_1+\gamma_2}.\delta_{-\gamma} = \left(\frac{r_2^2}{s_2^2}q\right)^{\gamma_1}(r_2^2q)^{\gamma_2}\delta_{-\gamma}.
\end{aligned} \tag{5.4}$$

If, in addition, $q^2 = 1$, then $\mathrm{Aut}(\Gamma, \lambda_f) = \mathrm{Aut}(\Gamma) \cong \mathrm{GL}_2(\mathbb{Z})$ (Lemma 5.6). For the involution

$$g_0 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we have $\mathrm{GL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \rtimes \langle g_0 \rangle$, and the elements g_0, g_1, g_2 satisfy

$$g_0g_1g_0 = g_1^{-1} = g_1^3 \quad \text{and} \quad g_0g_2g_0 = g_2^5 = g_2^{-1}.$$

To lift g_0 to an automorphism of A_q, we first note that $q^2 = 1$ implies that

$$(g_0^*f/f)(\gamma, \gamma') = q^{\gamma_2\gamma_1'-\gamma_1\gamma_2'} = q^{\gamma_2\gamma_1'+\gamma_1\gamma_2'},$$

which shows that each lift $\widehat{g}_0$ of g_0 is of the form $\widehat{g}_0.\delta_\gamma = r_0^{\gamma_1}s_0^{\gamma_2}q^{\gamma_1\gamma_2}\delta_{(\gamma_2,\gamma_1)}$ for some $r_0, s_0 \in \mathbb{K}^\times$. In view of

$$\widehat{g}_0^2.\delta_\gamma = r_0^{\gamma_1}s_0^{\gamma_2}q^{\gamma_1\gamma_2}\widehat{g}_0.\delta_{(\gamma_2,\gamma_1)} = r_0^{\gamma_1+\gamma_2}s_0^{\gamma_2+\gamma_1}q^{2\gamma_1\gamma_2}\delta_\gamma = (r_0s_0)^{\gamma_1+\gamma_2}\delta_\gamma,$$

$\widehat{g}_0^2 = \mathbf{1}$ is equivalent to $r_0s_0 = 1$. If this condition is satisfied, then

$$\widehat{g}_0.\delta_\gamma = r_0^{\gamma_1-\gamma_2}q^{\gamma_1\gamma_2}\delta_{(\gamma_2,\gamma_1)}.$$

Before we state the following theorem, we recall that for any split extension

$$\mathbf{1} \to A \to \widehat{G} \xrightarrow{q} G \to \mathbf{1}$$

of a group G by some (abelian) G-module A, the set of all splittings is parametrized by the group $Z^1(G, A) = \{f : G \to A : (\forall x, y \in G)\ f(xy) = f(x) + x.f(y)\}$ of A-valued 1-cocycles. This parametrization is obtained by choosing a homomorphic section $\sigma_0 : G \to \widehat{G}$ and then observing that any other homomorphic section $\sigma : G \to \widehat{G}$ is of the form $\sigma = f \cdot \sigma_0$, where $f \in Z^1(G, A)$.

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Theorem 5.9 *For each element $q \in \mathbb{K}^\times$ and $\lambda(\gamma, \gamma') = q^{\gamma_1\gamma_2' - \gamma_2\gamma_1'}$, the exact sequence*

$$\mathbf{1} \to \mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times) \to \mathrm{Aut}(A_q) \to \mathrm{Aut}(\mathbb{Z}^2, \lambda) \to \mathbf{1}$$

splits. For $q^2 = 1$, the homomorphisms $\sigma\colon \mathrm{GL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ splitting the sequence are parametrized by the abelian group

$$Z^1(\mathrm{GL}_2(\mathbb{Z}), \mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times)) \cong \{(r_0, r_1, r_2) \in (\mathbb{K}^\times)^3 : r_2^4 r_0^2 = r_1^2\},$$

and for $q^2 \neq 1$, the homomorphisms $\sigma\colon \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ splitting the sequence are parametrized by $Z^1(\mathrm{SL}_2(\mathbb{Z}), \mathrm{Hom}(\mathbb{Z}^2, \mathbb{K}^\times)) \cong (\mathbb{K}^\times)^2 \times \{z \in \mathbb{K}^\times : z^2 = 1\}$.

Proof First we consider the case $q^2 \neq 1$, where $\mathrm{Aut}(\mathbb{Z}^2, \lambda) = \mathrm{SL}_2(\mathbb{Z})$ (Remark 5.8). We shall use the description of the lifts of g_1, g_2 given in Remark 5.8. Since $\mathrm{SL}_2(\mathbb{Z})$ is presented by the relations $g_1^4 = g_2^6 = \mathbf{1}$ and $g_1^2 = g_2^3$ [13, p. 51], Remark 5.8 implies that a pair of elements $(\widehat{g}_1, \widehat{g}_2)$ lifting (g_1, g_2) leads to a lift $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{Aut}(A_q)$ if and only if $\widehat{g}_1^2 = \widehat{g}_2^3$. Comparing (5.3) and (5.4), we see that $\widehat{g}_1^2 = \widehat{g}_2^3$ is equivalent to

$$\frac{r_1}{s_1} = \frac{r_2^2}{s_2^2} q \quad \text{and} \quad r_1 s_1 = r_2^2 q,$$

which is equivalent to

$$s_1^2 = s_2^2 \quad \text{and} \quad s_1 = \frac{r_2^2 q}{r_1}. \tag{5.5}$$

These equations have the simple solution $r_1 = q, r_2 = s_1 = s_2 = 1$, showing that the action of the group $\mathrm{SL}_2(\mathbb{Z})$ on Γ lifts to an action on A_q. Moreover, for each pair (r_1, r_2), the set of all solutions is determined by the choice of sign in $s_2 := \pm s_1$, which is vacuous if $\mathrm{char}(\mathbb{K}) = 2$.

Next we consider the case $q^2 = 1$. We assume that the lift $\widehat{g}_0$ of g_0 satisfies $\widehat{g}_0^2 = \mathbf{1}$ (see Remark 5.8(iii)). Now the relation $\widehat{g}_0 \widehat{g}_1 \widehat{g}_0 = \widehat{g}_1^{-1}$ is equivalent to $(\widehat{g}_0 \widehat{g}_1)^2 = \mathbf{1}$. We calculate

$$\widehat{g}_0 \widehat{g}_1 . \delta_\gamma = r_1^{\gamma_1} s_1^{\gamma_2} q^{-\gamma_1\gamma_2} \widehat{g}_0 . \delta_{(\gamma_2, -\gamma_1)} = (r_0 r_1)^{\gamma_1} (r_0 s_1)^{\gamma_2} \delta_{(-\gamma_1, \gamma_2)}$$

to get

$$(\widehat{g}_0 \widehat{g}_1)^2 . \delta_\gamma = (r_0 r_1)^{\gamma_1} (r_0 s_1)^{\gamma_2} \widehat{g}_0 \widehat{g}_1 . \delta_{(-\gamma_1, \gamma_2)} = (r_0 s_1)^{2\gamma_2} \delta_\gamma .$$

Hence $\widehat{g}_0 \widehat{g}_1 \widehat{g}_0 = \widehat{g}_1^{-1}$ is equivalent to

$$r_0^2 s_1^2 = 1. \tag{5.6}$$

To see when $\widehat{g}_0 \widehat{g}_2 \widehat{g}_0 = \widehat{g}_2^{-1}$ holds, we first observe that

$$\widehat{g}_2^{-1} . \delta_\gamma = r_2^{\gamma_2} s_2^{-\gamma_1 - \gamma_2} q^{\frac{-\gamma_2}{2} - \gamma_2(\gamma_1 + \gamma_2)} \delta_{(-\gamma_2, \gamma_1 + \gamma_2)} .$$

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Further

$$\begin{aligned}\widehat{g}_0\widehat{g}_2.\delta_\gamma &= r_2^{\gamma_1}s_2^{\gamma_2}q^{-\binom{\gamma_1}{2}-\gamma_1\gamma_2}\widehat{g}_0.\delta_{(\gamma_1+\gamma_2,-\gamma_1)}\\ &= (r_0^2r_2)^{\gamma_1}(r_0s_2)^{\gamma_2}q^{\binom{\gamma_1}{2}+\gamma_1\gamma_2+(\gamma_1+\gamma_2)\gamma_1}\cdot\delta_{(-\gamma_1,\gamma_1+\gamma_2)}\\ &= (r_0^2r_2)^{\gamma_1}(r_0s_2)^{\gamma_2}q^{\binom{\gamma_1}{2}+\gamma_1^2}\cdot\delta_{(-\gamma_1,\gamma_1+\gamma_2)} = (r_0^2r_2q)^{\gamma_1}(r_0s_2)^{\gamma_2}q^{\binom{\gamma_1}{2}}\cdot\delta_{(-\gamma_1,\gamma_1+\gamma_2)},\end{aligned}$$

because $q^2=1$ implies $q^{n^2}=q^n=q^{-n}$ for each $n\in\mathbb{Z}$.

On the other hand, we have

$$\begin{aligned}\widehat{g_2}^{-1}\widehat{g}_0.\delta_\gamma &= r_0^{\gamma_1}r_0^{-\gamma_2}q^{\gamma_1\gamma_2}\widehat{g_2}^{-1}.\delta_{(\gamma_2,\gamma_1)}\\ &= r_0^{\gamma_1}r_0^{-\gamma_2}q^{\gamma_1\gamma_2}r_2^{\gamma_1}s_2^{-\gamma_2-\gamma_1}q^{\binom{-\gamma_1}{2}-\gamma_1(\gamma_1+\gamma_2)}\delta_{(-\gamma_1,\gamma_1+\gamma_2)}\\ &= (r_0r_2s_2^{-1})^{\gamma_1}(r_0s_2)^{-\gamma_2}q^{\binom{-\gamma_1}{2}-\gamma_1^2}\delta_{(-\gamma_1,\gamma_1+\gamma_2)}\\ &= (r_0r_2s_2^{-1})^{\gamma_1}(r_0s_2)^{-\gamma_2}q^{-\binom{\gamma_1}{2}}\delta_{(-\gamma_1,\gamma_1+\gamma_2)}\\ &= (r_0r_2s_2^{-1})^{\gamma_1}(r_0s_2)^{-\gamma_2}q^{\frac{\gamma_1}{2}}\delta_{(-\gamma_1,\gamma_1+\gamma_2)}.\end{aligned}$$

Therefore $\widehat{g}_0\widehat{g}_2\widehat{g}_0=\widehat{g_2}^{-1}$ is equivalent to $r_0r_2s_2^{-1}=r_0^2r_2q$ and $(r_0s_2)^2=1$, which is equivalent to

$$r_0s_2=q, \tag{5.7}$$

because this relation implies $(r_0s_2)^2=q^2=1$.

We conclude that the numbers r_0,r_1,r_2,s_1,s_2 which determine $\widehat{g}_0,\widehat{g}_1,\widehat{g}_2$ define a lift of $\mathrm{GL}_2(\mathbb{Z})$ to $\mathrm{Aut}(A_q)$ if and only if the equations (5.5), (5.6) and (5.7) are satisfied:

$$s_1^2=s_2^2,\quad s_1=\frac{r_2^2q}{r_1},\quad r_0^2s_1^2=1,\quad\text{and}\quad r_0s_2=q.$$

If r_0,r_1 and r_2 are given, we determine s_1 and s_2 by $s_1:=\frac{r_2^2q}{r_1}$ and $s_2:=\frac{q}{r_0}$. Then

$$\frac{s_1^2}{s_2^2}=\frac{r_2^4r_0^2}{r_1^2}=r_0^2s_1^2,$$

so that we obtain only the relation $r_2^4r_0^2=r_1^2$ for r_0,r_1,r_2. This completes the proof. ■

Remark 5.10 (i) From the proof of the preceding theorem, we see that if $q^2=1$, we obtain the particularly simple solution $r_0=r_1=r_2=1$, $s_1=s_2=q$.

(ii) For $\operatorname{char}\mathbb{K}=2$, the equation $q^2=1$ has the unique solution $q=1$, so that $A_q\cong\mathbb{K}[\mathbb{Z}^2]$, and the action of $\mathrm{GL}_2(\mathbb{Z})$ has a canonical lift to an action on A_q.

Problem 5.11 Does the sequence (5.2) always split? We have seen above, that this is true for $\Gamma=\mathbb{Z}^2$. If the answer is no, it would be of some interest to understand the cohomology groups $H^2(\mathrm{Aut}(\Gamma)_{[f]},\mathrm{Hom}(\Gamma,\mathbb{K}^\times))$ parametrizing the possible abelian extensions of $\mathrm{Aut}(\Gamma)_{[f]}$ by the module $\mathrm{Hom}(\Gamma,\mathbb{K}^\times)$.

Problem 5.12 Let $\lambda \in \mathrm{Alt}^2(\mathbb{Z}^n, Z)$, where Z is a cyclic group. Determine the structure of the group $\mathrm{Aut}(\mathbb{Z}^n, \lambda)$. It should have a semidirect product structure, where the normal subgroup is something like a Heisenberg group and the quotient is the automorphism group of $\mathbb{Z}^n/\operatorname{rad}(\lambda)$, endowed with the induced non-degenerate form. Can this group be described in a convenient way by generators and relations? Maybe the results in [15] can be used to deal with degenerate cocycles.

A The Group of Units If Γ Is Torsion Free

The following result is used in [20, Lemma 3.1] without reference. Here we provide a detailed proof.

Proposition A.1 *If the group Γ is torsion free and A a Γ-quantum torus, then $A^\times = A_h^\times$*, i.e., *each unit of A is graded.*

Proof Let $a \in A^\times$ be a unit and write $a = \sum_\gamma a_\gamma \delta_\gamma$ in terms of some graded basis. We do the same with its inverse $a^{-1} = \sum_\gamma (a^{-1})_\gamma \delta_\gamma$, and observe that the set $\operatorname{supp}(a) := \{\gamma \in \Gamma : a_\gamma \neq 0\}$ is finite. The same holds for $\operatorname{supp}(a^{-1})$, so that both sets generate a free subgroup F of Γ. Then $A_F := \operatorname{span}\{\delta_\gamma : \gamma \in F\}$ is an F-quantum torus with $a \in A_F^\times$. We may therefore assume that $\Gamma = \mathbb{Z}^d$ for some $d \in \mathbb{N}_0$.

We prove by induction on $k \in \{0, \ldots, d\}$ that the subalgebra

$$A_k := \operatorname{span}\{\delta_\gamma : \gamma \in \mathbb{Z}^k \times \{0\}\}$$

has no zero-divisors [21, Theorem 1.2] and that all its units are homogeneous. This holds trivially for $k = 0$.

Let $u_i := \delta_{e_i}$, where $e_1, \ldots, e_d$ is the canonical basis of $\mathbb{Z}^d$. We write $0 \neq x \in A$ as a finite sum $\sum_{k=k_0}^{k_1} x_k u_d^k$ with $x_k \in A_{d-1}$ and x_{k_0} and x_{k_1} non-zero. Likewise we write $0 \neq y \in A$ as $\sum_{m=m_0}^{m_1} y_m u_d^m$ with $y_m \in A_{d-1}$ and y_{m_0} and y_{m_1} non-zero. Then the lowest degree term with respect to u_d in xy is $x_{k_0} u_d^{k_0} y_{m_0} u_d^{m_0} = x_{k_0} \left(u_d^{k_0} y_{m_0} u_d^{-k_0} \right) u_d^{k_0+m_0}$, and the induction hypothesis implies $x_{k_0} u_d^{k_0} y_{m_0} u_d^{-k_0} \neq 0$ because conjugation with u_d preserves the subalgebra A_{d-1}. This implies that $xy \neq 0$.

Now assume that $x \in A$ is a unit and $y = x^{-1}$. Since A_{d-1} has no zero-divisors, $x_{k_0} u_d^{k_0} y_{m_0} u_d^{-k_0} \in A_{d-1} \setminus \{0\}$ leads to $k_0 + m_0 = 0$. A similar consideration for the highest order term implies $k_1 + m_1 = 0$, which leads to $k_0 = k_1$ and $m_0 = m_1$. Now we can argue by induction. ■

Corollary A.2 (**[20, Lemma 3.1]**) *If the group Γ is torsion free, then each automorphism of A is graded,* i.e., $\mathrm{Aut}(A) = \mathrm{Aut}_{\mathrm{gr}}(A)$. *See Definition* 5.1.

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References

[1] B. N. Allison, S. Berman, J. R. Faulkner, and A. Pianzola, *Realization of graded-simple algebras as loop algebras.* arXiv:math/RA.0511723.

[2] B. N. Allison, S. Azam, S. Berman, Y. Gao, and A. Pianzola, *Extended Affine Lie Algebras and Their Root Systems.* Memoirs of the Amer. Math. Soc. **126**(1997), no. 603.

[3] S. Berman, Y. Gao, and Y. S. Krylyuk, *Quantum tori and the structure of elliptic quasi-simple Lie algebras.* J. Funct. Anal. **135**(1996), no. 2, 339–389.

[4] F. Bonahon, and L. Xiaobo, *Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms.* arXiv:math.GT/0407086v3, 16.11.2004.

[5] B. A. Brenken, *Representations and automorphisms of the irrational rotation algebra.* Pacific J. Math. **111**(1984), no. 2, 257–282.

[6] K. S. Brown,*Cohomology of Groups.* Graduate Texts in Mathematics 87, Springer-Verlag, New York, 1982 .

[7] W. C. Brown, *Matrices over Commutative Rings.* Monographs and Textbooks in Pure and Applied Mathematics 169, Marcel Dekker, New York, 1993.

[8] C. de Concini, and C. Procesi, *Quantum groups.* In: $\mathcal{D}$-Modules, Representation Theory, and Quantum Groups. Lecture Notes in Math. 1565, Springer, Berlin, 1993, pp. 31–140.

[9] G. A. Elliott, *The diffeomorphism group of the irrational rotation C^*-algebra.* C. R. Math. Rep. Acad. Sci. Canada **8**(1986), no. 5, 329–334.

[10] L. Fuchs, *Infinite Abelian Groups. I.* Pure and Applied Math. 36, Academic Press, New York, 1970.

[11] P. Gille and A. Pianzola, *Galois cohomology and forms of algebras over Laurent polynomial rings*, Math. Ann. **338**(2007), no. 2, 497–543.

[12] J. M. Gracia-Bondía, J. C. Vasilly, and H. Figueroa, *Elements of Noncommutative Geometry.* Birkhäuser Boston, Boston, MA, 2001.

[13] P. de la Harpe, *Topics in Geometric Group Theory.* The University of Chicago Press, Chicago, 2000.

[14] N. J. S. Hughes, *The use of bilinear mappings in the classification of groups of class* 2. Proc. Amer. Math. Soc. **2**(1951), 742–747.

[15] R. S. Ismagilov, *The integral Heisenberg group as an infinite amalgam of commutative groups.* Math. Notes **74**(2003), no. 5, 630–636

[16] N. Jacobson, *Structure of Rings.* American Mathematical Society Colloquium Publications 37, American Mathematical Society, Providence, RI, 1956.

[17] E. Kirkman, C. Procesi, and L. Small, *A q-analog of the Virasoro algebra*, Comm. Alg. **22**(1994), no. 10, 3755–3774.

[18] S. Lang, *Algebra.* Third edition. Addison Wesley, London, 1993.

[19] M. Newman, *Integral Matrices.* Pure and Applied Mathematics 45, Academic Press, New York, 1972.

[20] J. M. Osborn and D. S. Passman, *Derivations of skew polynomial rings.* J. Algebra **176**(1995), 417–448.

[21] A. N. Panov, *Skew fields of twisted rational functions and the skew field of rational functions on* $\mathrm{GL}_q(n, \mathbb{K})$. St. Petersburg Math. J. **7**(1996), 129–143.

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