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SOME SPECTRAL PROPERTIES OF AN INTEGRAL OPERATOR IN POTENTIAL THEORY

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1. Introduction

In [7] Plemelj established some fundamental results in two- and three-dimensional potential theory about the eigenvalues of both the double layer potential operator and its adjoint, the normal derivative of the single layer potential operator. In [3] Blumenfeld and Mayer established some additional results concerning the eigenvalues of these integral operators in the case of $\mathbb{R}^2$. The spectral properties established by Plemelj [7] and by Blumenfeld and Mayer [3] have had a profound effect in the area of integral equation methods in scattering and potential theory in both $\mathbb{R}^2$ and $\mathbb{R}^3$.

Some applications that have been made of these results may be found in Colton and Kress [4]. A complete list, however, of all the different uses that have been made of the efforts of Plemelj [7] and of Blumenfeld and Mayer [3] would be a formidable task.

This paper arose from the author's long interest in the spectral properties of the double layer potential integral operator in both $\mathbb{R}^2$ and $\mathbb{R}^3$. For sufficiently smooth boundaries, it can be shown that for both $\mathbb{R}^2$ and $\mathbb{R}^3$ the point 0 lies in the spectrum of the integral operator. A fundamental question is what part of the spectrum does 0 lie in?

For the case of $\mathbb{R}^3$ some partial results on this topic are known. If the underlying boundary is either a sphere or a prolate spheroid, it can be shown (see [1]) that the eigenvalues of the double layer potential integral operator lie in the interval $[-1,0)$. Furthermore, it can be established that for both geometries 0 lies in the continuous spectrum of the integral operator. As for other geometries, the spectral classification of 0 remains an open question.

For the case of $\mathbb{R}^2$, it turns out that the situation is somewhat different for the double layer potential operator, which in this paper we denote by K and define in equation (2.1). After encountering some serious difficulties in an attempt to establish a general theorem about which part of the spectrum of K the point 0 lies in, the author looked at some specific examples, namely, the circle and the ellipse. For the case of the circle, the point 0 lies in the point spectrum of K. For the case of an ellipse, 0 lies in the continuous spectrum of K. Consequently, these examples demonstrate that 0 does not always lie in the same component of the spectrum of K for all sufficiently smooth boundaries.

In the next section we give some notation, definitions and basic results which we shall use. In Section 3, we consider the case of a circle and compute the spectrum of K. It is shown that 0 is an eigenvalue and moreover, that it has infinite geometric multiplicity.

In Section 4 we consider the case of an ellipse. There we compute all the eigenvalues for K and establish that every eigenvalue has a geometric multiplicity of 1. Furthermore, we prove that the continuous spectrum of K is equal to the set $\{0\}$.

2. Notation, definitions and basic results

In this section we give our notation and state some definitions and results which we shall require. Let D_i be a bounded, simply connected domain in $\mathbb{R}^2$ containing the origin with a C^2 boundary ∂D, and let D_e denote the region exterior to $\bar{D}_i$. Let $\hat{n}$ denote a unit normal directed out of D_i. Let x and y denote typical points in $\mathbb{R}^2$.

We now define the following integral operators of potential theory:

$$(K\psi)(x):=\frac{1}{\pi}\int_{\partial D}\psi(y)\frac{\partial}{\partial n(y)}\ln\frac{1}{|x-y|}ds_y, \quad x\in\partial D, \tag{2.1}$$

$$(D\psi)(x):=\frac{1}{\pi}\int_{\partial D}\psi(y)\frac{\partial}{\partial n(y)}\ln\frac{1}{|x-y|}ds_y, \quad x\in\mathbb{R}^2\backslash\partial D. \tag{2.2}$$

Here it is understood that the integration is taken with respect to arc length.

A standard result in two-dimensional potential theory (e.g. see [**9**, pp. 78–80]) states that for closed smooth curves ∂D

$$\lim_{\substack{x\to y\\ x,y\in\partial D}}\frac{\partial}{\partial n(y)}\ln\frac{1}{|x-y|}=-\frac{1}{2}\kappa(y), \tag{2.3}$$

where $\kappa(y)$ denotes the curvature of ∂D at y. Consequently, unlike the weakly singular nature of the double layer kernel in $\mathbb{R}^3$, the double layer kernel in $\mathbb{R}^2$ is continuous for all points x and y on ∂D, including when $x=y$.

Let $C(\partial D)$ denote the Banach space of complex-valued, continuous functions defined on ∂D equipped with the maximum norm. Since the integral operator K has a continuous kernel, it follows that K is a compact linear operator on $C(\partial D)$ (see [**4**, Theorem 1.10]).

Let $+$ and $-$ denote the limits obtained for the double layer potential $(D\psi)(x)$ by approaching the boundary ∂D from D_e and D_i, respectively, that is

$$(D_+u)(x)=\lim_{\substack{x_e\to x\\ x_e\in D_e}}(Du)(x_e), \quad x\in\partial D, \tag{2.4}$$

$$(D_-u)(x)=\lim_{\substack{x_i\to x\\ x_i\in D_i}}(Du)(x_i), \quad x\in\partial D. \tag{2.5}$$

It can be shown (e.g. see [**5**, p. 49] or [**8**, p. 392]) that

$$(D_\pm u)(x)=(Ku)(x)\pm u(x), \quad x\in\partial D. \tag{2.6}$$

Let A denote any bounded linear operator mapping a Banach space X into itself. By an eigenvalue of A we mean a complex number λ such that the nullspace $N(\lambda I-A)\neq\{0\}$ where I denotes the identity operator. Let $\rho(A)$ denote the resolvent set of A. Let $\sigma(A)$ denote the spectrum of A. Let $\sigma_C(A)$, $\sigma_P(A)$, and $\sigma_R(A)$ denote the continuous spectrum, point spectrum, and residual spectrum of A, respectively. It is known (e.g. see [2, Chapter 18] or [4, Theorem 1.34]) that if X is an infinite dimensional Banach space and if A is a compact linear operator then $\lambda=0$ lies in $\sigma(A)$ and $\sigma(A)\backslash\{0\}$ consists of at most a countable number of eigenvalues, with $\lambda=0$ the only possible limit point.

It can be shown (see [3]) that the eigenvalues of the integral operator K, defined in equation (2.1), lie in the interval $[-1,1)$ and are symmetric with respect to the origin. The only exception is the eigenvalue -1 corresponding to constant eigenfunctions.

Finally, we shall denote the set of positive integers by $\mathbb{N}$.

3. The Circle

In this section ∂D is taken to be a circle of radius a. Under this assumption, we compute the spectrum of the integral operator K and also determine the spectral properties of the point $\lambda=0$ for K.

With respect to polar coordinates, let the points x and y be given by (r_x,ϕ_x) and (r,ϕ), respectively. Then

$$\ln\frac{1}{|x-y|}=-\frac{1}{2}\ln\left[r_x^2+r^2-2r_xr\cos(\phi-\phi_x)\right]. \tag{3.1}$$

From equation (3.1), for $x,y\in\partial D$, we have the following known result (e.g. see [5, p. 52])

$$\begin{aligned}\frac{\partial}{\partial n(y)}\ln\frac{1}{|x-y|}&=\frac{\partial}{\partial r}\left\{-\frac{1}{2}\ln\left[a^2+r^2-2ar\cos(\phi-\phi_x)\right]\right\}_{r=a}\\&=-\frac{1}{2a}.\end{aligned} \tag{3.2}$$

Before proceeding to the stated purposes of this section, it is worthy of note to examine the result in equation (3.2) in the context of equation (2.3). It is a well known result in differential geometry that the curvature of a circle of radius a is $1/a$. Consequently, the results in equations (2.3) and (3.2) are seen to be compatible.

From equations (2.1) and (3.2) it follows that

$$K\psi(x)=-\frac{1}{2\pi}\int_0^{2\pi}\psi(a,\phi)\,d\phi. \tag{3.3}$$

Letting ψ equal 1, $\cos m\phi$, and $\sin m\phi, m\in\mathbb{N}$, respectively, in equation (3.3) we have

$$K(1)=-1, \tag{3.4}$$

$$K(\cos m\phi)=K(\sin m\phi)=0. \tag{3.5}$$

Thus $\lambda=-1$ is an eigenvalue of K with corresponding eigenfunction 1, and $\lambda=0$ is also an eigenvalue of K with corresponding eigenfunctions $\{\cos m\phi, \sin m\phi : m\in\mathbb{N}\}$. From the completeness of the orthogonal set of eigenfunctions $\{1, \cos m\phi, \sin m\phi : m\in\mathbb{N}\}$ in $L^2[0,2\pi]$, it follows, by a similar argument as used in [1], that with respect to the underlying Banach space $C(\partial D)$

$$\sigma_P(K)=\{-1,0\}. \tag{3.6}$$

That is, $\lambda=-1$ and $\lambda=0$ are the only eigenvalues of K for the case when ∂D is a circle.

In view of the fact that K is a compact linear operator on $C(\partial D)$, it follows that if $\lambda\neq 0$ then either $\lambda\in\rho(K)$ or $\lambda\in\sigma_P(K)$. Consequently, $\lambda=-1$ and $\lambda=0$ are the only elements of $\sigma(K)$. Furthermore, since $\cos m\phi$ and $\sin m\phi$, $m\in\mathbb{N}$, are all eigenfunctions of K corresponding to $\lambda=0$, it follows that

$$\dim N(K)=\infty. \tag{3.7}$$

4. The ellipse

The elliptical coordinates (μ,ϕ) are related to the rectangular Cartesian coordinates (y_1, y_2) by the transformation

$$y_1=\frac{1}{2}c\cosh\mu\cos\phi,$$

$$y_2=\frac{1}{2}c\sinh\mu\sin\phi, \tag{4.1}$$

where $0\leqq\mu<\infty$, $0\leqq\phi\leqq 2\pi$. The closed curves corresponding to $\mu=$constant, $0\leqq\phi\leqq 2\pi$ are confocal ellipses of interfocal distance c, eccentricity $e=(\cosh\mu)^{-1}$, major axis $c\cosh\mu$ and minor axis $c\sinh\mu$. The limiting case $\mu=0$ represents the line segment between the foci.

In this section ∂D will denote the ellipse corresponding to $\mu=b$, $0\leqq\phi\leqq 2\pi$, where b is some constant. To avoid the degenerate case, we will assume that $b>0$.

In terms of elliptical coordinates it can be shown that the gradient of a scalar function $\Phi(\mu,\phi)$ is given by

$$\nabla\Phi(\mu,\phi)=\frac{2}{c\tau}\left(\frac{\partial\Phi}{\partial\mu}\hat{e}_\mu+\frac{\partial\Phi}{\partial\phi}\hat{e}_\phi\right), \tag{4.2}$$

where

$$\tau := [\cosh^2\mu\sin^2\phi+\sinh^2\mu\cos^2\phi]^{1/2}, \tag{4.3}$$

and where $\hat{e}_\mu$ and $\hat{e}_\phi$ denote the orthonormal vectors

$$\hat{e}_\mu := (\sinh \mu \cos \phi\, \hat{i} + \cosh \mu \sin \phi\, \hat{j})/\tau,$$

$$\hat{e}_\phi := (-\cosh \mu \sin \phi\, \hat{i} + \sinh \mu \cos \phi\, \hat{j})/\tau. \tag{4.4}$$

Furthermore, it can be shown that the element of arc length ds is given by

$$ds = \frac{c}{2}\tau\, d\phi. \tag{4.5}$$

From [6, p. 1202] we have

$$\ln \frac{1}{|x-y|} = -\left(\mu_> + \ln \frac{c}{4}\right) + \sum_{n=1}^{\infty} \frac{2}{n} e^{-n\mu_>} [\cosh n\mu_< \cos n\phi \cos n\phi_x$$

$$+ \sinh n\mu_< \sin n\phi \sin n\phi_x], \tag{4.6}$$

where $\mu_> = \max\{\mu_x, \mu_y\}$, $\mu_< = \min\{\mu_x, \mu_y\}$, and (μ_x, ϕ_x) and (μ_y, ϕ) denote the elliptical coordinates of the points x and y, respectively.

At the point $(b, \phi) \in \partial D$ the unit tangent vector $\hat{T}$ and the outer unit normal vector $\hat{n}$ are given, respectively, by

$$\hat{T} = \hat{e}_\phi, \quad \hat{n} = \hat{e}_\mu. \tag{4.7}$$

For $y = (b, \phi) \in \partial D$ and $x = (\mu_x \phi_x) \in D_i$ it follows from equations (4.2) and (4.6) that

$$\frac{\partial}{\partial n(y)} \ln \frac{1}{|x-y|} = \frac{-2}{c\tau}\left\{1 + 2\sum_{n=1}^{\infty} e^{-nb} [\cosh n\mu_x \cos n\phi \cos n\phi_x\right.$$

$$\left. + \sinh n\mu_x \sin n\phi \sin n\phi_x]\right\}. \tag{4.8}$$

Consequently, from equations (2.4), (4.5) and (4.8) we have

$$D\psi(x) = -\frac{1}{\pi}\int_0^{2\pi} \psi(\mu, \phi)\left\{1 + 2\sum_{n=1}^{\infty} e^{-nb} [\cosh n\mu_x \cos n\phi \cos n\phi_x\right.$$

$$\left. + \sinh n\mu_x \sin n\phi \sin n\phi_x]\right\} d\phi, \quad x \in D_i. \tag{4.9}$$

Letting ψ equal 1, $\cos m\phi$, and $\sin m\phi$, where $m \in \mathbb{N}$, respectively, in equation (4.9), then using the orthogonality of the trigonometric functions, and finally letting

$\mu_x \to b$, we have from equation (2.6)

$$K(1) = -1, \tag{4.10}$$

$$K(\cos m\phi) = -e^{-2mb} \cos m\phi, \quad m \in \mathbb{N}, \tag{4.11}$$

$$K(\sin m\phi) = e^{-2mb} \sin m\phi, \quad \mathrm{m} \in \mathbb{N}. \tag{4.12}$$

Thus it is seen that $-1, -\mathrm{e}^{-2mb}$, and $e^{-2mb}, m \in \mathbb{N}$, are eigenvalues of K with corresponding eigenfunctions 1, $\cos m\phi$, and $\sin m\phi$, respectively. From the completeness of the orthogonal set of eigenfunctions $\{1, \cos m\phi, \sin m\phi : m \in \mathbb{N}\}$ in $L^2[0, 2\pi]$, it follows, by an argument similar to one used in [1], that with respect to the underlying Banach space $C(\partial D)$

$$\sigma_P(K) = \{-1, \pm e^{-2mb} : m \in \mathbb{N}\}. \tag{4.13}$$

That is, the above eigenvalues are the only eigenvalues of K. Consequently, unlike the situation when ∂D is a circle,

$$0 \notin \sigma_P(K), \tag{4.14}$$

when ∂D is an ellipse. Furthermore, it is seen that

$$\dim N(-I - K) = \dim N(\pm e^{-2mb} I - K) = 1 \tag{4.15}$$

for each $m \in \mathbb{N}$. Therefore each eigenvalue has a geometric multiplicity of 1.

To complete the analysis of this section we establish the following result which determines the spectral nature of the point $\lambda = 0$.

Theorem 4.1 *Let ∂D denote the ellipse corresponding to $\mu = b, 0 \leqq \phi \leqq 2\pi$, where b is some positive constant. Then*

$$\{0\} = \sigma_C(K).$$

Proof. Since K is a compact linear operator on $C(\partial D)$,

$$0 \in \sigma(K). \tag{4.16}$$

Furthermore, since each eigenfunction must necessarily lie in the range of K, $R(K)$, we have

$$\{1, \cos m\phi, \sin m\phi : m \in \mathbb{N}\} \subset R(K). \tag{4.17}$$

It follows that $R(K)$ is dense in $C(\partial D)$. Consequently, from equations (4.14) and (4.16) it follows that

$$0 \in \sigma_C(K). \tag{4.18}$$

Finally, by using the fact that K is a compact linear operator on $C(\partial D)$, we have from equation (4.18)

$$\{0\} = \sigma_C(K). \qquad \blacksquare$$

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