Bull. Aust. Math. Soc. **90** (2014), 469–475
doi:10.1017/S0004972714000501

A POSITIVE SOLUTION FOR A NONLOCAL SCHRÖDINGER EQUATION

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(Received 26 March 2014; accepted 7 June 2014; first published online 15 July 2014)

Abstract

We provide an existence result of radially symmetric, positive, classical solutions for a nonlinear Schrödinger equation driven by the infinitesimal generator of a rotationally invariant Lévy process.

2010 *Mathematics subject classification*: primary 35A01; secondary 35A15, 35J60.

Keywords and phrases: nonlocal Schrödinger equation, positive solution, mountain pass theorem.

1. Introduction

The purpose of this paper is to provide an existence result for radially symmetric, positive, classical solutions to the following problem:

$$\begin{cases} -2Au + \lambda u = |u|^{p-2}u \\ u \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $\lambda > 0$, $2 \le N \le 6$, $2 < p < 2^*$ with $2^* := +\infty$ if $N = 2$ and $2^* := 2N/(N-2)$ if $N > 2$, and A is the infinitesimal generator of a rotationally invariant Lévy process.

EXAMPLE 1.1. Consider the infinitesimal generator A of a Lévy process with jumps following a normal distribution:

$$Au(x) := \tfrac{1}{2}\Delta u(x) + \tfrac{1}{2}\int_{\mathbb{R}^N} (u(x+y) + u(x-y) - 2u(x))\varphi(y)\,dy,$$

where $\varphi(y) := (2\pi)^{-N/2}\exp(-|y|^2/2)$.

A basic motivation for the study of the problem (1.1) is the well-known nonlinear Schrödinger equation driven by the infinitesimal generator of a Brownian motion

$$-\Delta u + \lambda u = |u|^{p-2}u. \tag{1.2}$$

Many authors have investigated equation (1.2) (see, for example, [2–4, 9, 10]).

Note that the Brownian motion is a special rotationally invariant stable Lévy process. It is natural to consider the equation

$$(-\Delta)^{\alpha/2}u + \lambda u = |u|^{p-2}u, \tag{1.3}$$

where $0 < \alpha \le 2$, since $-(-\Delta)^{\alpha/2}$ is the infinitesimal generator of a rotationally invariant stable Lévy process with index α. Equation (1.3) has been studied by many authors (see, for example, [5–8]).

Naturally, we consider the (nonlocal) Schrödinger equation

$$-2Au + \lambda u = |u|^{p-2}u, \tag{1.4}$$

where A is the infinitesimal generator of a rotationally invariant Lévy process. In the present paper, we assume that the Lévy process is of N dimensions, where $2 \le N \le 6$, with nondegenerate diffusion terms and a finite Lévy measure.

Equation (1.4) also arises from looking for the standing waves of the following Schrödinger equation:

$$i\frac{\partial\psi}{\partial t} - 2A\psi = |\psi|^{p-2}\psi.$$

Before stating the main result of the present paper, let us make some comments on the operators $-(-\Delta)^{\alpha/2}$ and A. If $0 < \alpha < 2$, then the Lévy processes generated by $-(-\Delta)^{\alpha/2}$ are pure jump processes; in other words, these processes do not contain any diffusion term. In fact, their corresponding characteristics are given by $(0, 0, \mu)$ with

$$\mu(dx) = \frac{K(\alpha)\,dx}{|x|^{N+\alpha}} \quad \text{for some positive constant } K(\alpha).$$

Consequently, the Lévy measure μ is not finite. For the operator A, the corresponding characteristics are given by $(0, aI, \nu)$ for some positive number a and some finite rotationally invariant Lévy measure ν. Therefore, $-(-\Delta)^{\alpha/2}$ does not cover operators of type A and vice versa. In addition, equation (1.4) is an extension of equation (1.2).

Now we state the main result as follows.

THEOREM 1.2.

(1) *Any weak solution of the problem (1.1) in $H^1(\mathbb{R}^N)$ is a C^2 continuous function.*
(2) *There exists a radially symmetric, positive, classical solution of problem (1.1).*
(3) *The values of any positive solution of the problem (1.1) at maximum points are not less than $\lambda^{1/(p-2)}$.*

The rest of the paper is organised as follows. In Section 2 we present some preliminaries. The proof of Theorem 1.2 is given in Section 3.

2. Some preliminaries

This section serves as a preparation for the proof of Theorem 1.2. First, we state a compact embedding result. Second, a regularity result will be proved. Finally, we investigate the sign of solutions for a modified version of equation (1.4).

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Define

$$H^1_{\mathbf{O}(N)}(\mathbb{R}^N) := \{u \in H^1(\mathbb{R}^N) : u = gu, g \in \mathbf{O}(N)\}, \quad \text{where } gu := u \circ g^{-1}.$$

Then we have the following lemma.

Lemma 2.1 [13, page 18, Corollary 1.26]. *The following embedding is compact:*

$$H^1_{\mathbf{O}(N)}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N), \quad 2 < p < 2^*.$$

Lemma 2.2. *If u is a weak solution of the equation*

$$-2Au + \lambda u = (u^+)^{p-1}$$

in $H^1(\mathbb{R}^N)$, then $u \in C^2(\mathbb{R}^N)$.

Proof. (1) Note that the symbol σ_A of A is given by

$$\sigma_A(\xi) = -\frac{a}{2}|\xi|^2 + \int_{\mathbb{R}^N} [\cos(\xi \cdot x) - 1]\nu(dx),$$

where a is a positive number and ν is a finite $\mathbf{O}(N)$-invariant Lévy measure (see [1, page 128, Exercise 2.4.23 and pages 163–164, Theorem 3.3.3]).

Let A_2 be the operator with the symbol

$$\sigma_{A_2}(\xi) = -\frac{a}{2}|\xi|^2,$$

and A_0 be the operator with the symbol

$$\sigma_{A_0}(\xi) = \int_{\mathbb{R}^N} [\cos(\xi \cdot x) - 1]\nu(dx).$$

Then we have

$$-2A_2 u = h(\cdot)(1 + |u|),$$

where

$$h(x) := \frac{2A_0 u(x) + (u^+(x))^{p-1} - \lambda u(x)}{1 + |u(x)|} \quad \text{for } x \in \mathbb{R}^N.$$

(2) For any $u \in H^1(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (1 + |\xi|^2)\Big(\int_{\mathbb{R}^N} [\cos(\xi \cdot x) - 1]\nu(dx)\Big)^2 |\widehat{u}(\xi)|^2 \, d\xi < \infty, \tag{2.1}$$

where $\widehat{\ }$ denotes the Fourier transformation.

Thus $A_0 : H^1(\mathbb{R}^N) \to H^1(\mathbb{R}^N)$ is a bounded operator thanks to (2.1).

Furthermore, it follows that $h \in L^{N/2}_{\text{loc}}(\mathbb{R}^N)$. Consequently, we have $u \in L^q_{\text{loc}}(\mathbb{R}^N)$ for any $q \in [1, +\infty)$ by the Brézis–Kato theorem (see, for example, [12, page 270, B.3 Lemma]). Then, by the ellipticity of operator A, we find that $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^N)$ for any $q \in [1, +\infty)$. Now the Sobolev embedding theorem implies that $u \in C^1_{\text{loc}}(\mathbb{R}^N)$. Finally, also by the ellipticity of operator A, it follows that $u \in C^2(\mathbb{R}^N)$. □

Lemma 2.3. *If $u \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is a nontrivial solution of the equation*

$$-2Au + \lambda u = (u^+)^{p-1},$$

then $u > 0$.

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Proof. (1) First we have

$$\iint (u(x) - u(x+y))(u^-(x) - u^-(x+y))\nu(dy)\,dx$$
$$= \iint (u(x) - u(y))(u^-(x) - u^-(y))\nu(-x+dy)\,dx \le 0,$$

where we have used

$$\mathbb{R}^2 = \{x : u(x) \ge 0\} \times \{y : u(y) \ge 0\} \cup \{x : u(x) \ge 0\} \times \{y : u(y) < 0\}$$
$$\cup \{x : u(x) < 0\} \times \{y : u(y) \ge 0\} \cup \{x : u(x) < 0\} \times \{y : u(y) < 0\}$$

for the inequality. Then it follows that

$$(-2Au, -u^-)_{L^2} = a\|\nabla u^-\|^2_{L^2} - \iint (u(x) - u(x+y))(u^-(x) - u^-(x+y))\nu(dy)\,dx \ge 0.$$

Therefore, in light of $(-2Au, -u^-)_{L^2} + \lambda\|u^-\|^2_{L^2} = 0$, we have $u^- = 0$, which implies $u \ge 0$.
(2) Rewrite the equation $-2Au + \lambda u = (u^+)^{p-1}$ as

$$-2A_2u + (\lambda + 2\nu(\mathbb{R}^N))u = (u^+)^{p-1} + 2\int_{\mathbb{R}^N} u(\cdot + y)\nu(dy).$$

Then we find that

$$-2A_2u + (\lambda + 2\nu(\mathbb{R}^N))u \ge 0.$$

It follows from the strong maximum principle that $u > 0$. □

Corollary 2.4. *Assume that $u \in C^2(\mathbb{R}^N) \cap H^1(\mathbb{R}^N)$ is a nontrivial solution of the equation $-2Au + \lambda u = (u^+)^{p-1}$. If $x_0 \in \mathbb{R}^N$ is a maximum point of the function u, then $u(x_0) \ge \lambda^{1/(p-2)}$.*

Proof. (1) Since x_0 is a maximum point of the function u, we have $\Delta u(x_0) \le 0$.
(2) Note that Lemma 2.3 implies $u(x_0) > 0$. It follows from the positive maximum principle (see, for example, [11, page 283, (1.5) proposition] or [1, page 181, Theorem 3.5.2]) that $A_0u(x_0) \le 0$. This and $\Delta u(x_0) \le 0$ imply $Au(x_0) \le 0$. Therefore,

$$u(x_0)^{p-1} - \lambda u(x_0) = -2Au(x_0) \ge 0.$$

So the inequality $u(x_0) \ge \lambda^{1/(p-2)}$ holds. □

3. Proof of Theorem 1.2

In this section we provide a proof of Theorem 1.2 via the mountain pass theorem.

Observe that the operator $-A$ is positive self-adjoint (see [1, page 178, Theorem 3.4.10 and page 190, Theorem 3.6.1]). We define a new inner product on $H^1(\mathbb{R}^N)$ by

$$(v, w) := (-2Av, w)_{L^2} + \lambda(v, w)_{L^2}, \quad \text{for any } v, w \in C_0^\infty(\mathbb{R}^N),$$

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and denote its induced norm by $\|\cdot\|$. Since the operator $-A_0$ is also positive self-adjoint, it follows from $A = A_2 + A_0$ and (2.1) that the norm $\|\cdot\|$ is equivalent to $\|\cdot\|_{H^1}$.

Define a functional $E : H^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$E(u) := \frac{1}{2}\|u\|^2 - \frac{1}{p}\int_{\mathbb{R}^N} (u^+(x))^p \, dx.$$

Then it follows from [13, page 11, Corollary 1.13] that $E \in C^2(H^1(\mathbb{R}^N), \mathbb{R})$. In addition, the critical points of the functional E are weak solutions of the equation $-2Au + \lambda u = (u^+)^{p-1}$ in $H^1(\mathbb{R}^N)$, and vice versa.

Lemma 3.1. *The functional E is $\mathbf{O}(N)$-invariant.*

Proof. We only need to prove that the norm $\|\cdot\|$ is $\mathbf{O}(N)$-invariant.

Note that the symbol σ_A of A is given by

$$\sigma_A(\xi) = -\frac{a}{2}|\xi|^2 + \int_{\mathbb{R}^N} [\cos(\xi \cdot x) - 1]\nu(dx),$$

where a is a positive number and ν is a finite $\mathbf{O}(N)$-invariant Lévy measure (see [1, page 128, Exercise 2.4.23 and pages 163–164, Theorem 3.3.3]). We find that the symbol σ_A of A is $\mathbf{O}(N)$-invariant.

Therefore, for any $\varphi \in C_0^\infty(\mathbb{R}^N)$ and $g \in \mathbf{O}(N)$, we have

$$\begin{aligned}
\|g\varphi\|^2 &= (-2A(g\varphi), g\varphi)_{L^2} + \lambda\|g\varphi\|_{L^2}^2 \\
&= (-2\sigma_A \cdot \widehat{g\varphi}, \widehat{g\varphi})_{L^2} + \lambda\|g\varphi\|_{L^2}^2 \\
&= (-2g^{-1}\sigma_A \cdot \widehat{\varphi}, \widehat{\varphi})_{L^2} + \lambda\|g\varphi\|_{L^2}^2 \\
&= (-2\sigma_A \cdot \widehat{\varphi}, \widehat{\varphi})_{L^2} + \lambda\|\varphi\|_{L^2}^2 = \|\varphi\|^2,
\end{aligned}$$

which implies that the norm $\|\cdot\|$ is $\mathbf{O}(N)$-invariant. □

We need the following Lemma 3.2 in the verification of the Palais–Smale (PS) condition for the functional E restricted to $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$.

Lemma 3.2 [13, page 134, Theorem A.4]. *Assume that $1 \le p < \infty$, $1 \le q < \infty$, and $g \in C(\mathbb{R}^N)$ such that*

$$|g(u)| \le c|u|^{p/q} \quad \textit{for some constant } c.$$

Then the operator $L : L^p(\mathbb{R}^N) \to L^q(\mathbb{R}^N)$ defined by $u \mapsto g(u)$ is continuous.

Lemma 3.3 (Palais–Smale condition for the functional E restricted to $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$). *Any sequence $\{u_n\}_{n\in\mathbb{N}} \in H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$ such that*

$$d := \sup_{n\in\mathbb{N}}\{E(u_n)\} < \infty, \quad E'(u_n) \to 0, \text{ as } n \to \infty,$$

contains a convergent subsequence.

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Proof. The proof is the same as that of [13, page 15, Lemma 1.20].
(1) For n large enough, we have

$$d + \|u_n\| \geq E(u_n) - \frac{1}{p}\langle E'(u_n), u_n\rangle = \Big(\frac{1}{2} - \frac{1}{p}\Big)\|u_n\|^2.$$

It follows that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$ since $p > 2$.
(2) Without loss of generality, we assume that $u_n \rightharpoonup u$ in $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$. Then it follows from Lemma 2.1 that $u_n \to u$ in $L^p(\mathbb{R}^N)$. Consequently, by Lemma 3.2, we have $(u_n^+)^{p-1} \to (u^+)^{p-1}$ in $L^q(\mathbb{R}^N)$, where $q := p/(p-1)$.

Note that

$$\|u_n - u\|^2 = \langle E'(u_n) - E'(u), u_n - u\rangle + \int_{\mathbb{R}^N} (u_n^+(x)^{p-1} - u^+(x)^{p-1})(u_n(x) - u(x))\, dx. \tag{3.1}$$

For the first term of the right-hand side of the above equality, we see that

$$\langle E'(u_n) - E'(u), u_n - u\rangle \to 0, \quad \text{as } n \to \infty,$$

since $E'(u_n) \to 0$ as $n \to \infty$ and $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$.

And for the second term, it follows from the Hölder inequality that

$$\int_{\mathbb{R}^N} (u_n^+(x)^{p-1} - u^+(x)^{p-1})(u_n(x) - u(x))\, dx$$
$$\leq \|u_n^+(x)^{p-1} - u^+(x)^{p-1}\|_{L^q}\|u_n(x) - u(x)\|_{L^p} \to 0 \text{ as } n \to \infty,$$

because $u_n \to u$ in $L^p(\mathbb{R}^N)$ and $(u_n^+)^{p-1} \to (u^+)^{p-1}$ in $L^q(\mathbb{R}^N)$.

Therefore, $u_n \to u$ in $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$ as $n \to \infty$ by (3.1). □

Now we are in a position to give a proof of Theorem 1.2.

Proof of Theorem 1.2. (1) Consider the functional E restricted to $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$. Thanks to Lemma 2.1 or the Sobolev embedding theorem, there is a positive constant c such that $\|u\|_{L^p} \leq c\|u\|$ for any $u \in H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$. Then it follows from the definition of the functional E that

$$E(u) \geq \frac{1}{2}\|u\|^2 - \frac{c^p}{p}\|u\|^p.$$

Setting $r := (p/4c^p)^{1/(p-2)}$, we have

$$\inf_{\|u\|=r} E(u) \geq \frac{1}{4}\Big(\frac{p}{4c^p}\Big)^{2/(p-2)} > 0.$$

(2) Set $w(x) := \exp(-|x|^2)$. Then $w(x) \in H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$ and for any $t \in [0, +\infty)$,

$$E(tw) = \frac{t^2}{2}\|w\|^2 - \frac{t^p}{p}\|w\|^p_{L^p}.$$

Note that $p > 2$. We can take a positive number t such that $t\|w\| > r$ and $E(tw) < 0$.

(3) Now by the mountain pass theorem, there is a nontrivial critical point u of the functional E restricted to $H^1_{\mathbf{O}(N)}(\mathbb{R}^N)$. Note that the functional E is $\mathbf{O}(N)$-invariant. Thanks to the principle of symmetric criticality (see, for example, [13, page 18, Theorem 1.28]), it follows that the point u is also a critical point of the functional E. Consequently, the point u is a weak solution of the equation $-2Au + \lambda u = (u^+)^{p-1}$ in $H^1(\mathbb{R}^N)$.

(4) Finally, Lemmas 2.2 and 2.3 complete the proof. □

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