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NEWTON POLYHEDRA AND ESTIMATES FOR EXPONENTIAL SUMS

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The use of the Newton polygon method to establish certain properties concerning zeros of polynomials in one variable is well-known. For example in the proof of Puiseux's theorem on the power series development of algebraic functions. In the p-adic case where p is a prime the Newton polygon method yields complete information on the sizes and numbers of zeros of polynomials in one variable with coefficients in Ω_p, the algebraic closure of the field of p-adic numbers.

This thesis is concerned with building up the Newton polygon analogue for a polynomial in two variables with coefficients in Ω_p and to investigate relationships between properties of this analogue which we call the Newton polyhedron and the zeros of the associated polynomials. Secondly this thesis demonstrates the application of the Newton polyhedron method in estimating the exponential sums associated with certain polynomials in $Z_p[x,y]$, where Z_p denotes the ring of p-adic integers.

Chapter 2 investigates the Newton polygon of a polynomial in relation to estimating the exponential sum $S(f',p^\alpha)$, $\alpha \geqslant 1$, where f is not constant modulo p. For a polynomial g in $Z_p[x,y]$ and $\alpha \geqslant 1$ the exponential sums $S(g;\, p^\alpha)$ are defined by

$$S(g;\, p^\alpha) = \sum_{(x,y) \bmod p^\alpha} e_{p^\alpha}(g(x,y)) ,$$

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and the derivative of g is denoted by g'. The investigation of the Newton polyhedron for a polynomial in two variables over Ω_p begins in Chapter 3. If $f(x,y) = \sum_{i,j} a_{ij} x^i y^j$ is a polynomial in $\Omega_p[x,y]$, then the Newton polyhedron of f, denoted by N_f, is the lower convex hull of the set of points $(i,j, \operatorname{ord}_p a_{ij})$ in the Euclidean space R^3. The relationships between certain properties of N_f and the zeros of the associated polynomial f is subsequently examined.

Chapter 4 estimates the size of zeros of polynomials in $\Omega_p[x,y]$ and investigates the relationships between the Newton polyhedrons of a pair of polynomials in $Z_p[x,y]$ and their common zeros. It is shown that if f is a polynomial in $\Omega_p[x,y]$ and

$$\delta = \max_{r,s} \frac{1}{r+s} \left(\operatorname{ord}_p f(\alpha,\beta) - \operatorname{ord}_p \frac{f^{(r+s)}(\alpha,\beta)}{r!s!} \right)$$

for some α,β, where the maximum is taken over all pairs of (r,s) $0 \leqslant r, s < \deg f$, not both 0 or $\deg f$, then f has a zero (ξ_0,η_0) in Ω_p^2 with

$$\operatorname{ord}_p(\xi_0-\alpha,\eta_0-\beta) = \delta ,$$

and every zero (ξ,η) of f satisfies $\operatorname{ord}_p(\xi-\alpha,\eta-\beta) \leqslant \delta$. One other result is the following. If p is an odd prime, $\alpha \geqslant 0$ and

$$f(x,y) = ax^3 + bxy^2 + cx + dy + e$$

is a polynomial in $Z_p[x,y]$ with $\operatorname{ord}_p \frac{\partial f}{\partial x}(x_0,y_0)$, $\operatorname{ord}_p \frac{\partial f}{\partial y}(x_0,y_0) \geqslant \alpha > \delta$ (x_0,y_0) in Ω_p^2 and $\delta = \max\{\operatorname{ord}_p 3a, \frac{3}{2}\operatorname{ord}_p b\}$, then there are (ξ,η) in Ω_p^2 with $\frac{\partial f}{\partial x}(\xi,\eta) = 0$, $\frac{\partial f}{\partial y}(\xi,\eta) = 0$ and

$$\operatorname{ord}_p(x_0-\xi,y_0-\eta) \geqslant \frac{1}{2}(\alpha-\delta) .$$

Chapter 5 estimates the exponential sums $S(f;\, p^\alpha)$ with $\alpha \geqslant 1$ for certain polynomials f in $Z_p[x,y]$. Amongst other results in this chapter is the following. If $f(x,y) = ax^3 + bxy^2 + cx + dy + e$ and $\delta = \max\{\operatorname{ord}_p 3a, \frac{3}{2}\operatorname{ord}_p b\}$ then

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$$|S(f;p^{\alpha})| \leqslant 4p^{\min\{2\alpha,\frac{1}{2}(3\alpha+2\delta+1)\}}$$

The final chapter of this thesis considers the generalised form of the exponential sums associated with polynomials f in $Z_p[x_1,\ldots,x_n]$, with $n > 2$, and gives an estimate to $S(f;p^{\alpha})$.

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