A Theorem on the Contact of Circles leading up to the Theorems of Feuerbach and Hart.

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1. According to Feuerbach's theorem, *the inscribed and the three escribed circles of a plane triangle are all touched by a circle.*

Hart's Theorem extends the proposition to a spherical triangle, or, which comes to the same thing, to a plane triangle formed by three circular arcs.

The purpose of the present paper is to call attention to a theorem closely related to these two, but more fundamental or less highly specialised than either of them; and to deduce the celebrated results of Feuerbach and Hart from this new theorem.

2. *Theorem I.*

If the line EF meets the sides AC, AB of a plane triangle ABC in E, F, and if EF be either parallel to BC or antiparallel to BC (with respect to the sides AB and AC), then through E and F a circle can be drawn to touch the two escribed circles opposite B and C, and through E and F a circle can be drawn to touch the inscribed circle and the escribed circle opposite A. In both cases the tangent circle through E and F belongs to the same system as the common tangent BC.

Let the escribed circle opposite B touch the sides a, b, c at x_2, y_2, z_2, and that opposite C at x_3, y_3, z_3.

Let FE be $\parallel BC$.

We shall prove that a circle can be passed through E and F to touch the circles $x_2y_2z_2$ and $x_3y_3z_3$. To prove this, we show that Casey's well-known condition that four circles should be tangible by a circle is fulfilled for the circles $x_2y_2z_2$ and $x_3y_3z_3$ with the point circles E and F.

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If these four circles in order are called the circles 1, 2, 3, 4, then Casey's condition is that, for one choice of the ambiguous signs, we have

$$12 \,.\, 34 \pm 13 \,.\, 24 \pm 14 \,.\, 23 = 0,$$

where 12, 34 etc. denote the lengths of the common tangents of circles 1 and 2, circles 3 and 4, etc.

In the present case, in accordance with the last sentence in the theorem as stated above, 12 must be taken to be the length of the *direct* common tangent of 1 and 2.

Put $$AE = y,\ AC = z.$$

Then
$$12 = x_3 x_2 = b + c, \qquad 34 = EF,$$
$$13 = Ey_2 = \pm\{y - (s - c)\}, \qquad 24 = Fz_3 = \pm\{z - (s - b)\},$$
$$14 = Fz_2 = \pm\{z + (s - c)\}, \qquad 23 = Ey_3 = \pm\{y + (s - b)\}.$$

Now
$$(y + \overline{s-b})(z + \overline{s-c}) - (y - \overline{s-c})(z - \overline{s-b})$$
$$= y(\overline{s-c} + \overline{s-b}) + z(\overline{s-b} + \overline{s-c})$$
$$= a(y + z).$$

Also $$\frac{EF}{a} = \frac{y}{b} = \frac{z}{c},$$

so that $$EF(b + c) = a(y + z).$$

Hence Casey's condition holds.

If EF were antiparallel to BC, *i.e.* if B, C, E, F were concyclic, we would have

$$\frac{EF}{a} = \frac{y}{c} = \frac{z}{b},$$

so that $$EF(b + c) = a(y + z),$$

as before.

The first part of Theorem I. is thus proved, and the second part admits of similar proof.

3. *Feuerbach's Theorem.*

Take EF antiparallel to BC as to AB and AC. The circle through E and F touching the two escribed circles opposite B and

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C, now shown to exist, is uniquely determinate, for it is orthogonal to a certain fixed circle, viz. that circle of inversion of the two escribed circles which has its centre at their external centre of similitude, a point on BC.

The circle through E and F touching the inscribed circle and the escribed circle opposite A is similarly uniquely determinate, being orthogonal to that (imaginary) circle of inversion of the two circles which has its centre at their internal centre of similitude, a point on BC.

The two circles through E and F are in general distinct, but we shall now show that there is one position of EF for which they are identical. In other words, we have to show that a circle exists, orthogonal to both the circles of inversion just mentioned, and meeting AB and AC at points F and E such that B, C, E, F are concyclic.

Now for a circle to be orthogonal to two circles with centres on BC is the same thing as for it to pass through two definite points on BC, viz. the two limiting points L_1 and L_2 of the two circles.

Take the unique point S in BC such that

$$SL_1 \,.\, SL_2 = SB \,.\, SC.$$

Draw SEF antiparallel to BC to cut AC, AB in E, F.

Then the circle L_1L_2EF is the circle touching all four scribed circles.

It can be identified in various ways with the nine-point circle.

Another line of reasoning is interesting.

Let the circle through F and E tangent to the two escribed circles meet AB again in F_1; and let the circle through F and E tangent to the inscribed and the third escribed circle meet AB again in F_2.

When F is given, the circle FEF_1 is uniquely given, so that F_1 is uniquely given; also when F_1 is given, the circle FEF_1 is uniquely given, for it will pass through E_1 on AC where F_1E_1 is *parallel* to BC and it is orthogonal to a fixed circle. Hence there is a homographic relation connecting F and F_1, similarly one connecting F and F_2, and therefore one connecting F_1 and F_2. The homographic divisions F_1 and F_2 have two double points, of which one is B corresponding to F at infinity. If the second double point is F_0 and F_0E_0 is parallel to BC, a circle through F_0 and E_0 exists which touches all four scribed circles.

4. *Extension to a circular triangle.*

So far as I know, it is not possible to deduce Hart's Theorem from Feuerbach's by any of the ordinary methods of transformation. It so happens, however, that Theorem I. can be very easily extended to the general case in which the right lines BC, CA, AB are replaced by circles.

Starting from a rectilineal triangle ABC, let a circle tangent to the escribed circles opposite B and C cut AB in B', B'' and AC in C', C'' so that $B'C' \parallel BC$; and let another circle tangent to the same escribed circles cut AB in F, F_1 and AC in E, E_1 so that B, C, E, F are concyclic.

Then, obviously, B', C', E, F are concyclic, as also, we may note, are B'', C'', E_1, F_1.

If, then, we consider the triangle formed by the right lines AB', AC' and the circular arc $B'C'$, and if E, F are points on AC', AB' such that B', C', E, F are concyclic, then a circle can be drawn through E and F touching the two escribed circles opposite B' and C' of the triangle $AB'C'$, and belonging to the same system as $B'C'$.

Similarly with the inscribed circle and the escribed circle opposite A in such a triangle as $AB'C'$.

Practically the same reasoning as in Art. 3 can now be applied to deduce the extension of Feuerbach's Theorem to such a triangle as $AB'C'$. For we prove, as in Art. 3, that a circle passing through two definite points L_1', L_2' on the circle $B'C'$ will touch all four scribed circles of the triangle $AB'C'$, provided we can find two points E', F' in AC', AB' such that B', C', E', F' are concyclic, and also L_1', L_2', E', F'. But to find such points E', F' we have only to draw through the common point S' of the right lines $L_1'L_2'$ and $B'C'$ a line $S'E'F'$ antiparallel to $B'C'$ as to AB', AC'. The second line of reasoning in Art. 3 applies equally well.

Finally, since two intersecting circles can always be inverted into right lines by inversion from one of their common points, the general case of a triangle formed by three circular arcs can be reduced immediately to the case before us of a triangle formed by two right lines and one circular arc.

This proves Hart's Theorem.

5. The extension of Theorem I. at which we arrive in Art. 4 may be put thus:—

Theorem II.—If we take two circles of one system touching two given circles, and two circles of the other system touching the same two given circles, then the eight points of intersection of the two circles of the first system and the two circles of the second system lie on other two circles.

Two methods of proof of this theorem, independent of each other and of the former method, will now be indicated.

First Method.

Let the circles BC, EF of the first system, and the circles BF, CE of the second system form the curvilinear quadrilateral $BCEF$ where B, C, E, F are concyclic. Then, dealing with angles between circular arcs, we have

$$\angle B + \angle E = \angle C + \angle F,$$

as we see at once by forming the rectilineal quadrilateral $BCEF$.

Thus $$\angle B - \angle C = \angle F - \angle E.$$

Hence if we consider EF as a variable circle of the first system cutting two fixed circles of the second system, the difference of the angles which EF makes with the two fixed circles is constant.

Suppose now that a circle of the first system is defined by a parameter α, and one of the second system by a parameter β, then the angle between the circles α and β is a function of α and β. The above relation shows that this function has the form of the sum of a function of α alone and a function of β alone.

We shall now prove this somewhat remarkable result independently.

Let the two fixed circles be inverted into concentric circles, centre O, radii a and b, $b > a$.

A tangent circle of one system, centre P, has its radius $= \frac{1}{2}(b-a)$ and $OP = \frac{1}{2}(b+a)$.

A tangent circle of the other system, centre Q, has its radius $= \frac{1}{2}(b+a)$ and $OQ = \frac{1}{2}(b-a)$.

If these two tangent circles meet at X, the triangles QOP and QXP have their sides equal.

Hence the angle between the circles

$$= \angle QXP$$
$$= \angle QOP$$
$$= \angle AOP - \angle AOQ,$$

where OA is any fixed line through O.

This proves the result wanted.

Theorem I. might now be based on this.

6. *Second method for Theorem II.*

For this method, which is analytical, it is convenient to take the circles on a sphere.

Let A and B be two fixed circles on a sphere; $A = 0$, $B = 0$ the (linear) equations of the planes of the circles in any point coordinates; $\Sigma = 0$ the (quadric) equation of the sphere. Through A and B two quadric cones pass, and the sections of the sphere by tangent planes to these two cones are the circles of the two systems touching A and B.

Let $S_1 = 0$, $S_2 = 0$ be the equations of the cones. If $P_1 = 0$, $Q_1 = 0$ are two tangent planes to S_1, and $P_2 = 0$, $Q_2 = 0$ to S_2, we have, as in plane conics, identities of the form

$$S_1 + P_1Q_1 + \alpha_1^2 = 0,$$
$$S_2 + P_2Q_2 + \alpha_2^2 = 0,$$

where α_1 is the plane through the generators along which P_1 and Q_1 touch S_1; and similarly with α_2.

Again, since Σ passes through the intersection of S_1 and S_2, we have an identity which we may take to be

$$\Sigma + S_1 - S_2 = 0.$$

Eliminating S_1 and S_2 we have finally the identity

$$\Sigma - P_1Q_1 + P_2Q_2 + \alpha_2^2 - \alpha_1^2 = 0.$$

Thus the points at which $\Sigma = 0$, $P_1Q_1 = 0$ and $P_2Q_2 = 0$ lie on the planes $\alpha_1 \pm \alpha_2 = 0$.

This is Theorem II.

Since the planes $\alpha_1 \pm \alpha_2$ are coaxial with the planes α_1 and α_2, we see by this method that the two circles on which the eight points of Theorem II. lie are coaxial with the two circles, either of which passes through the four points of contact of two circles of one system.