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A LATTICE POINT PROBLEM RELATED TO SETS CONTAINING NO l-TERM ARITHMETIC PROGRESSION

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In 1927 van der Waerden [6] proved that given positive integers k and l, there exists an integer W such that if $1, 2, \ldots, W$ are partitioned into k or fewer classes, then at least one class contains an l-term arithmetic progression (l-progression). Let $W(k, l)$, be the smallest such integer W. It would be of interest to find a reasonable upper estimate for $W(k, l)$, say one that could be written down. Efforts to do this have included the study of $r_l(x)$, the largest number of integers that can be chosen from $1, 2, \ldots, x$ so as not to include any l-progression. If one could find x such that $r_l(x) < x/k$, then it would follow that $W(k, l) \le x$. It is known (see [4], [5]) that $r_3(x) = o(x)$, $r_4(x) = o(x)$, and the conjecture that $r_l(x) = o(x)$ for all $l \ge 3$ (due to Szekeres) has stood since the mid-1930s. On the other hand it has been shown that $r_l(x) > x^a$ for any a, $0 < a < 1$. The best result in this direction is that of Rankin [2] who showed that if $l > 2^h$ (h a positive integer), $\epsilon > 0$, and

$$c = (h+1)2^{h/2} (\log 2)^{h/(h+1)}(1+\epsilon),$$

then

$$(1) \qquad r_l(x) > x^{1-c/(\log x)^{h/(h+1)}}$$

provided x is sufficiently large.

In this paper we consider a related geometrical problem and find estimates for a function similar to $r_l(x)$ arising therein. A good upper estimate for this new function would similarly yield one for $W(k, l)$.

Consider the numbers $0, 1, 2, \ldots, l^n - 1$ written in the scale of l, and regard the digits of each number, taken in the usual order, as the coordinates of a point in n-space—for a number having m digits with $m < n$, the first $n - m$ coordinates of the corresponding point shall be 0. For example, with $n = 5$, $l = 3$, the number 11 has the representation 102 in base 3 and corresponds to the point (0, 0, 1, 0, 2) in 5-space. These points are all the lattice points in the cube $o \le x_i \le l - 1$, $i = 1, 2, \ldots, n$, and we shall call this set of l^n points the l^n-cube. We shall call a set of l collinear points in the l^n-cube a path, and let $M(l, n)$ be the cardinality of the largest path-free subset of the l^n-cube.

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When we consider (5) below, it will be evident that the numbers corresponding to the points of a path form an l-progression, and hence

$$(2)\qquad r_l(l^n) \le M(l, n).$$

Thus (1) provides a lower estimate for $M(l, n)$, and the main result is the following larger estimate.

THEOREM. *Let $l \ge 3$ and $n \ge 1$ be given, and let $r_0 = [(n+1)/l]$. Then*

$$(3)\qquad M(l, n) \ge \binom{n}{r_0}(l-1)^{n-r_0}.$$

The cases $l=1, 2$ are trivial since $M(1, n)=0$ and $M(2, n)=1$. For purposes of comparison with (1) we shall show later that (3) implies

$$(4)\qquad M(l, n) > \frac{1}{e^{3/2}\sqrt{2\pi}}\frac{l^{n+1}}{\sqrt{n(l-1)}}$$

provided $n > \max\{2(l-2), (l+8)/2\}$.

Proof of the theorem. If the l points of a path are $(x_1^{(i)}, x_2^{(i)}, \dots, x_n^{(i)})$, $i=1, 2, \dots, l$, and these n-tuples are written in a column, then the column l-tuples $(x_j^{(1)}, x_j^{(2)}, \dots, x_j^{(l)})'$, $j=1, 2, \dots, n$ are among the following $l+2$ columns:

$$(5)\qquad \begin{matrix} 0 & l-1 & 0 & 1 & 2 & \dots & l-1 \\ 1 & l-2 & 0 & 1 & 2 & \dots & l-1 \\ 2 & l-3 & 0 & 1 & 2 & \dots & l-1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots \\ l-1 & 0 & 0 & 1 & 2 & \dots & l-1 \end{matrix}$$

For the vector differences between consecutive points of a path must all be equal, and since there are only $l-1$ pairs of consecutive points the components of these difference vectors must be 0, 1, or -1. At least one of the first two columns in (5) must be included, for otherwise the l points would all be the same. Therefore at least one of the endpoints of a path contains more zeros among its coordinates than does any of the intermediate points. Hence a set of points that all have the same number of zeros among their coordinates will contain no path. The number of points of the l^n-cube each having exactly r zeros among its coordinates is $\binom{n}{r}(l-1)^{n-r}$, and this quantity is maximal for $r=r_0$. Hence the theorem.

By a further argument the estimates (3) and (4) can be increased by the factor $[(l-1)/2](1-\epsilon)$. This factor is obtained by observing that for $l \ge 5$ we can select more points for a path-free set than just those having r_0 zeros. In fact, if for each $i=0, 1, \dots, [(l-3)/2]$, R_i is the set of all points in the l^n-cube each of which has exactly r_0 i's among its coordinates, then the set of all points each of which is in

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exactly one of these R_i contains no path. The cardinality of this set of points is almost the sum of the cardinalities of the R_i. Since the details are rather lengthy for the improvement that they yield, we shall omit them.

To derive (4) we employ the inequality

$$e^{1/(12h+1)} < \frac{h!}{\sqrt{2\pi h}\,(h/e)^h} < e^{1/12h}, \tag{6}$$

which holds for any positive integer h (see Robbins [3]), and also

$$(n+2)/l-1 \le r_0 \le (n+1)/l. \tag{7}$$

Using (6) and (7) we find that

$$\binom{n}{r_0}(l-1)^{n-r_0} > \frac{l^{n+1}}{\sqrt{2\pi n(l-1)}}\cdot\frac{\exp\{1/(12n+1)-1/(12(n-r_0))-1/(12r_0)\}}{\{1+(l-2)/(n(l-1))\}^{n-r_0+1/2}\{1+1/n\}^{r_0+1/2}}.$$

Now, since for positive h and x, $(1+x/h)^h < e^x$, the second factor on the right exceeds

$$\exp\left\{\frac{1}{12n+1}-\frac{1}{12(n-r_0)}-\frac{1}{12r_0}-\frac{l-2}{l-1}\left(1-\frac{r_0-\frac{1}{2}}{n}\right)-\frac{r_0+\frac{1}{2}}{n}\right\},$$

and by (7) this can be shown to exceed

$$\exp\left\{-1+\frac{1}{l}-\frac{1}{n}\left[\frac{l-2}{l-1}\left(\frac{3}{2}-\frac{2}{l}\right)+\frac{1}{l}+\frac{1}{2}\right.\right.$$
$$\left.\left.+\frac{l}{12}\left(\frac{1}{l-1-1/n}+\frac{1}{1-(l-2)/n}-\frac{1}{l+l/(12n)}\right)\right]\right\}.$$

This in turn exceeds $e^{-3/2}$ if $n > \max\{2(l-2), (l+8)/2\}$. Thus we have (4), which can readily be shown to exceed the estimate for $r_l(l^n)$ provided by (1).

Upper estimates for $M(l, n)$ would be of interest, as they are in the case of $r_l(x)$, because of their possible use in estimating the van der Waerden numbers $W(k, l)$. If, given k and l, we could find n such that $M(l, n) < l^n/k$, then since a path corresponds to an l-progression, we would have $W(k, l) \le l^n$. However, since $M(l, n)$ exceeds $r_l(l^n)$, it may be harder to find good upper estimates for it. Be that as it may, corresponding to Szekeres's conjecture

$$r_l(x) = o(x) \quad (x \to \infty), \tag{8}$$

we make the conjecture

$$M(l, n) = o(l^n) \quad (n \to \infty). \tag{9}$$

This conjecture seems as reasonable as (8) in view of the results of Hales and Jewett [1] which say for an l^n-cube what van der Waerden's theorem says for a set $\{1, 2, \ldots, W\}$: if n is sufficiently large, then in any partition of the l^n-cube into k classes, at least one class contains a path.

The only upper estimate we have for $M(l, n)$ is the rather weak

$$M(l, n) \leq l^{n-1}(l-1). \tag{10}$$

To see this we observe that $M(l, 1) = l-1$, and since an l^n-cube consists of l disjoint l^{n-1}-cubes,

$$M(l, n) \leq lM(l, n-1). \tag{11}$$

We have equality in (10) in the case $n=2$. For, in a square lattice, it is necessary to remove at least one point from each horizontal and vertical path, and removing l diagonal points suffices, to obtain a set free of paths. (If l is even, the points removed cannot all be from the same diagonal.) Hence $M(l, 2) = l^2 - l$.

In closing we consider the case that n is fixed rather than l. We easily find

$$M(l, n) \sim l^n \quad (l \to \infty).$$

For, since there are no l collinear points in an $(l-1)^n$-cube, $M(l, n) \geq (l-1)^n$. Hence from (10)

$$\left(\frac{l-1}{l}\right)^n \leq \frac{M(l, n)}{l^n} \leq \frac{l-1}{l}.$$

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