ANZIAM J. 58(2016), 10–32
doi:10.1017/S1446181116000092

THE COST–TIME CURVE FOR AN OPTIMAL TRAIN JOURNEY ON LEVEL TRACK

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(Received 21 August, 2015; accepted 24 February, 2016; first published online 1 July 2016)

Abstract

In this paper, we show that the cost of an optimal train journey on level track over a fixed distance is a strictly decreasing and strictly convex function of journey time. The precise structure of the cost–time curves for individual trains is an important consideration in the design of energy-efficient timetables on complex rail networks. The development of optimal timetables for busy metropolitan lines can be considered as a two-stage process. The first stage seeks to find optimal transit times for each individual journey segment subject to the usual trip-time, dwell-time, headway and connection constraints in such a way that the total energy consumption over all proposed journeys is minimized. The second stage adjusts the arrival and departure times for each journey while preserving the individual segment times and the overall journey times, in order to best synchronize the collective movement of trains through the network and thereby maximize recovery of energy from regenerative braking. The precise nature of the cost–time curve is a critical component in the first stage of the optimization.

2010 *Mathematics subject classification*: primary 49K15; secondary 26A48, 26A51.

Keywords and phrases: optimal train control, cost–time curves, energy efficiency.

1. Introduction

1.1. Motivation and main contribution We consider the properties of the cost–time curves for an optimal train journey over a fixed distance on level track. Our study is motivated by the consideration of a typical metropolitan train journey which stops at a succession of intermediate stations while travelling from an initial station to a final station. It is a common practice to seek a driving strategy that minimizes the mechanical energy required to drive the train over each section of track between consecutive stops, subject to the associated section-specific maximum allowed running time. In order to minimize overall energy consumption for the entire journey, the

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sectional running times must be designed appropriately. To do this, it is necessary to understand the structure of the cost–time curves for the optimal driving strategies on each section of the track.

Our main contribution is to show that for a fixed-length journey on level track, the cost function for the optimal strategy is a strictly decreasing and strictly convex function of journey time. We shall refer to the corresponding graph of cost against journey time as the *cost–time curve*. Our contribution is aptly described by the remarkable formula

$$\frac{dJ}{dT} = -\psi(V_\mu) < 0, \tag{1.1}$$

where $V_\mu > 0$ is the optimal driving speed, T is the time taken for the journey and J is the cost of the journey. The function $\psi(v) = v^2 r'(v)$ is a nonnegative strictly increasing function that depends only on the resistive acceleration per unit mass $r(v)$ at speed v. The research in this paper is directly relevant to recent work on train separation [4, 5] where a *leading* train and *following* train are not permitted to occupy the same section of track at the same time, but are both required to use energy-efficient driving strategies. It is also relevant to unsolved problems of timetable optimization on busy metropolitan rail networks [8, 16, 17], where it is desirable to reduce energy consumption while retaining acceptable journey times.

1.2. Previous work For a given section of track and a fixed feasible journey time, the structure of the optimal driving strategy for a train is well understood. For details, we refer the reader to key papers by Albrecht et al. [1–3], Cheng and Howlett [6], Howlett et al. [9–12, 14], Khmelnitsky [15] and Liu and Golovitcher [18] and the book by Howlett and Pudney [13]. For each given journey time, the optimal strategy is characterized by a uniquely defined *optimal driving speed*. In general, the archetypal optimal strategy on a given section of the track for a journey that starts and finishes at rest with no intermediate stopping points is a *maximum acceleration–speedhold at the optimal driving speed with partial acceleration–coast–maximum brake* strategy, except that the *speedhold with partial acceleration* phase must be interrupted by phases of *maximum acceleration* to traverse steep uphill sections and phases of *coast* to traverse steep downhill sections. If energy can be recovered from regenerative braking, then there is also a uniquely defined *optimal regenerative braking speed* and, for sufficiently steep downhill sections, it is also necessary to interrupt the *coast* phase by switching to a phase of *speedhold at the optimal regenerative braking speed with partial brake*. For an optimal journey with several *speedhold* phases, it is important to understand that each phase of *speedhold with partial acceleration* uses the same *optimal driving speed* and that each phase of *speedhold with partial brake* uses the same *optimal regenerative braking speed* (see Albrecht et al. [1, 2] for more details). When speed limits are imposed, it is generally true that the restricted optimal strategy is similar in form to the corresponding unrestricted optimal strategy except that speed limits are followed whenever the speed on the unrestricted strategy exceeds the allowed limits.

In order to implement these strategies successfully, it is necessary to select optimal switching points for the *regular* control phases required to negotiate steep sections of track. It has been known for some time [10, 15, 18] that the optimal switching points for phases of regular control are determined by the evolution of a key adjoint variable, but recent work [1–3, 14] has established an alternative local energy minimization principle that finds the globally optimal switching points for each steep section of track by solving a local optimization problem. This principle has also enabled a constructive proof using perturbation analysis that shows that there is a uniquely defined optimal strategy for each journey. As a general rule, it is necessary to switch control from *speedhold with partial acceleration* before entering a steep uphill section to *maximum acceleration* while traversing the steep section before switching back to *speedhold with partial acceleration* when the train returns to the desired holding speed after leaving the steep section. A similar general rule applies when switching to *coast* to negotiate a steep downhill section. When regenerative braking is available, it may be necessary to switch to a phase of *speedhold with partial brake* if the downhill track becomes sufficiently steep (see the work of Albrecht et al. [1, 2] for an extended discussion).

The work in this paper is motivated by two closely related train timetable problems. The first problem involves the safe separation of trains travelling in the same direction on the same track, and the second involves the design of optimal timetables on busy rail networks.

If additional time constraints are imposed at intermediate points during the journey, but the train is not required to stop at these *intermediate timing points*, then the overall optimal strategy will consist of a sequence of linked optimal strategies over each timed section. The initial speed on the next section must be equal to the final speed on the current section, but these initial and final speeds are not known beforehand. The intermediate timing points are necessary, because railways commonly impose safety restrictions insisting that a following train must not enter a particular section of track until a leading train has left it. To solve this *two-train separation problem*, we must not only find the optimal initial and final speeds for both the leading train and the following train on each designated track section but we must also choose the optimal sectional running times. This is a very difficult problem, and a general solution has yet to be found. Recently, however, a complete solution to the two-train separation problem on level track has been given by Albrecht et al. [4, 5].

A key problem in the implementation of energy-efficient rail operations is the design of effective timetables for busy suburban rail networks. One potentially useful idea with electric trains is to construct a timetable that enables energy recovered from regenerative braking on one train to be transferred as driving power to another train. Li and Lo [16] proposed a genetic algorithm to synchronize the accelerating and braking actions of trains while simultaneously imposing a rudimentary optimal driving strategy to minimize the tractive energy consumption on each individual journey. In a subsequent paper [17], the same authors formulate an integrated energy-efficient timetable and speed profile optimization model which is solved approximately by transforming to a simplified convex optimization problem using linear approximations.

Gupta et al. [8] also solved a simplified problem by suggesting a two-stage process. In the first stage, the optimal running time for each train on each segment is determined subject to the usual trip-time, dwell-time, headway and connection constraints. In the second stage, the start and finish times for each journey are adjusted without changing the total journey time or the individual segment running times to synchronize, as far as possible, the arrivals and departures of colocated trains. In this way, they proposed to maximize the transfer of electrical energy generated by a braking train to an otherwise unrelated accelerating train that is powered from the same electrical substation. A key part of the first stage is the use of a characteristic cost–time curve for each train on each section of the track. The paper by Gupta et al. [8] uses a linear approximation to an empirical cost–time curve, but does not distinguish between different levels of energy recovery for different journeys.

Our paper provides some scope for improved solutions to these very difficult timetable problems.

1.3. Organization of the paper In Section 2, we review the necessary preliminary material from the known theory of optimal train control on level track [1, 6, 10, 12] and track with piecewise-constant gradient [10, 14, 18]. We describe the basic model for the motion of a point-mass train and the optimal strategies. We explain that on level track, there are only four possible modes of optimal control – *maximum acceleration*, *speedhold with partial acceleration*, *coast* and *maximum brake* – and we show how a *modified adjoint variable* is used to determine the optimal switching points. We also show that there are three distinct forms for the optimal strategy with the precise form depending on the length of the journey, the time allowed and the proportion of energy recovered during regenerative braking. We introduce several key ideas and some useful formulæ that are used later in the paper. In Section 3, we present the main results. For each fixed-length journey and each form of the optimal strategy, we show that the cost–time curve is strictly decreasing and strictly convex.

2. Preliminaries

Howlett and Pudney [13] showed that the motion of a train with distributed mass can be reduced to the motion of a point-mass train. Thus, we restrict our attention to point-mass trains. In general terms, the problem is to drive a train from $x = 0$ to $x = X$ within some prescribed time T in such a way that the energy consumption is minimized. It has been argued or assumed by all major contributors that in order to calculate the precise optimal strategy on nonlevel track, it is convenient to formulate the model with position $x \in [0, X]$ as the independent variable and with time $t = t(x) \in [0, T]$ and speed $v = v(x) \in [0, \infty)$ as the dependent state variables. In this paper, we restrict our attention to level track, but nevertheless use the general formulation. The material in this section is not new, but, since most readers are usually not familiar with the relevant results, we thought it prudent to include the details. For a more extensive treatment we refer readers to Albrecht et al. [1].

2.1. Model formulation The equations of motion are

$$t' = 1/v, \tag{2.1}$$
$$v' = [u - r(v) + g(x)]/v, \tag{2.2}$$

where $(t, v) = (t(x), v(x))$ and $(t', v') = (t'(x), v'(x)) = (dt/dx, dv/dx)$, and $u = u(x) \in \mathbb{R}$ is the known measurable control – the force per unit mass or acceleration for $x \in [0, X]$. In this paper, we only consider journeys with $v(0) = v(X) = 0$.

We assume that $v = v(x) > 0$ for all $x \in (0, X)$ and that $u(x)$ is bounded with $-K[v(x)] \le u(x) \le H[v(x)]$ for each $x \in (0, X)$. The bounds $H = H(v) \in (0, \infty)$ and $K = K(v) \in (0, \infty)$ for $v \in (0, \infty)$ are decreasing functions with $H(v) \downarrow 0$ as $v \uparrow \infty$. We suppose too that, for each $\epsilon > 0$, there exist constants $H'_\epsilon > 0$ such that $|H(v) - H(w)| \le H'_\epsilon |v - w|$ and K'_ϵ such that $|K(v) - K(w)| \le K'_\epsilon |v - w|$ for all $v, w \ge \epsilon$. The functions H and K define bounds for the maximum driving and braking forces per unit mass in a form that includes, as special cases, the bounds for a wide range of railway traction systems. The function $r(v)$ is a general resistance per unit mass with no specific formula assumed. We define auxiliary functions $\varphi(v) = vr(v)$ and $\psi(v) = v^2 r'(v)$, and assume only that $\varphi(v)$ is strictly convex with $\varphi(v) \ge 0$ for $v \ge 0$ and $\varphi(v)/v \to \infty$ as $v \to \infty$. It follows that both $r(v)$ and $\psi(v)$ are nonnegative and strictly increasing for $v \ge 0$. These properties capture the functional characteristics of the traditional quadratic resistance formula – the so-called Davis formula [7] – that has been used in practice by the rail industry for many years. The function $g(x)$ is nominally the component of gravitational acceleration due to track gradient but, in practice, it may also include additional position-dependent resistive forces due to track curvature. We will not consider details of such calculations, but note, in passing, that resistance due to curvature is often effectively modelled by calculating an equivalent gradient acceleration.

The cost of a control strategy is the net mechanical energy usage per unit mass, given by

$$J = \int_0^X \Big[\frac{(u + |u|)}{2} + \frac{\rho(u - |u|)}{2}\Big] dx,$$

where $\rho \in [0, 1]$ is the proportion of mechanical energy recovered during regenerative braking. For a more comprehensive discussion of the model, we refer the readers to Albrecht et al. [1, 2].

2.2. The minimum-time journey We assume that the track is level with $g(x) = 0$. For each fixed distance $X > 0$, the minimum-time journey uses a *maximum acceleration–maximum brake* strategy. Let V be the termination speed for the initial phase. In the minimum-time journey, V is the speed at which the control is switched from *maximum acceleration* with $u = H(v)$ to *maximum brake* with $u = -K(v)$. The speed $V = V_{\max}$ is uniquely defined by the distance constraint

$$X = \int_0^V \frac{v\,dv}{H(v) - r(v)} + \int_0^V \frac{v\,dv}{K(v) + r(v)},$$

https://doi.org/10.1017/S1446181116000092 Published online by Cambridge University Press

and the minimum possible journey time $T = T_{\min}$ is given by

$$T = \int_0^V \frac{dv}{H(v) - r(v)} + \int_0^V \frac{dv}{K(v) + r(v)}.$$

For each $T > T_{\min}$, there is an infinite collection of feasible strategies (see Albrecht et al. [1] for more details).

2.3. Optimal strategies We assume that the track is level with $g(x) = 0$. For each given distance $X > 0$ and time $T > T_{\min}$, it is known that the optimal strategy is uniquely defined [2, 15]. For each optimal strategy, there is a constant $\mu > 0$ with a corresponding *optimal driving speed* $v = V_\mu$ defined by the unique solution to the equation

$$\mu = \psi(V_\mu),$$

and an associated optimal braking speed $v = U_{\rho,\mu} \le V_\mu$ defined by the unique solution to the equation

$$\rho\varphi(v) = L_\mu(v) \iff \rho\varphi(v) = \varphi(V_\mu) + \varphi'(V_\mu)(v - V_\mu),$$

where $y = L_\mu(v)$ is the tangent to the strictly convex curve $y = \varphi(v)$ at the point $v = V_\mu$. If we write $U = U_{\rho,\mu}$, for convenience, then

$$\rho\,\varphi(U) = \varphi(V_\mu) + \varphi'(V_\mu)(U - V_\mu), \tag{2.3}$$

and differentiation gives

$$\rho\,\varphi'(U)\frac{dU}{dV_\mu} = \varphi'(V_\mu) + \varphi''(V_\mu)(U - V_\mu) + \varphi'(V_\mu), \left(\frac{dU}{dV_\mu} - 1\right)$$

from which it follows that

$$\frac{dU}{dV_\mu} = \frac{\varphi''(V_\mu)(V_\mu - U)}{\varphi'(V_\mu) - \rho\,\varphi'(U)} > 0. \tag{2.4}$$

This confirms that $U = U_{\rho,\mu}$ is uniquely determined by V_μ. For each optimal journey, there is an associated *energy density per unit mass function* $E_\mu : (0, \infty) \to (0, \infty)$ defined by

$$E_\mu(v) = \frac{\mu}{v} + r(v) = \frac{\psi(V_\mu)}{v} + r(v). \tag{2.5}$$

This function is strictly convex with a unique minimum turning point at $v = V_\mu$ (for more details we refer the readers to the detailed discussion by Albrecht et al. [1, Sections 3.4, 3.5 and 3.6]). On level track there are only four possible modes of optimal control: *maximum acceleration* with $u = H(v)$, *speedhold* at the optimal driving speed V_μ with control $u = r(V_\mu) > 0$, *coast* with $u = 0$ and *maximum brake* with $u = -K(v)$. The first phase of the optimal strategy is a phase of *maximum acceleration*. Let $v = V$ denote the termination speed for the initial phase. For $0 \le \rho < 1$, we either have $0 < V < V_\mu$ or $V = V_\mu$. When $\rho = 1$, we must have $V = V_\mu = U_{1,\mu}$. We will consider each of these cases in detail and, in so doing, will also review the relevant known material.

Case 1. In the case where $0 \le \rho < 1$ and $V < V_\mu$, the optimal strategy takes the form of a *maximum acceleration–coast–maximum brake* strategy. These are all regular control phases. The modified adjoint variable, $\eta : [0, \infty) \to \mathbb{R}$, defined by Albrecht et al. [1], is given by

$$\eta(v) = \begin{cases} \dfrac{E_\mu(v) - E_\mu(V)}{H(v) - r(v)} & \text{for } u = H(v) \\ \dfrac{E_\mu(v) - E_\mu(V)}{-r(v)} & \text{for } u = 0 \\ \dfrac{E_\mu(v) - E_\mu(U) + (1-\rho)[K(v) + r(U)]}{-K(v) - r(v)} & \text{for } u = -K(v), \end{cases}$$

where $U = U_{\rho,\mu}$ is the speed at which braking begins. The switching points are determined by the evolution of the modified adjoint variable. The switch from *maximum acceleration* to *coast* occurs when $\eta = 0$, and the switch from *coast* to *maximum brake* occurs when $\eta = \rho - 1$. The speed $U = U_{\rho,\mu}$ must satisfy the distance constraint

$$X = \int_0^V \frac{v\,dv}{H(v) - (v)} + \int_U^V \frac{v\,dv}{r(v)} + \int_0^U \frac{v\,dv}{K(v) + r(v)}.$$

In fact, $U = U(V)$ is uniquely determined by this constraint as a function of V. The time $T = T(V)$ for the journey is given by

$$T = \int_0^V \frac{dv}{H(v) - (v)} + \int_U^V \frac{dv}{r(v)} + \int_0^U \frac{dv}{K(v) + r(v)}.$$

The formulæ used above for the distance travelled and time taken during each phase are obtained by solving the equations of motion (2.1) and (2.2). Equation (2.2) is solved first by separation of variables to find $x = x(v)$, and then (2.1) is integrated directly to find $t = t(v)$. In this case, the optimal driving speed $v = V_\mu$ is never reached, but it can be determined retrospectively by solving the equation

$$\rho\,\varphi(U) = L_\mu(U)$$

to find μ. The procedure is straightforward, if we consider a graphical approach. We simply draw a straight line

$$y = \lambda(v - U) + \rho\varphi(U)$$

with slope λ passing through the point $(v, y) = (U, \rho\,\varphi(U))$. This point lies below the strictly convex curve $y = \varphi(v)$, so there is a unique value, $\lambda = \lambda_\mu$, such that the line is tangent to the curve $y = \varphi(v)$. The optimal driving speed is now defined by the equation $\lambda_\mu = \varphi'(V_\mu)$. We refer to the work of Albrecht et al. [1, Section 3.6] for an extended discussion.

Case 2. In the case where $0 \le \rho < 1$ and $V = V_\mu$, the optimal strategy is a *maximum acceleration–speedhold–coast–maximum brake* strategy. The *speedhold* phase is a phase of *singular* control with $v = V_\mu$ and $u = r(V_\mu) > 0$. The modified adjoint variable $\eta : [0, \infty) \to \mathbb{R}$, defined by Albrecht et al. [1], is given by

$$\eta(v) = \begin{cases} \dfrac{E_\mu(v) - E_\mu(V)}{H(v) - r(v)} & \text{for } u = H(v) \\ 0 & \text{for } u = r(V_\mu) \\ \dfrac{E_\mu(v) - E_\mu(V)}{-r(v)} & \text{for } u = 0 \\ \dfrac{E_\mu(v) - E_\mu(U) + (1-\rho)[K(v) + r(U)]}{-K(v) - r(v)} & \text{for } u = -K(v), \end{cases}$$

where we have written $V = V_\mu$ and $U = U_{\rho,\mu}$ for convenience. The switching points for the regular control phases of *maximum acceleration*, *coast* and *maximum brake* are determined by the evolution of the modified adjoint variable, but the modified adjoint variable is constant during the singular *speedhold* phase. Thus, we cannot determine the switching point from *speedhold* to *coast* by considering evolution of the modified adjoint variable. However, the holding speed V is equal to the optimal driving speed and hence U can be found by solving (2.3). Thus the initial and final speeds for each regular control phase are also known. Therefore, we can calculate the length of each regular phase. The total journey length must exceed the sum of the lengths of the three regular phases. Thus

$$X > \int_0^V \frac{v\,dv}{H(v) - r(v)} + \int_U^V \frac{v\,dv}{r(v)} + \int_0^U \frac{v\,dv}{K(v) + r(v)}.$$

Although the method of determination is different from the method used in the first case, we can, nevertheless, see from (2.3) that $U = U_{\rho,\mu}(V_\mu)$ is uniquely determined by $V = V_\mu$. Hence the length of the singular *speedhold* phase $X_s = X_s(V)$ is uniquely determined as a function of V with

$$X_s = X - \int_0^V \frac{v\,dv}{H(v) - r(v)} - \int_U^V \frac{v\,dv}{r(v)} - \int_0^U \frac{v\,dv}{K(v) + r(v)}.$$

It follows that the time $T = T(V)$ for the journey can also be regarded as a function of V with

$$T = \int_0^V \frac{dv}{H(v) - (v)} + \frac{X_s}{V} + \int_U^V \frac{dv}{r(v)} + \int_0^U \frac{dv}{K(v) + r(v)}.$$

Once again, the formulæ for distance travelled and time taken in each phase are obtained by solving (2.1) and (2.2).

Case 3. In the case where $\rho = 1$ and $V = V_\mu$, the optimal strategy takes the form of a *maximum acceleration–speedhold–maximum brake* strategy. The modified adjoint

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variable, $\eta : [0, \infty) \to \mathbb{R}$, defined by Albrecht et al. [1], is given as

$$\eta(v) = \begin{cases} \dfrac{E_\mu(v) - E_\mu(V)}{H(v) - r(v)} & \text{for } u = H(v) \\ 0 & \text{for } u = r(V_\mu) \\ \dfrac{E_\mu(v) - E_\mu(V)}{-K(v) - r(v)} & \text{for } u = -K(v), \end{cases}$$

where $\rho = 1$ means that the phase of *maximum brake* begins when $\eta(v) = 0$, and the speed $U = U_{1,\mu}$ at which braking begins is given by $U = V$. Also

$$X > \int_0^V \frac{v\,dv}{H(v) - r(v)} + \int_0^V \frac{v\,dv}{K(v) + r(v)},$$

so the length of the speedhold phase $X_s = X_s(V)$ is given by

$$X_s = X - \int_0^V \frac{v\,dv}{H(v) - r(v)} - \int_0^V \frac{v\,dv}{K(v) + r(v)}.$$

The time taken $T = T(V)$ is given by

$$T(V) = \int_0^V \frac{dv}{H(v) - r(v)} + \frac{X_s}{V} + \int_0^V \frac{dv}{K(v) + r(v)}.$$

Example 2.1. We use the model described in Section 2.1 with $H(v) = H/v$ and $K(v) = K$, where $H = 3$ and $K = 0.3$. We assume that $r(v) = r_0 + r_2 v^2$ with $r_0 = 6.75 \times 10^{-3}$ and $r_2 = 5 \times 10^{-5}$ and that $\rho = 0$. Distance is measured in metres and time is measured in seconds. The units of H are watts per kilogram ($\text{m}^2\ \text{s}^{-3}$). The units of K are newtons per kilogram (m s^{-2}). We consider a full range of optimal strategies with $X = 2000$ by nominating selected equally-spaced values for the speed $V \in \{3, 4, \ldots, 21\}$ and two special values $V \in \{V_{\text{crit}} = 5.7088, V_{\max} = 21.5564\}$ and calculating the corresponding phase lengths and phase times. The critical journey is the journey where $V = V_\mu$, but the *speedhold* phase is degenerate. For $V > V_{\text{crit}}$, $V < V_\mu$ and, for $V \le V_{\text{crit}}$, $V = V_\mu$. We focus on the following four representative journeys:

- the minimum-time *maximum acceleration–maximum brake* (ab*) strategy, defined by $V = V_{\max} \approx 21.5564$ with $T = T_{\min} \approx 154.95$ and $J \approx 259.11$;
- a *maximum acceleration–coast–maximum brake* (acb*) strategy, defined by $V = 15$, $U_{0,\mu} \approx 13.4422$ and $V_\mu \approx 22.0325$ with $T \approx 175.15$ and $J \approx 117.88$;
- the critical *maximum acceleration–coast–maximum brake* (crit*) strategy, defined by $V = V_{\text{crit}} = V_\mu \approx 5.7088$ and $U_{\text{crit}} = U_{0,\mu} \approx 1.5986$ with $T = T_{\text{crit}} \approx 561.46$ and $J = J_{\text{crit}} \approx 16.46$; and
- a *maximum acceleration–speedhold–coast–maximum brake* (ahcb*) strategy, defined by $V = V_\mu = 3$ and $U_{0,\mu} \approx 0.3333$ with $T \approx 699.22$ and $J \approx 14.91$.

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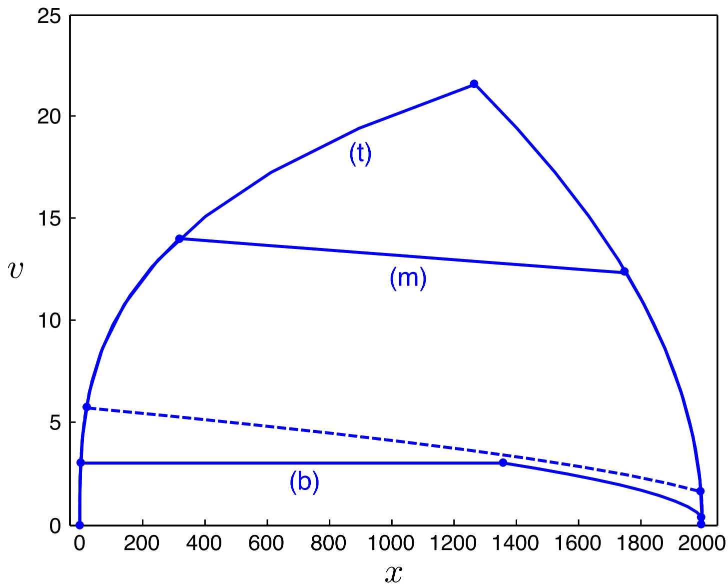

FIGURE 1. Speed profiles for Example 2.1. Minimum-time *maximum acceleration–maximum brake* (ab*) speed profile shown at the top (t), typical *maximum acceleration–coast–maximum brake* (acb*) speed profile shown in the middle (m) and typical *maximum acceleration–speedhold–coast–maximum brake* (ahcb*) speed profile shown at the bottom (b). The critical *maximum acceleration–coast–maximum brake* (crit*) speed profile with degenerate *speedhold* phase is shown as a dashed curve. Distance on the horizontal axis is measured in metres and speed on the vertical axis is measured in metres per second.

The speed profiles $v = v(x)$ for these four strategies are shown in Figure 1. The speeds V at which the *maximum acceleration* phase is terminated, the speeds $U_{0,\mu}$ at which braking begins and the optimal driving speeds V_μ, are shown for all strategies in Table 1. Note that the optimal driving speed is never reached for the *maximum acceleration–maximum brake* strategy and for the *maximum acceleration–coast–maximum brake* strategies. The distances travelled for each phase are shown in Table 2. The times taken for each phase and the total journey times are shown in Table 3. Details for the strategies corresponding to the speed profiles in Figure 1 are marked with an asterisk. All calculations were performed using MATLAB.

The cost–time curve is shown in Figure 2. The curve comprises two parts, with the shorter journey times ($T < T_{\text{crit}} \approx 561.46$) associated with the *maximum acceleration–coast–maximum brake* strategies and the longer journey times ($T > T_{\text{crit}} \approx 561.46$) associated with the *maximum acceleration–speedhold–coast–maximum brake* strategies. All calculations were performed using MATLAB. Detailed cost calculations can be obtained from the times shown in Table 3 using the formula

$$J = HT_a + \varphi(V)T_h,$$

where T_a is the time spent using *maximum acceleration* and T_h is the time spent using *speedhold* at speed V.

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TABLE 1. The speeds V, $U_{0,\mu}$ and V_μ in metres per second for Example 2.1.

Strategy form	V	$U_{0,\mu}$	V_μ
ab*	21.5564	21.5564	33.6218
acb	21.0000	20.7872	32.5085
acb	20.0000	19.4545	30.5856
acb	19.0000	18.1766	28.7494
acb	18.0000	16.9441	26.9866
acb	17.0000	15.7489	25.2859
acb	16.0000	14.5838	23.6375
acb*	15.0000	13.4422	22.0325
acb	14.0000	12.3178	20.4625
acb	13.0000	11.2041	18.9191
acb	12.0000	10.0938	17.3930
acb	11.0000	8.9783	15.8729
acb	10.0000	7.8460	14.3433
acb	9.0000	6.6792	12.7795
acb	8.0000	5.4467	11.1351
acb	7.0000	4.0793	9.3015
acb	6.0000	2.3385	6.8610
crit*	5.7088	1.5986	5.7088
ahcb	5.0000	1.1905	5.0000
ahcb*	4.0000	0.6995	4.0000
ahcb	3.0000	0.3333	3.0000

3. The cost–time curves

We assume that X is fixed and consider the cost–time curves $y = J(T)$ for each of the three possible forms of optimal strategy. In each case, we show that the cost–time curve is strictly decreasing and strictly convex.

3.1. The cost–time curve for $0 \le \rho < 1$ and $V < V_\mu$ In this case, the optimal strategy is a *maximum acceleration–coast–maximum brake* strategy. Let V be the speed at which the strategy switches from *maximum acceleration* to *coast* and let U be the speed at which the strategy switches from *coast* to *maximum brake*. We wish to show that the cost of the optimal strategy decreases as the time taken for the journey increases. We use an indirect argument by finding explicit expressions for the journey cost $J = J(V)$ and journey time $T = T(V)$ in terms of the speed V at the end of the initial phase of *maximum acceleration*. This will allow us to show that

$$\frac{dJ}{dT} = \frac{J'(V)}{T'(V)} < 0.$$

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TABLE 2. The distances in metres travelled during each phase for Example 2.1.

Strategy form	V	X_a	X_h	X_c	X_b
ab*	21.5564	1269.9	0	0	730.1
acb	21.0000	1163.8	0	155.6	680.6
acb	20.0000	990.8	0	410.6	598.6
acb	19.0000	838.5	0	637.0	524.5
acb	18.0000	704.6	0	838.0	457.4
acb	17.0000	587.4	0	1016.3	396.3
acb	16.0000	485.1	0	1174.1	340.8
acb*	15.0000	396.4	0	1313.3	290.3
acb	14.0000	319.9	0	1435.8	244.3
acb	13.0000	254.4	0	1543.0	202.6
acb	12.0000	198.9	0	1636.4	164.7
acb	11.0000	152.4	0	1717.0	130.5
acb	10.0000	114.0	0	1786.0	99.8
acb	9.0000	82.8	0	1844.8	72.5
acb	8.0000	57.9	0	1893.8	48.2
acb	7.0000	38.7	0	1934.2	27.1
acb	6.0000	24.3	0	1966.8	8.9
crit*	5.7088	20.9	0	1974.9	4.2
ahcb	5.0000	14.0	389.1	1594.6	2.3
ahcb*	4.0000	7.2	908.2	1083.9	0.8
ahcb	3.0000	3.0	1359.6	637.2	0.2

The distance constraint is

$$X = \int_0^V \frac{v\,dv}{H(v) - r(v)} + \int_U^V \frac{v\,dv}{r(v)} + \int_0^U \frac{v\,dv}{K(v) + r(v)}. \tag{3.1}$$

If we regard V as the primary variable, then we can show that U depends implicitly on V. By differentiating the distance constraint, we obtain

$$\frac{dU}{dV} = \frac{H(V)V}{[H(V) - r(V)]r(V)} \frac{[K(U) + r(U)]r(U)}{K(U)U} > \frac{r(U)}{r(V)} > 0. \tag{3.2}$$

Since U increases when V increases, it follows that $U = U(V)$ is uniquely determined by V. The time for the journey is given by

$$T(V) = \int_0^V \frac{dv}{H(v) - r(v)} + \int_U^V \frac{dv}{r(v)} + \int_0^U \frac{dv}{K(v) + r(v)}.$$

If we differentiate with respect to V and use (3.2), then

$$T'(V) = \frac{H(V)}{[H(V) - r(V)]r(V)}\Big(1 - \frac{V}{U}\Big) < 0, \tag{3.3}$$

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TABLE 3. The times taken in seconds for each phase and total times for Example 2.1.

Strategy form	V	T_a	T_h	T_c	T_b	T
ab*	21.5564	86.37	0	0	68.58	154.95
acb	21.0000	81.38	0	7.45	66.24	155.07
acb	20.0000	72.95	0	20.81	62.16	155.93
acb	19.0000	65.14	0	34.28	58.22	157.64
acb	18.0000	57.91	0	47.98	54.40	160.29
acb	17.0000	51.21	0	62.11	50.67	163.92
acb	16.0000	45.02	0	76.85	47.00	168.87
acb*	15.0000	39.29	0	92.46	43.40	175.15
acb	14.0000	34.02	0	109.28	39.83	183.13
acb	13.0000	29.17	0	127.75	36.28	193.20
acb	12.0000	24.74	0	148.48	32.73	205.94
acb	11.0000	20.69	0	172.39	29.14	222.23
acb	10.0000	17.04	0	200.90	25.49	243.43
acb	9.0000	13.75	0	236.39	21.72	271.87
acb	8.0000	10.83	0	283.38	17.73	311.94
acb	7.0000	8.27	0	352.16	13.29	373.72
acb	6.0000	6.06	0	478.65	7.62	492.33
crit*	5.7088	5.49	0	550.76	5.21	561.46
ahcb	5.0000	4.20	77.82	523.76	3.88	609.66
ahcb*	4.0000	2.69	227.04	467.22	2.28	699.22
ahcb	3.0000	1.51	453.21	385.58	1.09	841.38

since $U < V$. The cost of the journey is given by

$$J(V) = \int_0^V \frac{H(v)v\,dv}{H(v) - r(v)} - \int_0^U \frac{\rho K(v)v\,dv}{K(v) + r(v)},$$

so, using (3.2), we get

$$J'(V) = \frac{H(V)V}{H(V) - r(V)}\Big(1 - \rho\frac{r(U)}{r(V)}\Big). \tag{3.4}$$

We can now use (3.3) and (3.4) to deduce that

$$\begin{aligned} \frac{dJ}{dT} = \frac{J'(V)}{T'(V)} &= \frac{H(V)V}{H(V) - r(V)}\Big(1 - \rho\frac{r(U)}{r(V)}\Big)\frac{[H(V) - r(V)]r(V)}{H(V)(1 - V/U)} \\ &= \frac{\varphi(V)}{(1 - V/U)}\Big(1 - \rho\frac{r(U)}{r(V)}\Big), \end{aligned} \tag{3.5}$$

from which it follows that $(dJ/dT) < 0$. Hence the cost–time curve is locally strictly decreasing when $V < V_\mu$.

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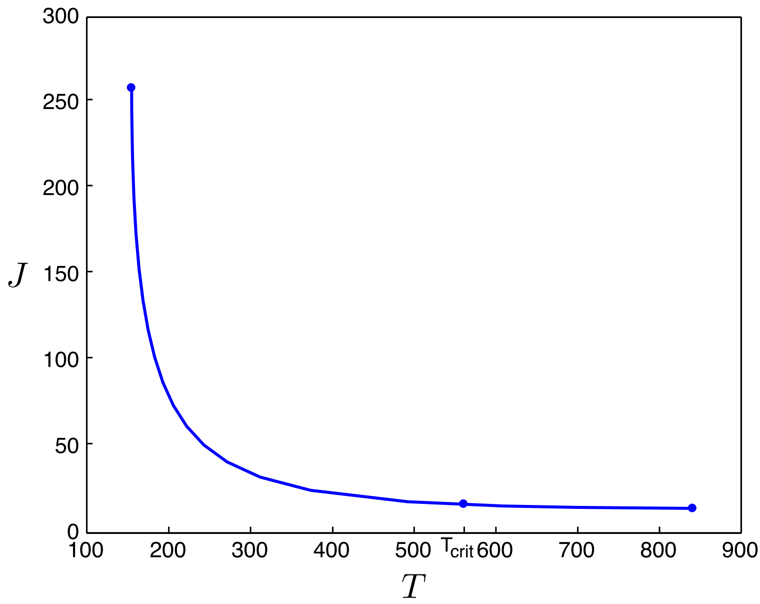

FIGURE 2. Cost–time curve for Example 2.1. The highest cost is for the minimum-time *maximum acceleration–maximum brake* strategy. The cost–time curve for the *maximum acceleration–coast–maximum brake* strategy is bounded on the left by the cost–time point for the minimum-time journey and on the right by the cost–time point for the critical journey with $V = V_\mu$ and $U = U_{0,\mu}$ but with no *speedhold* phase. The cost–time curve for the *maximum acceleration–speedhold–coast–maximum brake* strategies is bounded on the left by the cost–time point for the critical journey. Time on the horizontal axis is measured in seconds (s) and cost on the vertical axis is measured in joules per kilogram ($\mathrm{m}^2\ \mathrm{s}^{-2}$).

We now show that $J = J(V)$ is a strictly convex function of $T = T(V)$. We rewrite (3.5) in the form

$$\frac{dJ}{dT} = \frac{\varphi(V)U - \rho\,\varphi(U)V}{U - V}, \tag{3.6}$$

but there is a further simplification that gives more insight. During the coast phase, the modified adjoint variable is given by

$$\eta(v) = \frac{E_\mu(v) - E_\mu(V)}{-r(v)}$$

for $U_{\rho,\mu} \le v \le V$. The switch to *maximum brake* occurs when $\eta = \rho - 1$ and $v = U = U_{\rho,\mu}$ and hence

$$\rho - 1 = \frac{E_\mu(U) - E_\mu(V)}{-r(U)},$$

which we can rearrange using (2.5) into the equivalent form

$$\psi(V_\mu) = \frac{\varphi(V)U - \rho\,\varphi(U)V}{V - U}. \tag{3.7}$$

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Therefore, (3.6) becomes

$$\frac{dJ}{dT} = -\psi(V_\mu). \tag{3.8}$$

This is the key formula (1.1) in the case, where the termination speed for the initial phase is less than the optimal driving speed, that is, when $V < V_\mu$. We now wish to show that $J = J(V)$ is a strictly convex function of $T = T(V)$. From the strict convexity of $\varphi(v)$ we obtain

$$\begin{aligned}
\frac{d}{dV}\Big(\frac{\varphi(V)U}{U-V}\Big) &= \frac{\varphi'(V)U}{U-V} + \frac{\varphi(V)}{U-V}\frac{dU}{dV} - \frac{\varphi(V)U}{(U-V)^2}\Big(\frac{dU}{dV} - 1\Big)\\
&= \frac{[\varphi(V) + \varphi'(V)(U-V)]U}{(U-V)^2} - \frac{\varphi(V)V}{(U-V)^2}\frac{dU}{dV}\\
&< \frac{\varphi(U)U}{(U-V)^2} - \frac{\varphi(V)V}{(U-V)^2}\frac{dU}{dV} \quad \text{and}\\
-\frac{d}{dV}\Big(\frac{\rho\,\varphi(U)V}{U-V}\Big) &= -\frac{\rho\,\varphi'(U)V}{U-V}\frac{dU}{dV} - \frac{\rho\,\varphi(U)}{U-V} + \frac{\rho\,\varphi(U)V}{(U-V)^2}\Big(\frac{dU}{dV} - 1\Big)\\
&= \frac{\rho\,[\varphi(U) + \varphi'(U)(V-U)]V}{(U-V)^2}\frac{dU}{dV} - \frac{\rho\,\varphi(U)U}{(U-V)^2}\\
&< \frac{\rho\,\varphi(V)V}{(U-V)^2}\frac{dU}{dV} - \frac{\rho\,\varphi(U)U}{(U-V)^2}.
\end{aligned}$$

It follows, from (3.2), (3.7), (3.8) and the previous two inequalities, that

$$\begin{aligned}
\frac{d}{dV}\Big(\frac{dJ}{dT}\Big) &= \frac{d}{dV}(-\psi(V_\mu))\\
&< -\frac{(1-\rho)\varphi(V)V}{(U-V)^2}\frac{dU}{dV} + \frac{(1-\rho)\varphi(U)U}{(U-V)^2}\\
&< -\frac{(1-\rho)\varphi(V)V}{(U-V)^2}\frac{r(U)}{r(V)} + \frac{(1-\rho)\varphi(U)U}{(U-V)^2}\\
&= \frac{(1-\rho)(U+V)r(U)}{U-V}\\
&< 0,
\end{aligned}$$

where we have used $\varphi(V) = Vr(V)$ at the penultimate step. Therefore,

$$\frac{d^2J}{dV\,dT} < 0 \quad \Rightarrow \quad \frac{d^2J}{dT^2} = \frac{d^2J}{dV\,dT}\Big(\frac{dT}{dV}\Big)^{-1} > 0,$$

so the cost–time curve is locally strictly convex when $V < V_\mu$.

For the *maximum acceleration–coast–maximum brake* strategy, we have established that the slope of the cost–time curve decreases as V increases. We can now find the full range of values for the slope by investigating what happens at the endpoints of the curve. Consider the shortest possible journey time for this type of optimal strategy. As $V \uparrow V_{\max}$, we also have $U = U_{\rho,\mu} \uparrow V_{\max}$ and $T \downarrow T_{\min}$. Since $U < V < V_{\max}$, it follows that $U \uparrow V_{\max}$ implies

$$U\varphi(V) - \rho\, V\varphi(U) \uparrow (1-\rho)V_{\max}\varphi(V_{\max}) > 0,$$

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and hence (3.7) shows that $\psi(V_\mu) \uparrow \infty$. Now (3.8) shows that

$$\frac{dJ}{dT} \downarrow -\infty.$$

Consider the longest possible journey time for an optimal strategy of this type. Suppose we could choose $V > 0$ so small that the train coasts to a stop at $x = X$. Thus $U_{\rho,\mu} = U(V) = 0$. Now (2.3) shows that

$$0 = \varphi(V_\mu) - \varphi'(V_\mu)V_\mu.$$

The strict convexity of $\varphi(v)$ means that $V_\mu = 0 < V$, which contradicts our assumption that $V < V_\mu$. Thus, it is not possible to have an optimal strategy in the form *maximum acceleration–coast–maximum brake* with a degenerate *brake* phase. It follows that when $U = U_{\rho,\mu}$ is sufficiently small, we must have $V = V_\mu$ and the optimal strategy must be a *maximum acceleration–speedhold–coast–maximum brake* strategy.

Now imagine that the *maximum acceleration–coast–maximum brake* strategy is not necessarily optimal, but V is chosen so that

$$X = \int_0^V \frac{v\,dv}{H(v) - r(v)} + \int_{U_{\rho,\mu}}^V \frac{v\,dv}{r(v)} + \int_0^{U_{\rho,\mu}} \frac{v\,dv}{K(v) + r(v)}. \tag{3.9}$$

This identity means that if $U = U_{\rho,\mu}$ increases, then V increases and the journey time decreases. We know that when $U_{\rho,\mu} = V_{\max}$, $V = V_{\max} < V_\mu$ and when $U_{\rho,\mu} = 0$, $V > 0 = V_\mu$. We argue that, in practice, there is a unique value $\mu = \mu_{\text{crit}}$ and a corresponding *critical* optimal strategy with $V = V_\mu = V_{\text{crit}}$ and $U = U_{\rho,\mu} = U_{\text{crit}}$ such that

$$X = \int_0^{V_{\text{crit}}} \frac{v\,dv}{H(v) - r(v)} + \int_{U_{\text{crit}}}^{V_{\text{crit}}} \frac{v\,dv}{r(v)} + \int_0^{U_{\text{crit}}} \frac{v\,dv}{K(v) + r(v)}. \tag{3.10}$$

In this case, it follows from (3.8) that

$$\frac{dJ}{dT}(V_{\text{crit}}) = -\psi(V_{\text{crit}}).$$

Thus, when $V_{\text{crit}} \le V \le V_\mu$, the slope of the cost–time curve varies from $-\psi(V_{\text{crit}})$ to $-\infty$. In practice, a simple model with

$$H(v) = \frac{P}{v}, \quad K(v) = Q, \quad r(v) = r_0 + r_1 v + r_2 v^2,$$

where $r_0 > 0$, $r_1 \ge 0$ and $r_2 > 0$ can be used to give excellent results. If we again write $U = U_{\rho,\mu}$ then, for this model, (2.3) gives

$$U > \frac{r_1 V_\mu^2 + 2r_2 V_\mu^3}{r_0(1-\rho) + 2r_1 V_\mu + 3r_2 V_\mu^2} \uparrow \frac{2V_\mu}{3}$$

as $V_\mu \uparrow \infty$. Hence U essentially increases at roughly the same rate as V_μ. On the other hand, if V is defined by (3.9), then it follows, from (3.1) and (3.2), that

$$\frac{dV}{dU} = \frac{QU}{r(U)[Q + r(U)]} \frac{\varphi(V)[P - \varphi(V)]}{PV^2} < \frac{U\varphi(V)[P - \varphi(V)]}{V^2 r(U)} \downarrow 0$$

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as $U \uparrow V^*$, where V^* is the limiting speed under *maximum acceleration* defined by $\varphi(V^*) = P$. Note that $\varphi(V)[P - \varphi(V)] \le P^2/4$. Thus V ultimately increases much more slowly than U and hence also much more slowly than V_μ. Therefore, it is reasonable to expect that there will be a unique value $\mu = \mu_{\rm crit}$ where $V = V_\mu = V_{\rm crit}$, and (3.10) is satisfied. This is certainly true in Examples 2.1 and 3.1.

3.2. The cost–time curve for $0 \le \rho < 1$ and $V = V_\mu$ In this case, the optimal strategy is a *maximum acceleration–speedhold–coast–maximum brake* strategy. Let $V = V_\mu$ be the hold speed – it is also the speed at which the phase of *maximum acceleration* ends and the *coast* phase begins – and let $U = U(V) = U_{\rho,\mu}$ be the speed at which the strategy switches from *coast* to *maximum brake*. First, we recall from (2.4) that

$$U'(V) = \frac{\varphi''(V)(V-U)}{\varphi'(V) - \rho\,\varphi'(U)} > 0.$$

The distance constraint

$$X = \int_0^V \frac{v\,dv}{H(v) - r(v)} + X_s(V) + \int_U^V \frac{v\,dv}{r(v)} + \int_0^U \frac{v\,dv}{K(v) + r(v)}$$

shows that the length of the *speedhold* phase is given by

$$X_s(V) = X - \Big[\int_0^V \frac{v\,dv}{H(v) - r(v)} + \int_U^V \frac{v\,dv}{r(v)} + \int_0^U \frac{v\,dv}{K(v) + r(v)}\Big].$$

It follows that

$$X_s'(V) = -\frac{H(V)V}{[H(V) - r(V)]r(V)} + \frac{K(U)U}{[K(U) + r(U)]r(U)}\frac{dU}{dV}. \tag{3.11}$$

The time for the journey is

$$T(V) = \int_0^V \frac{dv}{H(v) - r(v)} + \frac{X_s(V)}{V} + \int_U^V \frac{dv}{r(v)} + \int_0^U \frac{dv}{K(v) + r(v)},$$

and, if we differentiate with respect to V and use (3.11), we obtain

$$T'(V) = \frac{K(U)}{[K(U) + r(U)]r(U)}\Big(\frac{U}{V} - 1\Big)\frac{dU}{dV} - \frac{X_s(V)}{V^2} < 0, \tag{3.12}$$

because $U < V$. The cost of the journey is

$$J(V) = \int_0^V \frac{H(v)v\,dv}{H(v) - r(v)} + r(V)X_s(V) - \rho\int_0^U \frac{K(v)\,dv}{K(v) + r(v)},$$

so

$$J'(V) = r'(V)X_s + \frac{K(U)U}{[K(U) + r(U)]}\Big[\frac{r(V)}{r(U)} - \rho\Big]\frac{dU}{dV} > 0, \tag{3.13}$$

because $r(V)/r(U) > 1 > \rho$. Now it follows, from (3.12) and (3.13), that

$$\frac{dJ}{dT} = \frac{J'(V)}{T'(V)} < 0.$$

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Thus the cost–time curve is locally strictly decreasing for $V = V_\mu$. It also follows, from (3.12) and (3.13), that

$$\frac{dJ}{dT} = \Big[r'(V)X_s(V) + \frac{K(U)U}{[K(U)+r(U)]}\Big(\frac{r(V)}{r(U)} - \rho\Big)\frac{dU}{dV}\Big] \div \Big[\frac{K(U)}{[K(U)+r(U)]r(U)}\Big(\frac{U}{V} - 1\Big)\frac{dU}{dV} - \frac{X_s(V)}{V^2}\Big].$$

Now we note that

$$\begin{aligned}\frac{Ur(U)[\{r(V)/r(U)\} - \rho]}{(U/V) - 1} &= \frac{U\varphi(V) - \rho V\varphi(U)}{U - V} \\ &= \frac{U\varphi(V) - V[\varphi(V) + \varphi'(V)(U - V)]}{U - V} \\ &= \varphi(V) - V\varphi'(V) \\ &= -V^2 r'(V) \\ &= -\psi(V),\end{aligned}$$

which means that

$$\begin{aligned}r'(V)X_s(V) &+ \frac{K(U)U}{[K(U)+r(U)]}\Big[\frac{r(V)}{r(U)} - \rho\Big]\frac{dU}{dV} \\ &= \psi(V)\Big[\frac{X_s(V)}{V^2} - \frac{K(U)}{[K(U)+r(U)]r(U)}\Big(\frac{U}{V} - 1\Big)\frac{dU}{dV}\Big],\end{aligned}$$

and hence

$$\frac{dJ}{dT} = -\psi(V) = \frac{U\varphi(V) - \rho V\varphi(U)}{U - V}.$$

Thus we have once again established the fundamental formula (1.1). Now

$$\frac{d}{dV}\Big(\frac{dJ}{dT}\Big) = -\psi'(V) < 0,$$

so

$$\frac{d^2J}{dT^2} = \frac{d^2J}{dVdT}\Big(\frac{dT}{dV}\Big)^{-1} > 0.$$

This shows that the cost–time curve is locally strictly convex when $V = V_\mu$.

3.3. The combined cost–time curve for $0 \le \rho < 1$ The cost–time curve is locally strictly decreasing and locally strictly convex in each of the regions $V < V_\mu$ and $V = V_\mu$. Since

$$\frac{dJ}{dT} = -\psi(V_\mu)$$

in each case, it follows that dJ/dT is continuous at all junction points, and hence the combined cost–time curve is strictly decreasing and strictly convex everywhere in the region $T > T_{\min}$.

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Let V be the speed at the end of the initial phase of *maximum acceleration*. If there is a uniquely defined critical speed $V_{\rm crit} \in (0, V_{\max})$ for which (3.10) is satisfied, then, for the corresponding critical optimal strategy, we define

$$T_{\rm crit} = \int_0^{V_{\rm crit}} \frac{dv}{H(v) - r(v)} + \int_{U_{\rm crit}}^{V_{\rm crit}} \frac{dv}{r(v)} + \int_0^{U_{\rm crit}} \frac{dv}{K(v) + r(v)}.$$

For *fast* journeys with $T_{\min} \le T \le T_{\rm crit}$ and $V_\mu > V > V_{\rm crit}$, the optimal strategy is a *maximum acceleration–coast–maximum brake* strategy, whereas, for *slow* journeys with $T \ge T_{\rm crit}$ and $V_\mu = V < V_{\rm crit}$, the optimal strategy is a *maximum acceleration–speedhold–coast–maximum brake* strategy. If V^* denotes the limiting speed under *maximum acceleration*, then we can see that the total distance travelled during the *coast* and *maximum brake* phases is bounded above by

$$\int_U^{V^*} \frac{v\,dv}{r(v)} + \int_0^U \frac{v\,dv}{K(v) + r(v)} < \int_0^{V^*} \frac{v\,dv}{r(v)} = X_c(V^*)$$

and the total time taken during these phases is bounded above by

$$\int_U^{V^*} \frac{dv}{r(v)} + \int_0^U \frac{dv}{K(v) + r(v)} < \int_0^{V^*} \frac{dv}{r(v)} = T_c(V^*).$$

Hence a *fast* journey over a long distance means that the train must use *maximum acceleration* for a large portion of the journey. This is extremely inefficient. It is better to increase the allowed journey time so that $T > T_{\rm crit}$, in which case a so-called *slow* journey is feasible. The major part of the journey will now be a phase of *speedhold at the optimal driving speed*.

3.4. The cost–time curve for $\rho = 1$ In this case, $V = V_\mu = U_{1,\mu}$ and the optimal strategy is a *maximum acceleration–speedhold–maximum brake* strategy. Let V be the hold speed – it is also the speed at which the phase of *maximum acceleration* ends and the phase of *maximum brake* begins. The distance constraint

$$X = \int_0^V \frac{v\,dv}{H(v) - r(v)} + X_s(V) + \int_0^V \frac{v\,dv}{K(v) + r(v)}$$

shows that the length of the *speedhold* phase is given by

$$X_s(V) = X - \int_0^V \frac{v\,dv}{H(v) - r(v)} - \int_0^V \frac{v\,dv}{K(v) + r(v)}.$$

It follows that

$$X_s'(V) = -\frac{V}{H(V) - r(V)} - \frac{V}{K(V) + r(V)}. \tag{3.14}$$

The time for the journey is

$$T(V) = \int_0^V \frac{dv}{H(v) - r(v)} + \frac{X_s(V)}{V} + \int_0^V \frac{dv}{K(v) + r(v)},$$

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and, if we differentiate with respect to V and use (3.14), we obtain

$$T'(V) = -\frac{X_s(V)}{V^2} < 0. \tag{3.15}$$

Since $\rho = 1$, the cost of the journey is

$$J(V) = \int_0^V \frac{H(v)v\,dv}{H(v) - r(v)} + r(V)X_s(V) - \int_0^V \frac{K(v)v\,dv}{K(v) + r(v)},$$

so

$$\begin{aligned} J'(V) &= \frac{H(V)V}{H(V) - r(V)} + r'(V)X_s(V) \\ &\quad - r(V)\Big[\frac{V}{H(V) - r(V)} + \frac{V}{K(V) + r(V)}\Big] - \frac{K(V)V}{K(V) + r(V)} \\ &= r'(V)X_s(V) \\ &> 0. \end{aligned} \tag{3.16}$$

It follows, from (3.15) and (3.16), that

$$\frac{dJ}{dT} = \frac{dJ}{dV}\Big(\frac{dT}{dV}\Big)^{-1} = -v^2 r'(V) = -\psi(V) < 0.$$

Thus the cost–time curve is locally strictly decreasing when $\rho = 1$. Now

$$\frac{d}{dV}\Big(\frac{dJ}{dT}\Big) = -\psi'(V) < 0 \quad \text{and so,} \quad \frac{d^2J}{dT^2} = \frac{d^2J}{dVdT}\Big(\frac{dT}{dV}\Big)^{-1} > 0.$$

Hence the cost–time curve is locally strictly convex when $\rho = 1$.

EXAMPLE 3.1. We use the same model as in Example 2.1 but, in this case, we consider a much longer journey with $X = 20\,000$ m. As the length of the journey increases, the critical speed increases and the optimal strategy is more likely to be a *maximum acceleration–speedhold–coast–maximum brake* strategy. Indeed, on long journeys, one can see that it is intuitively reasonable to maintain a constant optimal driving speed for a large portion of the journey. We focus on four representative journeys:

- the minimum-time *maximum acceleration–maximum brake* (ab*) strategy, defined by $V = V_{\max} \approx 37.2088$ with $T = T_{\min} \approx 706.32$ and $J \approx 1779.25$;
- a *maximum acceleration–coast–maximum brake* (acb*) strategy, defined by $V = 36.5$, $U_{0,\mu} \approx 27.6877$ and $V_\mu \approx 42.5632$ with $T \approx 724.53$ and $J \approx 1452.99$;
- the critical *maximum acceleration–coast–maximum brake* (crit*) strategy, defined by $V = V_{\text{crit}} = V_\mu \approx 35.8105$ and $U_{\text{crit}} = U_{0,\mu} \approx 23.0644$ with $T = T_{\text{crit}} \approx 756.46$ and $J = J_{\text{crit}} \approx 1260.36$; and
- a *maximum acceleration–speedhold–coast–maximum brake* (ahcb*) strategy, defined by $V = V_\mu = 25$ and $U_{0,\mu} \approx 15.5473$ with $T \approx 947.66$ and $J \approx 766.39$.

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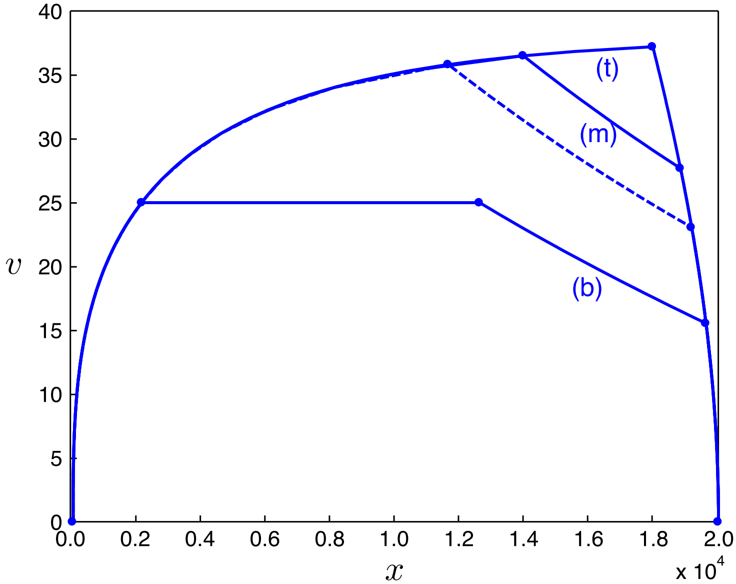

FIGURE 3. Speed profiles for Example 3.1. Minimum-time *maximum acceleration–maximum brake* (ab*) speed profile shown at the top (t), typical *maximum acceleration–coast–maximum brake* (acb*) speed profile shown in the middle (m) and typical *maximum acceleration–speedhold–coast–maximum brake* (ahcb*) speed profile shown at the bottom (b). The critical *maximum acceleration–coast–maximum brake* (crit*) speed profile with degenerate *speedhold* phase is shown as a dashed curve. Distance on the horizontal axis is measured in metres and speed on the vertical axis is measured in metres per second.

The speed profiles $v = v(x)$ for these four strategies are shown in Figure 3. The cost–time curve is shown in Figure 4. The curve comprises two parts, with the shorter journey times ($T < T_{\text{crit}} \approx 756.46$) associated with the *maximum acceleration–coast–maximum brake* strategies and the longer journey times ($T > T_{\text{crit}} \approx 756.46$) associated with the *maximum acceleration–speedhold–coast–maximum brake* strategies. All calculations were performed in MATLAB.

4. Conclusions

For level track over a fixed distance, we have shown that the cost–time curve for an optimal strategy is strictly decreasing and strictly convex. Importantly, we have found a fundamental formula (1.1) which describes this dependence. Although it is intuitively reasonable to expect that the cost–time curve may be strictly decreasing and possibly even convex on track with nonzero gradient, it is likely that any proof will be much more difficult, because the explicit formulæ used in this paper for the distance travelled and the time taken as functions of the speed V are no longer valid.

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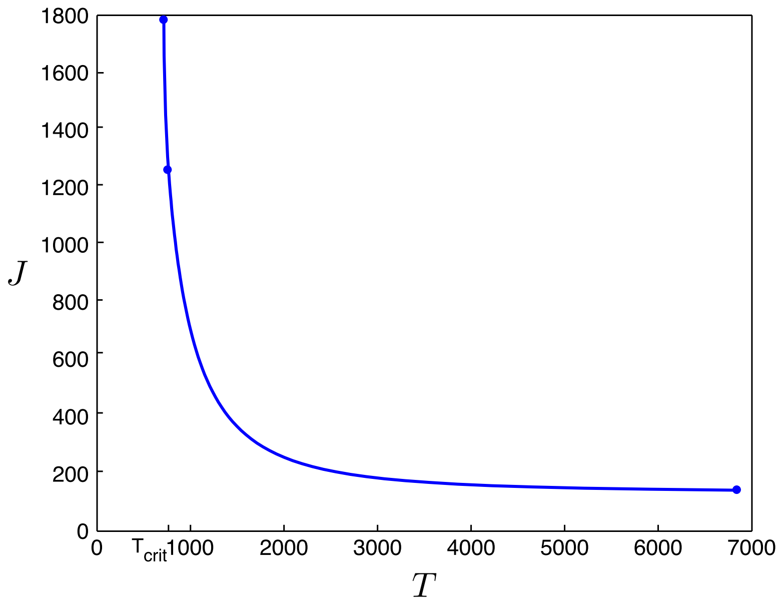

FIGURE 4. Cost–time curve for Example 3.1. The highest cost is for the minimum-time *maximum acceleration–maximum brake* strategy. The cost–time curve for the *maximum acceleration–coast–maximum brake* strategy is bounded on the left by the cost–time point for the minimum-time journey and on the right by the cost–time point for the critical journey with $V = V_\mu$ and $U = U_{0,\mu}$ but with no *speedhold* phase. The cost–time curve for the *maximum acceleration–speedhold–coast–maximum brake* strategies is bounded on the left by the cost–time point for the critical journey. Time on the horizontal axis is measured in seconds (s) and cost on the vertical axis is measured in joules per kilogram ($m^2\ s^{-2}$).

Acknowledgements

We acknowledge the strong support given to the Scheduling and Control Group by Dale Coleman, Managing Director of Sydney Company TTG Transportation Technology – our long-term industry partner. We also acknowledge the support of the Australian Research Council on our previous ARC Linkage Grant LP110100136 when much of this work was done and on our new ARC Linkage Grant LP150100749 in which we hope to use the work in this paper to assist development of timetables on a congested rail network that better coordinate train movements, improve timekeeping, increase capacity and reduce energy usage.

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